

# Theory of Quantum Crystals II

I. E. Dzyaloshinskii, P. S. Kondratenko, and V. S. Levchenkov

*Institute of Theoretical Physics, USSR Academy of Sciences*

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A microscopic justification of the phenomenological theory of Fermi-type quantum crystals, developed earlier in I, is given. The results are derived by the methods of quantum field theory, without using any assumptions concerning the nature of the interaction between the particles.

## 1. INTRODUCTION

IN a previous paper<sup>[1]</sup> (referred to below as I), we developed a phenomenological theory of Fermi-type quantum crystals possessing gapless Fermi excitations. Equations were found determining the spectrum of the long-wave vibrations of the crystal, and expressions for the thermodynamic quantities of the crystal were obtained. In the present paper, a microscopic justification of the phenomenological theory proposed in I is given. The results will be derived by the methods of quantum field theory, without using any assumptions concerning the interaction between the particles.

The presence of gapless Fermi excitations leads to a significant analogy between the properties of a quantum crystal and those of a Fermi liquid. In particular, as a result of the interaction between the quasi-particles at the Fermi surface, a quantum crystal, like a Fermi liquid, can possess Bose excitations of the zero-sound type. At the same time, the absence of spatial uniformity of the quantum crystal leads to a fundamental difference between it and a quantum liquid. According to Goldstone's theorem<sup>[2]</sup>, in a system whose ground state possesses lower symmetry than that of its Hamiltonian, provided that the broken symmetry has a continuous group, a gapless Bose excitation, restoring the broken symmetry, must exist. Clearly, in a crystal, the long-wave acoustic phonons are excitations of this type. We shall carry out an investigation of the vibration spectrum of a quantum crystal by studying the singularities of the two-particle vertex part.

In the second section, the properties of the single-particle Green's function of a quantum crystal will be considered. In the third and fourth sections, the singularities of the two-particle vertex part with respect to the momentum transfer will be determined. In the fifth section, a connection between the lattice deformation and the parameters characterizing the vibration spectrum of the crystal will be established. In the Appendix, a number of relations necessary for the derivation of the basic results are obtained.

## 2. THE GREEN'S FUNCTION

One of the basic field-theory quantities characterizing the properties of a many-particle system at  $T = 0$  is the single-particle Green's function<sup>[3]</sup>:

$$G(x, x') = -i \langle T(\psi(x)\psi^+(x')) \rangle, \quad (2.1)$$

where  $\psi(x)$  and  $\psi^+(x)$  are the Heisenberg particle annihilation and creation operators,  $T$  is the time-ordering operator, and  $\langle \dots \rangle$  denotes averaging over the ground state at  $T = 0$ . For brevity, here and in the following,

we omit the spin indices. We can assume formally that these are contained in the definition of the coordinates  $x = \{\mathbf{r}, t, \alpha\}$ , the integration over  $d^4x$  including a summation over  $\alpha$ . Unlike the case of an isotropic medium, for which the  $G$ -function depends on the space variables through their difference, the Green's function in the crystal conserves invariance only with respect to the transformations of the space group of the lattice. In addition, in the absence of external varying fields, the  $G$ -function depends only on  $t - t'$ . The latter fact and, also, the translational symmetry make it possible to write  $G$  in the form<sup>[4,5]</sup>

$$G(x, x') = \int \frac{dp}{(2\pi)^4} e^{ipx} G(p, \mathbf{r}, \mathbf{r}'), \quad (2.2)$$

where  $G(p, \mathbf{r}, \mathbf{r}')$  is a function that is periodic both in  $\mathbf{r}$  and in  $\mathbf{r}'$ ,  $p = \{\mathbf{p}, \epsilon\}$  is the quasi-momentum 4-vector,  $dp = d\mathbf{p}d\epsilon$ , and the integration over  $\mathbf{p}$  is bounded by the volume of a unit cell of the reciprocal lattice (the Brillouin zone). If we represent  $G(p, \mathbf{r}, \mathbf{r}')$  in the form of an expansion in a set of functions  $\psi_{\mathbf{n}\mathbf{p}}(\mathbf{r})$  that are periodic in the lattice, then  $G(x, x')$  takes the form

$$G(x, x') = \int \frac{dp}{(2\pi)^4} G_{nm}(p) \psi_{n\mathbf{p}}(\mathbf{r}) \psi_{m\mathbf{p}}^*(\mathbf{r}') e^{ip(x-x')} \quad (2.3)$$

(here and below, we use the convention that repeated indices are to be summed over).

For arbitrary values of  $\mathbf{p}$ , the matrix  $G_{nm}(\mathbf{p})$  is non-Hermitian. Therefore, it cannot be brought to diagonal form by a unitary transformation. However, as follows from the Lehmann expansion (see, e.g.,<sup>[3]</sup>), for  $\epsilon = 0$  the anti-Hermitian part of  $G_{nm}(\mathbf{p})$  vanishes. We shall find it convenient to take as the basis for the expansion of  $G(x, x')$  the system of functions  $\varphi_{\mathbf{n}\mathbf{p}}(\mathbf{r})$  defining the representation in which  $G_{nm}(\mathbf{p})$  is diagonal for  $\epsilon = 0$ . Below, we shall call this the band representation, and the labels  $n$  of the functions  $\varphi_{\mathbf{n}\mathbf{p}}(\mathbf{r})$  will be called the band indices.

As follows from Luttinger's theorem<sup>[5]</sup> (see also<sup>[3]</sup>), the number of particles per unit cell of the crystal, to within an even integer (we are considering a system consisting of spin- $1/2$  particles), is proportional to the sum of the volumes enclosed within the surfaces at which  $G_n(0, \mathbf{p})$  goes either to infinity or to zero. We shall assume that  $G_n(0, \mathbf{p})$  becoming infinite for values of  $\mathbf{p}$  lying on a certain surface corresponds to a pole of  $G_n(\epsilon, \mathbf{p})$  for  $\epsilon \rightarrow 0$  and for values of  $\mathbf{p}$  close to the indicated surface (we denote the index of such a band by  $f$ ):

$$G_f(\epsilon, \mathbf{p}) \approx \frac{a(\mathbf{p})}{\epsilon - \epsilon(\mathbf{p}) + i\delta \text{sign } \epsilon(\mathbf{p})}. \quad (2.4)$$

This means that, for  $\epsilon \rightarrow 0$ , the crystal possesses weakly attenuating Fermi excitations with dispersion law  $\epsilon = \epsilon(\mathbf{p})$ ;  $\epsilon(\mathbf{p})$  vanishes at the above-mentioned surface,

which we shall call the Fermi surface.<sup>1)</sup> The simplest example of a surface at which the G-function vanishes at  $\epsilon = 0$  is the Fermi surface of electrons in a superconducting metal. The vanishing of the pole at  $\epsilon = 0$  in this case is connected with the formation of a gap in the Fermi spectrum, due to the appearance of bound Cooper pairs. The crystal as a whole is also a bound state, and, therefore, the presence of a gap in certain branches of the single-particle spectrum is completely natural. Such a possibility enables us to explain the presence of defect Fermi surfaces of small volume in a crystal with an odd number of sites per unit cell and with a small difference between the numbers of sites and particles.

### 3. THE VERTEX PART FOR $\mathbf{k} = 0$

Whereas the poles of the single-particle Green's function determine the spectrum of the Fermi-type excitations, information on the Bose branches of the spectrum of a Fermi system is contained in the two-particle vertex part  $\Gamma(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4)$ . It is connected by a standard relation<sup>[3]</sup> with the two-particle Green's function

$$G(x_1, x_2; x_3, x_4) = \langle T(\psi(x_1)\psi(x_2)\psi^+(x_3)\psi^+(x_4)) \rangle$$

and, like the single-particle G-function, is invariant under a simultaneous displacement of all its space variables by an arbitrary translation vector of the crystal lattice. This fact, together with the invariance with respect to time translations, enables us to represent the function  $\Gamma(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4)$  in the form

$$\Gamma(x_1, x_2; x_3, x_4) = \int \frac{d\mathbf{p} d\mathbf{p}' d\mathbf{k}}{(2\pi)^{12}} \Gamma_{nmn'm'}(p, p'; k) \varphi_{n\mathbf{p}}(\mathbf{r}_1) \varphi_{m\mathbf{p}+\mathbf{k}}(\mathbf{r}_2) \times \varphi_{n'\mathbf{p}+\mathbf{k}}(\mathbf{r}_3) \varphi_{m'\mathbf{p}}(\mathbf{r}_4) \exp[ip(x_1 - x_3) + ip'(x_2 - x_4) + ik(x_2 - x_3)]. \quad (3.1)$$

Here,  $\mathbf{k} = \{\mathbf{k}, \omega\}$  is the quasi-momentum-transfer 4-vector. As in (2.2) also, the integrals over  $\mathbf{p}$ ,  $\mathbf{p}'$  and  $\mathbf{k}$  are taken over the volume of the Brillouin zone; in the case when  $\mathbf{p} + \mathbf{k}$  or  $\mathbf{p}' + \mathbf{k}$  extends beyond its limits, we propose to use the periodicity condition of the Bloch functions  $\varphi_{n\mathbf{p}}(\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{r}}$  in the space of the reciprocal lattice.

We proceed to investigate the singularities of the vertex part at small quasi-momentum transfers. First of all, it is clear that, in view of the presence of weakly attenuating Fermi excitations, the function  $\Gamma(p, p'; k)$ , as in the case of an isotropic Fermi liquid<sup>[6]</sup>, can possess singularities of the zero-sound type. With these one associates diagrams containing Green's-function loops, for which the difference in the quasi-momenta is equal to  $\mathbf{k}$ . One can convince oneself of this by considering the equation connecting  $\Gamma$  with the irreducible vertex  $\tilde{\Gamma}$ <sup>[3]</sup>.

$$\Gamma_{nmn'm'}(p, p'; k) = \Gamma_{nmn'm'}^{(1)}(p, p'; k) - i \int \frac{d\mathbf{p}_1}{(2\pi)^4} \Lambda_{nm_1n'm_1'}^{(1)}(p, p_1; k) \times G_{m_1'm_1}(p_1) G_{m_2'm_2}(p_1 + k) \Gamma_{m_2m_2'm_1'}(p_1, p'; k). \quad (3.2)$$

$\Gamma^{(1)}$  is determined by the set of graphs for  $\Gamma$  which cannot be cut into two G-lines with quasi-momentum difference equal to  $\mathbf{k}$ . A singularity of  $\Gamma$  arises as a result of the non-regularity of the kernel of Eq. (3.2), which is due to the integration over  $p_1$ , in the vicinity of the Fermi

surface, of the term with  $m_1' = m_2 = m_2' = m_1 = \mathbf{f}$ , when the poles of the G-functions come together as  $\mathbf{k} \rightarrow 0$ . On the other hand, as already noted, in a quantum crystal, excitations of the type of long-wave acoustic phonons should occur. The presence of the corresponding pole in  $\Gamma$  stems from the relation (A.6) obtained in the Appendix

$$\omega^{-1}[P_{nm'}^i(p)G_{m'm}^{-1}(p + \tilde{\mathbf{k}}) - G_{nm'}^{-1}(p)P_{m'm}^i(p)] = Q_{nm'}^i(p, \tilde{\mathbf{k}}), \quad (3.3)$$

where

$$Q_{nm'}^i(p, \tilde{\mathbf{k}}) = P_{nm'}^i(p) - i \int \frac{d\mathbf{p}'}{(2\pi)^4} \Gamma_{nn'mm'}(p, p'; \tilde{\mathbf{k}}) \times G_{m'm'}(p') P_{m''n''}^i(p') G_{n''n'}(p' + \tilde{\mathbf{k}}), \quad (3.4)$$

$$P_{nm'}^i(p) = \int \frac{d\mathbf{r}}{v_c} \varphi_{n\mathbf{p}}^*(\mathbf{r}) e^{-i\mathbf{p}\cdot\mathbf{r}} \left( -i \frac{\partial}{\partial r_i} \right) \varphi_{m\mathbf{p}}(\mathbf{r}) e^{i\mathbf{p}\cdot\mathbf{r}}, \quad (3.5)$$

$\tilde{\mathbf{k}} = \{0, \omega\}$ , and  $v_c$  is the volume of a unit cell of the lattice. It can be seen from these equalities that the vertex part  $\Gamma$  for  $\mathbf{k} = 0, \omega \rightarrow 0$  has a singularity of the type  $\omega^{-1}$  or stronger. This singularity is not of the zero-sound type, since, as follows from the theory of a Fermi liquid<sup>[6]</sup>, the latter is absent in the limit  $\mathbf{k} = 0, \omega \rightarrow 0$ .

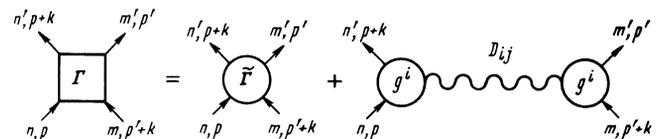
We stress that the residue at the pole of  $\Gamma(p, p'; \tilde{\mathbf{k}})$  vanishes in the transition to a uniform system, as a result of the fact that, in this case,  $P_{nm}^i(p), G_{nm}(p) \sim \delta_{nm}$ . This means that the pole under consideration is associated with inter-band transitions and, thus, being a specific property of the crystalline state, is a result of the broken symmetry of the ground state.

It is clear that, at small but non-zero  $\mathbf{k}$ , the pole of  $\Gamma$  occurring at  $\mathbf{k} = 0$  remains, having been displaced to one side ( $\omega \neq 0$ ). Therefore, we write  $\Gamma$  in the form of a sum of two terms, one of which ( $\tilde{\Gamma}$ ) is regular for  $\mathbf{k} = 0, \omega \rightarrow 0$ , while the other part contains the singularity of interest:

$$\Gamma_{nmn'm'}(p_1, p_2; k) = \tilde{\Gamma}_{nmn'm'}(p_1, p_2; k) + g_{nm'}^i(p_1, k) D_{ij}(k) g_{mm'}^j(p_2 + k, -k), \quad (3.6)$$

where  $g_{nm}^i(p, k)$  is regular at  $\mathbf{k} = 0, \omega \rightarrow 0$ , and  $D_{ij}(k)$  contains the singularity. The indices  $i$  and  $j$  in  $D_{ij}$  have arisen in connection with the fact that the relation (3.3) has three independent components. Therefore,  $g_{nm}^i(p, k)$  has a vector character, and  $D_{ij}(k)$  has the properties of a tensor. The relation (3.6) is depicted graphically in the figure. Because of the vector character of the residues at the pole and by virtue of the character of the broken symmetry of the ground state of the crystal, the function  $D_{ij}(k)$ , reduced to diagonal form, clearly determines the spectrum of the three branches of the acoustic vibrations of the crystal. Therefore, below, we shall call the function  $D_{ij}$  the phonon Green's function, and shall call  $g_{nm}^i(p, k)$  the vertex of the fermion-phonon interaction.

We shall study how to determine the functions  $g_{nm}^i(p, k)$  and  $D_{ij}(k)$  for  $\mathbf{k} = 0$ . We introduce for this the quantity  $\tilde{Q}_{nm}^i(p, \tilde{\mathbf{k}})$ , which differs from  $Q_{nm}^i(p, \tilde{\mathbf{k}})$  by the replacement of  $\Gamma(p, p'; k)$  by  $\tilde{\Gamma}(p, p'; k)$  in (3.4), and the functions  $\tilde{R}_{ij}^i(\tilde{\mathbf{k}})$  and  $A_{ij}^i(\tilde{\mathbf{k}})$ , defined by the following relations:



<sup>1)</sup>Unlike in I, the chemical potential  $\mu$  is assumed here to be an independent variable. Therefore, the energy of the quasiparticles is reckoned from  $\mu$ .

$$\tilde{R}_{ij}(\tilde{k}) = -i \int \frac{dp}{(2\pi)^4} G_{nn'}(p + \tilde{k}) P_{n'm'}^i(p) G_{m'm}(p) \tilde{Q}_{mn}^i(p, \tilde{k}), \quad (3.7)$$

$$A_{ij}(\tilde{k}) = -i \int \frac{dp}{(2\pi)^4} g_{nm}^i(p + \tilde{k}, -\tilde{k}) G_{m'm'}(p) P_{m'n'} G_{n'n}(p + \tilde{k}). \quad (3.8)$$

Expressed in terms of  $\tilde{Q}_{nm}^i(p, \tilde{k})$  and  $A_{ij}(\tilde{k})$ , the relation (3.3) takes the form

$$\begin{aligned} \omega^{-1} [P_{nm'}^i(p) G_{m'm}^{-1}(p + \tilde{k}) - G_{nm'}^{-1}(p) P_{m'n'}^i(p)] \\ = \tilde{Q}_{nm}^i(p, \tilde{k}) + g_{nm}^i(p, \tilde{k}) D_{ij}(\tilde{k}) A_{ij}(\tilde{k}). \end{aligned} \quad (3.9)$$

Hence, after multiplying by  $G_{mm'}(p + \tilde{k}) P_{m'n'}^j(p) G_{n'n}(p)$  and integrating over  $p$ , we obtain the equality

$$\tilde{R}_{ij}(\tilde{k}) + A_{ij}(-\tilde{k}) D_{ij}(\tilde{k}) A_{ij}(\tilde{k}) = 0. \quad (3.10)$$

The quantities  $\tilde{R}_{ij}(\tilde{k})$ ,  $A_{ij}(\tilde{k})$  and  $D_{ij}(\tilde{k})$  do not contain the space components of the momentum transfer  $k$  (we recall that  $\tilde{k} = \{0, \omega\}$ ) and are second-rank tensors. The symmetry requirements of the system lead to the result that they are symmetric; in the case of cubic crystals they are proportional to the unit tensor, and in uniaxial crystals they have two independent components each and are commutative.<sup>2)</sup>

Because of the fact that, according to (3.9), to within a factor independent of  $p$ ,  $g_{nm}^i(p, 0)$  is equal to

$$P_{nm'}^i(p) G_{m'm}^{-1}(p) - G_{nm'}^{-1}(p) P_{m'n'}^i(p),$$

it follows from (3.8) that  $A_{ij}(0) = 0$ . Therefore,  $A_{ij}(\tilde{k}) \sim \omega$  and, in accordance with (3.10),  $D_{ij}(\tilde{k}) \sim \omega^{-2}$ :

$$D_{ij}(\tilde{k}) = \rho_{ij}^{-1} \omega^{-2}. \quad (3.11)$$

The structure of the relation (3.6) is such that the choice of the functions  $g_{nm}^i(p, k)$  and  $D_{ij}(k)$  is arbitrary to within multiplicative tensor constants, which are such that the second term of the right-hand side of (3.6) remains fixed. We put

$$\rho_{ij} = -\tilde{R}_{ij} \equiv \lim_{k \rightarrow 0, \omega \rightarrow 0} \tilde{R}_{ij}(k). \quad (3.12)$$

(Here and in the following, quantities with the index  $\omega$  denote the limit corresponding to  $k = 0$ ,  $\omega \rightarrow 0$ . Analogously, the index  $k$  will denote the limit of a function of the momentum transfer for  $\omega = 0$ ,  $k \rightarrow 0$ .) Taking (3.12) into account, from (3.9) and (3.10) we obtain

$$g_{nm}^i(p, \tilde{k}) = -i [P_{nm'}^i(p) G_{m'm}^{-1}(p + \tilde{k}) - G_{nm'}^{-1}(p) P_{m'n'}^i(p) - \omega \tilde{Q}_{nm}^{i(\omega)}(p)]. \quad (3.13)$$

In the expression (3.13), terms quadratic and of higher order in the frequency have been omitted, since, from the very nature of the definition of the  $g_{nm}^i(p, k)$ , it is meaningful to treat them only to this level of exactness. In the case  $n = m = f$  with  $\epsilon, \epsilon(p) \rightarrow 0$ , using formula (2.4) we obtain

$$g_f^i(p, \tilde{k}) = -i\omega \left[ \frac{P_i(p)}{a(p)} - \tilde{Q}_f^{i(\omega)}(p) \right], \quad (3.14)$$

where  $P_i(p) = P_{nm}^i(p)$  for  $n = m = f$ . The connection between the quantities  $\tilde{Q}_f^{i(\omega)}(p)$ ,  $\rho_{ij} = -\tilde{R}_{ij}^{\omega}$  and the parameters characterizing the properties of the quasi-particles at the Fermi surface will be established in the next Section.

#### 4. THE VERTEX PART FOR $k \neq 0$

In order to determine the nature of the singularities of the vertex part for  $k \neq 0$ , we make use of Eq. (3.2). In analogy with the case of the isotropic Fermi liquid,<sup>[3]</sup> the extraction of the singularity in the kernel of the equation can be performed by a formal decomposition of the product of G-functions into a regular and a singular term:

$$\begin{aligned} G_{nm}(p) G_{n'm'}(p + k) = \overline{G}_{nm}(p) G_{n'm'}(p + k) \\ + 2\pi i a^2(p) \delta(\epsilon) \delta(\epsilon(p)) \frac{vk}{\omega - vk} \delta_{nj} \delta_{m'} \delta_{n'} \delta_{m'}, \end{aligned} \quad (4.1)$$

where  $\mathbf{v} \equiv \mathbf{v}(p) = \partial \epsilon(p) / \partial \mathbf{p}$  is the velocity of the Fermi excitations at the Fermi surface. The second term of (4.1) corresponds to the contribution of the product of G-functions in an integral over the vicinity of the surface  $\epsilon(p) = 0$ ,  $\epsilon = 0$ . We introduce the quantity  $\overline{\Gamma}_{nn'mm'}(p_1, p_2; k)$  satisfying Eq. (3.2) in which the replacement  $GG \rightarrow \overline{G}\overline{G}$  has been made. Since, as follows from (4.1),

$$\overline{G}(p) \overline{G}(p + \tilde{k}) = G(p) G(p + \tilde{k}),$$

we have

$$\overline{\Gamma}(p_1, p_2; \tilde{k}) = \Gamma(p_1, p_2; \tilde{k}).$$

We write out the equations for  $\Gamma(p_1, p_2; k)$ ,  $\overline{\Gamma}(p_1, p_2; k)$  and  $\Gamma(p_1, p_2; \tilde{k})$  stemming from (3.2):

$$\begin{aligned} \Gamma_{nn'mm'}(p_1, p_2; k) = \Gamma_{nn'mm'}^{(1)}(p_1, p_2; k) - i \int \frac{dp'}{(2\pi)^4} \Gamma_{n'm's}^{(1)}(p_1, p'; k) \cdot \\ \cdot \overline{G_{s'l'}(p') G_{s'l'}(p' + k)} \Gamma_{l'n's'm'}(p', p_2; k) + \int dS' \Gamma_{n'm'l'}^{(1)}(p_1, p'; k) \cdot \\ \cdot a^2(p') \frac{v'k}{\omega - v'k} \Gamma_{l'n's'm'}(p', p_2; k), \end{aligned} \quad (4.2)$$

$$\begin{aligned} \overline{\Gamma}_{nn'mm'}(p_1, p_2; k) = \Gamma_{nn'mm'}^{(1)}(p_1, p_2; k) - i \int \frac{dp'}{(2\pi)^4} \Gamma_{n'm's}^{(1)}(p_1, p'; k) \cdot \\ \cdot \overline{G_{s'l'}(p') G_{s'l'}(p' + k)} \Gamma_{l'n's'm'}(p', p_2; k), \end{aligned} \quad (4.3)$$

$$\begin{aligned} \Gamma_{nn'mm'}(p_1, p_2; \tilde{k}) = \Gamma_{nn'mm'}^{(1)}(p_1, p_2; \tilde{k}) - i \int \frac{dp'}{(2\pi)^4} \Gamma_{n'm's}^{(1)}(p_1, p'; \tilde{k}) \cdot \\ \cdot G_{s'l'}(p') G_{s'l'}(p' + \tilde{k}) \Gamma_{l'n's'm'}(p', p_2; \tilde{k}). \end{aligned} \quad (4.4)$$

(Here we have used the notation  $(2\pi)^{-4} \int dp \delta(\epsilon) \delta(\epsilon(p)) = \int dS$ .)

From a comparison of (4.3) with (4.4) and (3.6), we find that the solution for  $\overline{\Gamma}$  can be represented in the form

$$\overline{\Gamma}_{nn'mm'}(p_1, p_2; k) = \tilde{\Gamma}_{nn'mm'}^{\omega}(p_1, p_2) + g_{nn'}^i(p_1, k) D_{ij}^0(k) g_{mm'}^j(p + k, -k). \quad (4.5)$$

For small  $k$ , the quantity  $g_{nm}^i(p, k)$  differs from  $g_{nm}^i(p, \tilde{k})$  by a term proportional to the vector  $k$ , and the function  $D_{ij}^0(k)$  is expressed in terms of  $D_{ij}(\tilde{k})$  through the relation

$$[D_{ij}^0(k)]^{-1} = D_{ij}^{-1}(\tilde{k}) - \lambda_{i1s} k_i k_s, \quad (4.6)$$

where  $\lambda_{i1s}$  is a fourth-rank tensor satisfying the symmetry requirements of the crystal, so that

$$\lambda_{i1s} = \lambda_{s1i} = \lambda_{j1t}. \quad (4.7)$$

These relations follow from the actual structure of the vertex part. The tensor  $\lambda_{i1s}$  and the term in  $\overline{\Gamma}_{nm}^i(p, k)$  that is linear in the wave vector  $k$  could be expressed in terms of  $\overline{\Gamma}(p_1, p_2; \tilde{k})$  and  $\Gamma^{(1)}(p_1, p_2; k)$ , but since  $\Gamma^{(1)}$  is an unobservable quantity, we shall leave them as independent parameters.

<sup>2)</sup>It is known that crystalline He<sup>3</sup> exists either in one of the cubic modifications or in the hexagonal modification.

It is easily seen that from (4.2) and (4.4) we can obtain an equation connecting  $\Gamma$  with  $\bar{\Gamma}$ . Since the singularities of the vertex  $\Gamma_{nmn'm'}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{k})$  in the momentum transfer have a factorizable form, i.e., do not depend on the band indices or the variables  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , we write out the required equation for the component of  $\Gamma$  with  $n = n' = m = m' = f$  for  $\epsilon_1 = \epsilon_2 = 0$  and for  $\mathbf{p}_1$  and  $\mathbf{p}_2$  lying on the Fermi surface (i.e.,  $\epsilon(\mathbf{p}_1) = 0, \epsilon(\mathbf{p}_2) = 0$ ):

$$\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k) = \mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k) + \int dS' \mathcal{F}(\mathbf{p}_1, \mathbf{p}'_1; k) \frac{v'k}{\omega - v'k} \mathcal{F}(\mathbf{p}'_1, \mathbf{p}_2; k). \quad (4.8)$$

Here, we have used the notation

$$\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k) = a(\mathbf{p}_1) a(\mathbf{p}_2) \Gamma_{ffff}(\mathbf{p}_1, \mathbf{p}_2; k) |_{\epsilon_1 = \epsilon_2 = 0}, \quad (4.9)$$

$$\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k) = a(\mathbf{p}_1) a(\mathbf{p}_2) \bar{\Gamma}_{ffff}(\mathbf{p}_1, \mathbf{p}_2; k) |_{\epsilon_1 = \epsilon_2 = 0}. \quad (4.10)$$

If we also use the notation

$$f(\mathbf{p}_1, \mathbf{p}_2) = a(\mathbf{p}_1) a(\mathbf{p}_2) \bar{\Gamma}_{ffff}^{\omega}(\mathbf{p}_1, \mathbf{p}_2) |_{\epsilon_1 = \epsilon_2 = 0}, \quad (4.11)$$

$$i(-\omega \xi_i(\mathbf{p}) + k_i \zeta_i(\mathbf{p})) = a(\mathbf{p}) g_{ff'}(\mathbf{p}, k) |_{\epsilon=0}, \quad (4.12)$$

and take (4.5) into account, the function  $\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k)$  takes the form

$$\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k) = f(\mathbf{p}_1, \mathbf{p}_2) + (\omega \xi_i(\mathbf{p}_1) - k_i \zeta_i(\mathbf{p}_1)) \times D_{ij}^{\omega}(k) (\omega \xi_j(\mathbf{p}_2) - k_m \zeta_{mj}(\mathbf{p}_2)). \quad (4.13)$$

If we use the equality (4.13), the solution of Eq. (4.8) for  $\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k)$  can be written in the form

$$\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k) = \tilde{\mathcal{F}}(\mathbf{p}_1, \mathbf{p}_2; k) + (\omega \xi_i(\mathbf{p}_1, k) - k_i \zeta_i(\mathbf{p}_1, k)) D_{ij}(k) (\omega \xi_j(\mathbf{p}_2, k) - k_m \zeta_{mj}(\mathbf{p}_2, k)), \quad (4.14)$$

where  $\tilde{\mathcal{F}}(\mathbf{p}_1, \mathbf{p}_2; k)$ ,  $\xi_i(\mathbf{p}, k)$  and  $\zeta_{ij}(\mathbf{p}, k)$  satisfy the equations

$$\tilde{\mathcal{F}}(\mathbf{p}_1, \mathbf{p}_2; k) = f(\mathbf{p}_1, \mathbf{p}_2) + \int dS' f(\mathbf{p}_1, \mathbf{p}'_1) \frac{v'k}{\omega - v'k} \tilde{\mathcal{F}}(\mathbf{p}'_1, \mathbf{p}_2; k), \quad (4.15)$$

$$\xi_i(\mathbf{p}, k) = \xi_i(\mathbf{p}) + \int dS' f(\mathbf{p}, \mathbf{p}') \frac{v'k}{\omega - v'k} \xi_i(\mathbf{p}', k), \quad (4.16)$$

$$\zeta_{ii}(\mathbf{p}, k) = \zeta_{ii}(\mathbf{p}) + \int dS' f(\mathbf{p}, \mathbf{p}') \frac{v'k}{\omega - v'k} \zeta_{ii}(\mathbf{p}', k), \quad (4.17)$$

and, if we take (4.6) and (3.11) into account,  $D_{ij}(k)$  is determined by the relations

$$D_{ij}^{-1}(k) = \rho_{ij} \omega^2 - T_{ij}(k), \quad (4.18)$$

$$T_{ij}(k) = \lambda_{iim} k_i k_m + \int dS \left\{ (\omega \xi_i(\mathbf{p}) - k_i \zeta_{ii}(\mathbf{p})) \frac{v\mathbf{k}}{\omega - v\mathbf{k}} (\omega \xi_j(\mathbf{p}, k) - k_m \zeta_{mj}(\mathbf{p}, k)) \right\}. \quad (4.19)$$

It can be shown that, since the singularities in  $\mathbf{k}$ , determined by Eqs. (4.15)–(4.17), of the functions  $\tilde{\mathcal{F}}(\mathbf{p}_1, \mathbf{p}_2; k)$ ,  $\xi_i(\mathbf{p}, k)$  and  $\zeta_{ij}(\mathbf{p}, k)$  are common to all three functions, this leads to their cancellation in the expression (4.14) for  $\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k)$ . But the true singularities of  $\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k)$  correspond to poles of the function  $D_{ij}(k)$ . The dispersion law, i.e., the dependence  $\omega = \omega(\mathbf{k})$ , at the pole is determined by the secular equation

$$[\rho_{ij} \omega^2 - T_{ij}(k)] u_j(k) = 0. \quad (4.20)$$

This equation, and also the dependences on  $\mathbf{k}$  of the quantities  $T_{ij}(k)$ ,  $\xi_i(\mathbf{p}, k)$  and  $\zeta_{ij}(\mathbf{p}, k)$  coincide with Eq.

(I.3.11), obtained by a phenomenological method in I, and with the dependences determined by Eqs. (I.3.13) and (I.3.14).

Another form of the dispersion equation for the Bose branches of the excitation spectrum of a quantum crystal can be obtained from (4.8), if we note that, in the vicinity of the pole of  $\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k)$ , we can neglect the term  $\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k)$  in the right-hand side of (4.8), and, in view of the fact that the variable  $\mathbf{p}_2$  in this equation plays the role of a parameter, represent  $\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k)$  in the form of a product:

$$\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k) = \frac{\omega - v_1 k}{v_1 k} v(\mathbf{p}_1, k) \frac{\omega - v_2 k}{v_2 k} v(\mathbf{p}_2, k).$$

After this, we obtain the following homogeneous equation for  $v(\mathbf{p}, k)$ :

$$(\omega - v\mathbf{k}) v(\mathbf{p}, k) = v\mathbf{k} \int dS' \mathcal{F}(\mathbf{p}, \mathbf{p}'; k) v(\mathbf{p}', k). \quad (4.21)$$

This equation coincides exactly with the dispersion equation (I.3.8). According to (4.13), (4.6), (3.11), (I.3.9) and (I.3.10), the  $\mathbf{k}$ -dependences of the function  $\mathcal{F}(\mathbf{p}_1, \mathbf{p}_2; k)$  appearing in (4.21) and (I.3.8) are identical.

Thus, on the basis of a microscopic treatment, we have proved the correctness of the dispersion equations obtained in I for the Bose-excitation branches of a Fermi crystal, which correspond to coupled oscillations of the zero-sound and phonon types. According to (4.11) and (4.5), the Landau function introduced in I for the Fermi quasi-particles of the crystal is proportional to the  $\omega$ -limit of the irreducible (with respect to the phonons) vertex part  $\bar{\Gamma}^{\omega}(\mathbf{p}_1, \mathbf{p}_2)$ .

To conclude this section, we shall express the quantities  $\xi_i(\mathbf{p})$  and  $\rho_{ij}$  in terms of parameters characterizing the properties of the quasi-particles at the Fermi surface. We note that the function  $\bar{Q}_f^{\omega}(\mathbf{p})$ , with which the quantity  $\xi_i(\mathbf{p})$  is connected through the relations (4.12) and (3.14), can be expressed in terms of  $\bar{Q}_f^{\omega}(\mathbf{p})$  by the following formula:

$$a(\mathbf{p}) \bar{Q}_f^{\omega}(\mathbf{p}) = a(\mathbf{p}) \bar{Q}_f^{(h)}(\mathbf{p}) + \int dS' f(\mathbf{p}, \mathbf{p}') a(\mathbf{p}') \bar{Q}_f^{(h)}(\mathbf{p}'), \quad (4.22)$$

which follows from (4.15) and the definition (A.4) of  $\bar{Q}_{nm}^{\omega}(\mathbf{p}, k)$  with the replacement  $\Gamma \rightarrow \bar{\Gamma}$ . Substituting (4.22) into (3.13) and (4.12) and taking into account the equality

$$a(\mathbf{p}) \bar{Q}_f^{(h)}(\mathbf{p}) = a(\mathbf{p}) Q_f^{(h)}(\mathbf{p}) = m v_i, \quad (4.23)$$

which stems from (A.7) and (2.4), we obtain an expression for  $\xi_i(\mathbf{p})$  that coincides with the formula (I.2.7).

We now study the transformation of the tensor  $\rho_{ij}$ , which, according to (3.11) and (A.13), is equal to

$$\rho_{ij} = -\bar{R}_{ij}^{\omega} = i \lim_{k \rightarrow 0, \omega \rightarrow 0} \int \frac{d\mathbf{p}}{(2\pi)^3} G_{mm'}(p+k) P_{m'n'}^i(\mathbf{p}) G_{n'n}(p) \bar{Q}_{nm}^{\omega}(p, k). \quad (4.24)$$

By making use of the relation between the two limiting values ( $\mathbf{k} = 0, \omega \rightarrow 0$  and  $\omega = 0, \mathbf{k} \rightarrow 0$ ), of the product of G-functions, which follows from (4.1), and using the relations (4.23) and (4.24), and also the equality

$$\bar{R}_{ij}^h = R_{ij}^h = -m N_0 \delta_{ij}, \quad (4.25)$$

which is a consequence of (A.7) and (A.18), we shall have a formula for  $\rho_{ij}$  that coincides with (I.2.9). We remark, finally, that the functions  $f(\mathbf{p}, \mathbf{p}')$ ,  $\zeta_{ik}^k(\mathbf{p})$  and  $\lambda_{iklm}$  are related to the forward scattering amplitude of

the quasi-particles at the Fermi surface; this amplitude, as in the case of a Fermi liquid<sup>[6]</sup>, is expressed in terms of the  $k$ -limit of the vertex  $\Gamma(p, p'; k)$ .

### 5. CONNECTION BETWEEN THE CHARACTERISTICS OF THE PHONON SPECTRUM AND THE LATTICE DEFORMATION

With the aim of further elucidating the physical meaning of the quantities appearing in the microscopic theory, we shall determine the linear response of a crystal-lattice structure to a weak long-wave (and low-frequency) external scalar field, given by the potential  $\delta\varphi e^{-ikx}$  ( $kx = \mathbf{k} \cdot \mathbf{r} - \omega t$ ). The lattice deformation arising under the action of the external field can be found using the following formal device. We introduce an arbitrary integral operator with a difference kernel  $\mathcal{L}(\mathbf{r} - \mathbf{r}')$  that falls off at distances of the order of the dimensions of the lattice unit cell; this operator has the property of orthogonality to a constant:

$$\hat{\mathcal{L}}\text{const} \equiv \int d\mathbf{r} \mathcal{L}(\mathbf{r} - \mathbf{r}') \text{const} = 0.$$

The result of the action of an operator with such properties on a periodic function is a function that is also periodic, its integral over its period being equal to zero. In particular,

$$\int_{\tilde{v}_c} d\mathbf{r} \hat{\mathcal{L}}n(\mathbf{x}) = 0, \quad (5.1)$$

where

$$n(\mathbf{x}) = -i \int \frac{dp}{(2\pi)^4} e^{i\epsilon_0 G_{nm}(p)} \varphi_{np}(\mathbf{r}) \varphi_{m\mathbf{p}}^*(\mathbf{r}), \quad \delta \rightarrow +0 \quad (5.2)$$

is the microscopic particle-number density of the crystal, and  $v_c$  is the volume of a unit cell of the lattice. In the case of a weak perturbation with nonuniformity of characteristic scale substantially greater than the period of the lattice, the relation (5.1) can be written:

$$\int_{\tilde{v}_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}}n_0(\mathbf{r}) + \int_{v_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}}\delta n(\mathbf{r}, t) = 0, \quad (5.3)$$

where  $\tilde{v}_c(\mathbf{R}, t)$  is the volume of a unit cell of the deformed lattice; the position of this cell is defined by the macroscopic coordinate  $\mathbf{R}$  at time  $t$ .

We shall transform the first term of (5.3). If we introduce the deformation tensor (unsymmetrized)

$$w_{ik}(\mathbf{R}, t) = \frac{\partial u_k(\mathbf{R}, t)}{\partial R_i}, \quad (5.4)$$

where  $u_i(\mathbf{R}, t)$  is the displacement vector of the lattice sites (and not of the particles), then, on changing to a system of coordinates  $\mathbf{r}'$  by the formula  $\mathbf{r}_i = (\delta_{ik} + w_{ik})\mathbf{r}'_k$ , the integration in the new variables in the first term of (5.3) will be over a region whose shape and volume are the same as those of the unit cell of the unperturbed lattice. Taking into account the fact that the Jacobian of the transformation in the lowest order in  $w_{ik}$  is equal to  $1 + w_{ll}$ , we obtain

$$\int_{\tilde{v}_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}}n_0(\mathbf{r}) = \int_{v_c} d\mathbf{r}' \int d\mathbf{r}'' \mathcal{L}[(\hat{1} + \hat{w}(\mathbf{R}, t))(\mathbf{r}' - \mathbf{r}'')] n_0[(\hat{1} + \hat{w}(\mathbf{R}, t))\mathbf{r}'] \times (1 + w_{ll}(\mathbf{R}, t))^2. \quad (5.5)$$

Retaining only the terms of first order in  $w_{ik}$ , we shall have

$$\int_{\tilde{v}_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}}n_0(\mathbf{r}) = C_{lm} w_{lm}(\mathbf{R}, t), \quad (5.6)$$

$$C_{lm} = \int_{v_c} d\mathbf{r} \hat{\mathcal{L}}r_m \frac{\partial n_0(\mathbf{r})}{\partial r_l}. \quad (5.7)$$

We shall study the transformation of the second term in (5.3):

$$\int_{v_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}}\delta n(\mathbf{r}, t) = -i \int \frac{dp}{(2\pi)^4} e^{i\omega t} \delta G_{nm}(p, k) \int_{v_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}} e^{-i\mathbf{k}\mathbf{r}} \varphi_{np}(\mathbf{r}) \varphi_{m\mathbf{p}+\mathbf{k}}^*(\mathbf{r}). \quad (5.8)$$

Since  $\varphi_{np}(\mathbf{r}) \varphi_{m\mathbf{p}+\mathbf{k}}^*(\mathbf{r})$  is a function that is periodic in the unperturbed lattice,

$$\begin{aligned} \int_{v_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}} \varphi_{np}(\mathbf{r}) \varphi_{m\mathbf{p}+\mathbf{k}}^*(\mathbf{r}) &= 0, \\ \int_{v_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}} e^{-i\mathbf{k}\mathbf{r}} \varphi_{np}(\mathbf{r}) \varphi_{m\mathbf{p}+\mathbf{k}}^*(\mathbf{r}) &= \int_{v_c(\mathbf{R}, t)} d\mathbf{r} \hat{\mathcal{L}} e^{-i\mathbf{k}\mathbf{r}} [\varphi_{np}(\mathbf{r}) \varphi_{m\mathbf{p}+\mathbf{k}}^*(\mathbf{r})] \\ &= -ie^{-i\mathbf{k}\mathbf{R}} \int_{v_c} d\mathbf{r} \hat{\mathcal{L}}(\mathbf{k}\mathbf{r}) \varphi_{np}(\mathbf{r}) \varphi_{m\mathbf{p}}^*(\mathbf{r}) + O(k^2). \end{aligned}$$

Therefore, the part of  $\delta G_{nm}(p, k)$  that is regular as  $k \rightarrow 0$  leads in (5.8) to an expression proportional to at least the first power of  $|\mathbf{k}|$ . We shall be interested in two limiting cases of the relation between the frequency and the wave vector of the external field:  $\omega \gg s_0|\mathbf{k}|$  and  $\omega \ll s_0|\mathbf{k}|$  ( $s_0$  is the characteristic speed of the zero-sound excitations). In both cases, the irreducible (in the phonons) vertex  $\tilde{\Gamma}(p_1, p_2; k)$  is regular and can therefore be omitted by virtue of what was stated above in the calculation of the  $G$ -function response necessary for our purposes. Consequently, corresponding to (A.3) and (3.6),

$$\delta G_{nm}(p, k) \cong -i G_{n'n'}(p) G_{m'm}(p) g_{n'm'}^i(p, 0) D_{ij}(k) \eta_{js}(k) k_s \delta\varphi, \quad (5.9)$$

$$\begin{aligned} \eta_{is}(k) k_s &= \int \frac{dp}{(2\pi)^4} g_{nm}^i(p+k, -k) G_{m'm}(p) \\ &\times \langle m', \mathbf{p} | e^{-i\mathbf{k}\mathbf{r}} | n', \mathbf{p} + \mathbf{k} \rangle G_{n'm}(p+k). \end{aligned} \quad (5.10)$$

Substituting (3.13) into (5.9) and then into (5.8), we obtain

$$\begin{aligned} \int_{v_c} d\mathbf{r} \hat{\mathcal{L}}\delta n(\mathbf{r}, t) &= -e^{i\mathbf{k}\mathbf{R} + i\omega t} \int_{v_c} d\mathbf{r} \hat{\mathcal{L}}(\mathbf{k}\mathbf{r}) \frac{\partial n_0(\mathbf{r})}{\partial r_i} D_{ij}(k) \eta_{js}(k) k_s \delta\varphi \\ &= -C_{lm} k_m D_{ij}(k) \eta_{js}(k) k_s \delta\varphi e^{-i\mathbf{k}\mathbf{R} + i\omega t}. \end{aligned} \quad (5.11)$$

Substitution of (5.6) and (5.11) into (5.3) gives

$$C_{lm} (w_{lm} - k_m D_{ij}(k) \eta_{js}(k) k_s \delta\varphi e^{-i\mathbf{k}\mathbf{R} + i\omega t}) = 0. \quad (5.12)$$

In view of the arbitrariness in the choice of the operator  $\hat{\mathcal{L}}$ , which defines the matrix  $C_{lm}$  (5.7), it follows from (5.12) that

$$w_{lm}(\mathbf{R}, t) = k_m D_{ij}(k) \eta_{js}(k) k_s \delta\varphi e^{-i\mathbf{k}\mathbf{R} + i\omega t}. \quad (5.13)$$

This is the formula sought for the lattice deformation due to an external scalar field.

It is clear that, for  $\omega \gg s_0|\mathbf{k}|$ , the function  $\eta_{ij}(\mathbf{k})$  reduces, to within quantities of order  $(s_0|\mathbf{k}|/\omega)^2$ , to a constant, which we denote by  $\eta_{ij}$ :

$$\eta_{ij} = \lim_{k \rightarrow 0, \omega \rightarrow 0} \eta_{ij}(k) \equiv \eta_{ij}^0. \quad (5.14)$$

Analogously, in the region  $\omega \ll s_0|\mathbf{k}|$ , to within terms of order  $(\omega/s_0|\mathbf{k}|)^2$ , the quantity  $\eta_{ij}(\mathbf{k})$  is also equal to a constant, for which we introduce the notation  $\bar{\eta}_{ij}$ :

$$\bar{\eta}_{ij} = \lim_{\omega \rightarrow 0, k \rightarrow 0} \eta_{ij}(k) \equiv \eta_{ij}^A. \quad (5.15)$$

Using the definition (5.10), and also the relation (4.1) and Eqs. (4.15) and (4.17), by separating out the Fermi-liquid singularities we obtain the connection between  $\bar{\eta}_{ij}$  and  $\eta_{ij}$ :

$$\bar{\eta}_{ij} = \eta_{ij} - \int dS \bar{\zeta}_{ij}(\mathbf{p}) \quad (5.16)$$

( $\bar{\zeta}_{ij}(\mathbf{p}) = \zeta_{ij}^{\mathbf{k}}(\mathbf{p})$ ), which coincides with formula (I.4.12).

In fact, the tensor  $\eta_{ij}$  is not an independent quantity. Its connection with the other parameters of the theory can be established by means of the relation (A.19) obtained in the Appendix. After substitution of the expression for the vertex part into the definition (A.12) of  $R_{00}(\mathbf{k})$  and extraction of the Fermi-liquid singularities, we arrive at a formula coinciding with (I.4.22).

In conclusion, we shall consider the linear response of the lattice structure and of the macroscopic particle-number density of the crystal to a static field tending to uniformity. In this limit, in the formula (5.13) for the deformation tensor the function  $\eta_{jS}(\mathbf{k})$  must be replaced by  $\bar{\eta}_{jS}$ , and  $D_{l_j}(\mathbf{k})$  by the static expression for the phonon Green's function, which follows from (4.18), (4.19) and (4.17). After passing to the limit  $\mathbf{k} \rightarrow 0$ , taking into account the effect of the finite size of the system, we obtain

$$w_{ij} = -\Lambda_{ijlm} \bar{\eta}_{lm} \delta\varphi, \quad (5.17)$$

where  $\Lambda_{ijlm}$  coincides with the tensor introduced by the formulas (4.14) and (4.16) in I.

If we take into account formula (A.2) for the response of the G-function and formulas (3.1) and (3.6), the response  $\delta N(\mathbf{R}, t)$  of the macroscopic particle-number density of the crystal to an external field takes the form

$$\delta N(\mathbf{R}, t) = [R_{00}(k) + k_i \eta_{il}(k) D_{lm} \eta_{ms}(k) k_s] \delta\varphi e^{-i\mathbf{k}\mathbf{R} + i\omega t}, \quad (5.18)$$

where  $\tilde{R}_{00}(\mathbf{k})$  is defined by the expression

$$\begin{aligned} \tilde{R}_{00}(k) = & -i \int \frac{dp}{(2\pi)^4} B_{n_1 n_2}(\mathbf{p}, \mathbf{k}) G_{n_2 m_1}(p) G_{m_2 n_1}(p+k) B_{m_1 m_2}(\mathbf{p}, \mathbf{k}) \\ & - \int \frac{dp dp'}{(2\pi)^8} B_{n_1 n_2}(\mathbf{p}, \mathbf{k}) G_{n_2 n_1}(p) G_{n' n_1}(p+k) \bar{\Gamma}_{n' m' n'' m''}(p, p'; k) \\ & \times G_{m'' m_1}(p') G_{m_2 n''}(p'+k) B_{m_1 m_2}(\mathbf{p}, \mathbf{k}), \\ B_{m_1 m_2}(\mathbf{p}, \mathbf{k}) = & \int \frac{d\mathbf{r}}{v_c} \varphi_{m_1 \mathbf{p}}^*(\mathbf{r}) \varphi_{m_2 \mathbf{p}+\mathbf{k}}(\mathbf{r}). \end{aligned} \quad (5.19)$$

The response to the static field in the uniform limit can be obtained by separating out the Fermi-liquid singularities and then passing to the limit  $\omega = 0$ ,  $\mathbf{k} \rightarrow 0$ , taking into account the effect of the finite size of the system in the second term of (5.18). The final result takes the form

$$\left( \frac{\delta N}{\delta\varphi(\mathbf{r})} \right)_{\delta\varphi \rightarrow \text{const}} = - \left[ \left( \frac{\partial N}{\partial \mu} \right)_F + \bar{\eta}_{ik} \Lambda_{iklm} \bar{\eta}_{lm} \right], \quad (5.20)$$

where  $(\partial N / \partial \mu)_F$  coincides with the quantity defined by formulas (I.4.18) and (I.4.8). The formula (I.4.19) for the compressibility follows from (5.20).

## APPENDIX

We shall prove a number of relations connecting the single-particle Green's function with the vertex part. For this, we shall make use of the fact that, on switching on an infinitesimally small external field, with the corresponding interaction Hamiltonian

$$\hat{H}_{int} = \gamma \int d\mathbf{r} \psi^+(x) e^{-i\mathbf{k}\mathbf{x}} \hat{u}_\alpha(\mathbf{r}) \psi(x), \quad \gamma \rightarrow 0 \quad (A.1)$$

(we shall consider the case  $\hat{u}_0 = 1$ ,  $\hat{u}_i(\mathbf{r}) = -i\partial/\partial r_i$ ), the resulting linear change of the Green's function is determined by the equality<sup>[3]</sup>

$$\begin{aligned} \delta \hat{G}^{-1}(x, x') = & \gamma \left\{ e^{-i\mathbf{k}\mathbf{x}} \hat{u}_\alpha(\mathbf{r}) \delta(x-x') - i \int d^4 x_1 d^4 x_2 \Gamma(x, x_1; x', x_1') \right. \\ & \left. \times G(x_1', x_2) e^{-i\mathbf{k}\mathbf{x}_2} \hat{u}_\alpha(\mathbf{r}_2) G(x_2, x_1) \right\}, \end{aligned} \quad (A.2)$$

which, in the band representation, takes the form

$$\delta G_{nm}^{-1}(p, p+k) = \gamma Q_{nm}^\alpha(p, k), \quad (A.3)$$

$$Q_{nm}^\alpha(p, k) = \langle n, \mathbf{p} | e^{-i\mathbf{k}\mathbf{r}} \hat{u}_\alpha(\mathbf{r}) | m, \mathbf{p} + \mathbf{k} \rangle - i \int \frac{d^4 p'}{(2\pi)^4} \Gamma_{n'n''m''}(p, p'; k) \quad (A.4)$$

$$\times G_{m''m''}(p') \langle m'', \mathbf{p}' | e^{-i\mathbf{k}\mathbf{r}} \hat{u}_\alpha(\mathbf{r}) | n'', \mathbf{p}' + \mathbf{k} \rangle G_{n''n''}(p' + k).$$

Here,

$$\langle n, \mathbf{p} | e^{-i\mathbf{k}\mathbf{r}} \hat{u}_\alpha(\mathbf{r}) | m, \mathbf{p} + \mathbf{k} \rangle = \int \frac{d\mathbf{r}}{v_c} \varphi_{n\mathbf{p}}^*(\mathbf{r}) \hat{u}_\alpha(\mathbf{r}) \varphi_{m\mathbf{p}+\mathbf{k}}(\mathbf{r})$$

( $v_c$  is the volume of a unit cell of the crystal). On the other hand, in the case of a special form of the operator  $\hat{u}_\alpha(\mathbf{r})$ , the response of the Green's function to an external field can be calculated directly using the symmetry properties. A comparison of such a result with (A.3) enables us to obtain the necessary relations.

We find the first relation by assuming that the Hamiltonian  $\hat{H}_{int}$  corresponds to a uniform scalar field with non-zero frequency ( $\alpha = 0$ ,  $u_\alpha(\mathbf{r}) = 1$ ,  $\mathbf{k} = 0$ ):

$$\hat{H}_{int} = \gamma \hat{N} e^{i\omega t}, \quad \hat{N} = \int d\mathbf{r} \psi^+(x) \psi(x)$$

is the operator of the total number of particles. By virtue of the fact that  $\hat{N}$  commutes with the Hamiltonian  $\hat{H}_0$  of the unperturbed system, the Heisenberg operators (with respect to  $\hat{H}_0 + \hat{H}_{int}$ )  $\tilde{\psi}(\mathbf{x})$  and  $\tilde{\psi}^*(\mathbf{x})$  are expressed in terms of the unperturbed Heisenberg operators  $\psi(\mathbf{x})$  and  $\psi^*(\mathbf{x})$  as follows:

$$\begin{aligned} \tilde{\psi}(x) = & \exp \left( i \int_{-\infty}^t dt' \gamma \hat{N} e^{i\omega t'} \right) \psi(x) \exp \left( -i \int_{-\infty}^t dt' \gamma \hat{N} e^{i\omega t'} \right) \\ = & \psi(x) \exp \left( -\gamma \frac{e^{i\omega t}}{\omega} \right), \quad \tilde{\psi}^+(x) = \psi^+(x) \exp \left( \gamma \frac{e^{i\omega t}}{\omega} \right). \end{aligned}$$

From this we find the connection between the Green's function in the presence of the external field and the unperturbed G-function:

$$\tilde{G}(x, x') = G(x, x') \exp \left[ -\frac{\gamma}{\omega} (e^{i\omega t_1} - e^{i\omega t_2}) \right].$$

After expanding this expression in  $\gamma$ , going over to the band representation, and substituting it into (A.3), we shall have

$$\omega^{-1} [G_{nm}^{-1}(p+k) - G_{nm}^{-1}(p)] = Q_{nm}^0(p, \tilde{k}), \quad (A.5)$$

where  $Q_{nm}^0(\mathbf{p}, \tilde{\mathbf{k}})$  is defined in accordance with (A.4) with the substitution

$$u_\alpha(\mathbf{r}) = u_0(\mathbf{r}) = 1, \quad \tilde{k} = (0, \omega).$$

We obtain the second relation by considering the change of the Green's function due to a transformation to a coordinate frame moving with velocity  $\gamma \mathbf{n}_i e^{i\omega t}$  ( $\mathbf{n}_i$  is the unit vector in the direction of the  $i$ -th coordinate axis). The corresponding addition to the Hamiltonian is, clearly,

$$\hat{H}_{int} = \gamma \hat{\mathcal{P}}_i e^{i\omega t}$$

( $\hat{\mathcal{P}}_i = \int d\mathbf{r} \psi^*(\mathbf{x}) (+i\partial/\partial r_i) \psi(\mathbf{x})$  is the operator of the total momentum). Performing calculations analogous to the

preceding ones and using the property that the total momentum of the system is conserved, we obtain

$$\omega^{-1} [P_{nm}^i(p) G_{\alpha m}^{-1}(p + \tilde{k}) - G_{nm}^{-1}(p) P_{m'n}^i(p)] = Q_{nm}^i(p, \tilde{k}),$$

$$P_{nm}^i(p) = \int \frac{d\mathbf{r}}{v_c} \varphi_{n\mathbf{p}}^*(\mathbf{r}) \left( -i \frac{\partial}{\partial r_i} \right) \varphi_{m\mathbf{p}}(\mathbf{r}), \quad (\text{A.6})$$

where  $Q_{nm}^i(p, \tilde{k})$  is defined by the equality (A.4) with the substitutions

$$\hat{u}_\alpha(\mathbf{r}) = \hat{u}_\alpha(\mathbf{r}) = -i\partial / \partial r_i, \quad \tilde{k} = (0, \omega).$$

We find the third relation by specifying the interaction Hamiltonian to be of the form

$$\hat{H}_{int} = n_i \gamma \lim_{k \rightarrow 0} \int d\mathbf{r} \psi^+(x) e^{-i\mathbf{k}\mathbf{r}} \left( -i \frac{\partial}{\partial r_i} \right) \psi(x),$$

which corresponds to a change of gauge of the electromagnetic field under the assumption that the particles are provided with infinitesimally small charge (proportional to  $\gamma$ ). It follows from the gauge invariance that, in this case,

$$G(x, x') = G(x, x') \exp[-im\gamma(r_i - r'_i)n_i]$$

( $m$  is the mass of a single particle). Expansion in  $\gamma$  and substitution into (A.3) give

$$-m \frac{\partial G_{nm}^{-1}(p)}{\partial p_i} + m [G_{nm}^{-1}(p) B_{m'm}^i(p) - B_{nm'}^i(p) G_{m'm}^{-1}(p)] = Q_{nm}^{i(k)}(p), \quad (\text{A.7})$$

where

$$B_{nm}^i(p) = \int \frac{d\mathbf{r}}{v_c} \varphi_{n\mathbf{p}}^* e^{i\mathbf{p}\mathbf{r}} \frac{\partial}{\partial p_i} (\varphi_{m\mathbf{p}}(\mathbf{r}) e^{-i\mathbf{p}\mathbf{r}}). \quad (\text{A.8})$$

The following series of relations follows from the continuity equation for the particle-number density:

$$\frac{\partial}{\partial t} (\psi^+(x)\psi(x)) + \frac{\partial}{\partial r_i} \left\{ \frac{1}{2im} \left[ \psi^+(x) \frac{\partial}{\partial r_i} \psi(x) - \left( \frac{\partial}{\partial r_i} \psi^+(x) \right) \psi(x) \right] \right\} = 0, \quad (\text{A.9})$$

from which follows a condition for the two-particle Green's function:

$$\frac{\partial}{\partial t_2} [G^{11}(x_1, x_2; x_3, x_4) |_{r_2=r_4, t_2=t_3+0}] + \frac{1}{2im} \frac{\partial}{\partial r_{2i}} \left[ \left( \frac{\partial}{\partial r_{2i}} - \frac{\partial}{\partial r_{4i}} \right) \right] \quad (\text{A.10})$$

$$\times G^{11}(x_1, x_2; x_3, x_4) |_{r_2=r_4, t_2=t_3+0} = iG(x_1, x_3) [\delta(x_2 - x_3) - \delta(x_1 - x_2)].$$

Multiplying this equality by  $\exp(-i\mathbf{k}\mathbf{x}_2)$ , integrating over  $\mathbf{x}_2$  and then going over to the band representation in  $\mathbf{x}_1$  and  $\mathbf{x}_3$ , we obtain

$$\left( \omega + \frac{k^2}{2m} \right) Q_{nm}^0(p, k) - \frac{k_i}{m} Q_{nm}^i(p, k) = \langle n, \mathbf{p} | e^{-i\mathbf{k}\mathbf{r}} | m', \mathbf{p} + \mathbf{k} \rangle G_{m'm}^{-1}(p + k) - G_{nm'}^{-1}(p) \langle m', \mathbf{p} | e^{-i\mathbf{k}\mathbf{r}} | m, \mathbf{p} + \mathbf{k} \rangle. \quad (\text{A.11})$$

It is easily seen that the relations (A.5) and (A.7) are a particular case of (A.11). Multiplying (A.11) by

$$-iG_{m'm'}(p + k) \langle m', \mathbf{p} + \mathbf{k} | e^{i\mathbf{k}\mathbf{r}} | n', \mathbf{p} \rangle G_{n'n}(p),$$

integrating over  $\mathbf{p}$  and defining the quantity  $R_{\alpha\beta}(k)$ :

$$R_{\alpha\beta}(k) = -i \int \frac{d^4p}{(2\pi)^4} G_{m'm'}(p + k) \langle m', \mathbf{p} + \mathbf{k} | e^{i\mathbf{k}\mathbf{r}} \hat{u}_\alpha(\mathbf{r}) | n', \mathbf{p} \rangle$$

$$\times G_{n'n}(p) Q_{nm}^\beta(p, k) \equiv -i \int \frac{d^4p}{(2\pi)^4} G_{m'm'}(p + k) \langle m', \mathbf{p} + \mathbf{k} | e^{i\mathbf{k}\mathbf{r}} \hat{u}_\alpha(\mathbf{r}) | n', \mathbf{p} \rangle$$

$$\times G_{n'n}(p) \left[ \langle n, \mathbf{p} | e^{-i\mathbf{k}\mathbf{r}} \hat{u}_\beta(\mathbf{r}) | m, \mathbf{p} + \mathbf{k} \rangle - i \int \frac{d^4p'}{(2\pi)^4} \Gamma_{n'm\alpha}(p, p'; k) G_{s's'}(p') \right]$$

$$\times \langle s', \mathbf{p}' | e^{-i\mathbf{k}\mathbf{r}} \hat{u}_\beta(\mathbf{r}) | l', \mathbf{p}' + \mathbf{k} \rangle G_{l'l}(p' + k) \quad (\text{A.12})$$

(we recall that  $\hat{u}_0(\mathbf{r}) = 1$ ,  $\hat{u}_i(\mathbf{r}) = -i\partial / \partial r_i$ ), we find that

$$\left( \omega + \frac{k^2}{2m} \right) R_{\alpha\alpha}(k) - \frac{k_i}{m} R_{\alpha i}(k) = 0. \quad (\text{A.13})$$

In an analogous way, multiplying (A.11) by

$$-iG_{m'm}(p + k) \langle m', \mathbf{p} + \mathbf{k} | e^{i\mathbf{k}\mathbf{r}} \left( -i \frac{\partial}{\partial r_i} \right) | n', \mathbf{p} \rangle G_{n'n}(p)$$

and integrating over  $\mathbf{p}$ , we shall have

$$\left( \omega + \frac{k^2}{2m} \right) R_{i\alpha}(k) + \frac{k_j}{m} R_{ij}(k) = k_i N_0 \quad (\text{A.14})$$

( $N_0$  is the number of particles in unit volume). It is obvious that

$$R_{\alpha\beta}(k) = R_{\beta\alpha}(-k).$$

Therefore, on the basis of (A.13) and (A.14), we obtain

$$\left[ \omega^2 - \left( \frac{k^2}{2m} \right)^2 \right] R_{00}(k) = \frac{k^2}{m} N_0 + \frac{k_i k_j}{m^2} R_{ij}(-k). \quad (\text{A.15})$$

From this it follows that

$$R_{00}(\tilde{k}) = 0 \quad (\tilde{k} = (0, \omega)), \quad (\text{A.16})$$

$$R_{ij}^\dagger \equiv \lim_{\omega \rightarrow 0, k \rightarrow 0} R_{ij}(k) = -m N_0 \delta_{ij}. \quad (\text{A.17})$$

Using (A.15), and taking into account (A.16), (A.17) and also the relation

$$R_{ij}(\tilde{k}) = 0 \quad (\text{A.18})$$

which follows from (A.6), we obtain

$$\lim_{k \rightarrow 0} \frac{\partial^2 R_{00}(k)}{\partial k_i \partial k_j} = 2 \frac{N_0}{m\omega^2} \delta_{ij}. \quad (\text{A.19})$$

The relations established are a generalization, to the case of a crystal, of known identities for uniform systems. In particular, relations of the type (A.5) and (A.6) for  $\omega \rightarrow 0$  and (A.7) for isotropic Fermi liquids were obtained by Pitaevskii<sup>[7]</sup>.

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