

Theory of Quantum Crystals I. Phenomenological Theory

I. E. Dzyaloshinskii, P. S. Kondratenko, and V. S. Levchenkov

L. D. Landau Institute of Theoretical Physics of the USSR Academy of Sciences

Submitted November 19, 1971

Zh. Eksp. Teor. Fiz. 62, 1574–1584 (April, 1972)

A phenomenological theory of Fermi-type quantum crystals with gapless Fermi excitations is constructed. Assumptions analogous to those used in the Landau theory of Fermi liquids form the basis of the theory. A system of equations is obtained for the spectrum of the long-wavelength and low-frequency vibrations of the crystal. Expressions are found for the compressibility, specific heat and spin susceptibility.

1. The ground state of a classical crystal corresponds to an assembly of stationary atoms, localized at the lattice sites. Quantum effects lead to zero-point lattice vibrations and, associated with these, delocalization of the atoms. In the case when the interaction between the atoms is weak, and their mass small, the amplitude of the zero-point vibrations can become comparable with the lattice constant. In these conditions, overlap of the wave functions of neighboring atoms is important and, because of the principle of indistinguishability of identical particles, the lattice sites can no longer be identified with the atoms of the crystal. The properties of these systems must depend on the statistics by which the atoms composing them are described. The above effects make classical methods of treating these systems invalid. Such crystals are called quantum crystals. The most striking examples of these are solidified He^3 and He^4 .

The development of a microscopic theory of quantum crystals has been associated principally with the self-consistent field approximation. In this, the starting point used has been either a system of self-consistent single-particle levels, on the basis of which the vibrational spectrum of the crystal has been determined^[1]; or an aggregate of interacting phonon vibrations^[2]. It is characteristic that, in both cases, phonons turn out to be the only type of weakly excited states of the system, as in a classical insulating and nonmagnetic crystal also. Andreev and Lifshitz^[3] first expressed the idea of a possible quantum-crystal state in which gapless single-particle excitations (defectons) occur. Their presence results in the number of sites being different from the number of atoms in the lattice, and leads to a qualitatively new classification of the low excited states of the crystal.

In the present paper, a phenomenological theory of a Fermi-type quantum crystal (i.e., consisting of Fermi-particles) possessing gapless Fermi excitations will be constructed. As the basis of the theory, we make assumptions analogous to those used in the Landau theory of an isotropic Fermi liquid^[4].

In the second section, we shall formulate the basic postulates of the theory and obtain relations, stemming from the Galilean invariance, between the parameters introduced.

In the third section, we shall find a system of equations describing the long-wavelength (and low-frequency) vibrations of a quantum crystal. In essence, these vibrations are coupled excitations of the zero-sound and phonon types.

In the fourth section, we shall obtain expressions for some thermodynamic quantities of a quantum

crystal—the compressibility, specific heat and spin susceptibility.

2. In constructing a phenomenological theory of a quantum crystal of the Fermi type, we shall start from the assumption that any weakly excited state of the system can be characterized by an assembly of Fermi quasi-particles with quasi-momenta \mathbf{p} lying in the vicinity of a certain surface, called the Fermi surface, and by the lattice deformation. We shall describe the latter by the deformation vector $u_i(\mathbf{x})$ ($\mathbf{x} = \{\mathbf{r}, t\}$ is the space-time coordinate) defining the displacement of the lattice sites, which, as already mentioned, cannot be identified with the atoms of the crystal. In view of the fact that we are interested in perturbed states with inhomogeneities of dimensions large compared with the period of the lattice, $u_i(\mathbf{x})$ depends on a continuous space variable.

Unlike the Fermi surface of an isotropic Fermi liquid^[4], the Fermi surface of a quantum crystal is, in general, anisotropic. According to the theorem proved by Luttinger^[5], the volume of the Fermi surface determines, to within an even integer (the particle spin $S = \frac{1}{2}$), the number of particles per unit cell of the crystal. We shall not use this relation here. We require only that the change in the number of quasi-particles for fixed lattice sites be equal to the change in the number of particles. By analogy with isotropic Fermi systems, the ground state of a quantum crystal can be represented conveniently in the form of a combination of the quasi-particles occupying all the states with quasi-momenta \mathbf{p} lying within the Fermi surface and a subsystem, whose degrees of freedom are characterized by the vector $u_i(\mathbf{x})$ (in the ground state, $u_i(\mathbf{x}) = 0$), interacting with the quasi-particles. In the following, by convention we shall call this subsystem the lattice.

We shall assume that the interaction between the quasi-particles is described by the self-consistent field acting on a quasi-particle and arising from the other quasi-particles. In this case, the energy of the system must obviously be a functional of the quasi-particle distribution function $n(\mathbf{p}, \mathbf{x})$ and the deformation vector $u_i(\mathbf{x})$: $E = E\{n(\mathbf{p}, \mathbf{x}), u_i(\mathbf{x})\}$. For small deviations from equilibrium, the energy can be represented in the form of a functional expansion in $\delta n(\mathbf{p}, \mathbf{x}) = n(\mathbf{p}, \mathbf{x}) - n_0(\mathbf{p})$ ($n_0(\mathbf{p})$ is the equilibrium distribution function) and in the derivatives $\partial u_i / \partial t$, $\partial u_k / \partial r_l$ of the deformation vector (because of the invariance of the energy with respect to arbitrary uniform displacements that are independent of time, the quantities $u_i(\mathbf{x})$ themselves do not appear in the expansion). The most general expansion of the energy, exact to terms quadratic in $\delta n(\mathbf{p}, \mathbf{x})$, $\partial u_i(\mathbf{x}) / \partial t$ and $\partial u_k(\mathbf{x}) / \partial r_l$, if we take into

account the invariance under inversion of the space and time coordinates, has the form

$$E = \int d\mathbf{r} \left\{ \mathcal{E}_0 + \int \frac{d\mathbf{p}}{(2\pi)^3} \varepsilon(\mathbf{p}) \delta n(\mathbf{p}, x) + \frac{1}{2} \int \frac{d\mathbf{p} d\mathbf{p}'}{(2\pi)^6} f(\mathbf{p}, \mathbf{p}') \delta n(\mathbf{p}, x) \right. \\ \times \delta n(\mathbf{p}', x) + \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\xi_i(\mathbf{p}) \frac{\partial u_i(x)}{\partial t} + \zeta_{\alpha}(\mathbf{p}) \frac{\partial u_{\alpha}(x)}{\partial r_i} \right] \delta n(\mathbf{p}, x) \\ \left. + \frac{1}{2} \rho_{ik} \frac{\partial u_i(x)}{\partial t} \frac{\partial u_k(x)}{\partial t} + \frac{1}{2} \lambda_{iklm} \frac{\partial u_k(x)}{\partial r_i} \frac{\partial u_m(x)}{\partial r_l} \right\}. \quad (2.1)$$

As in the Landau theory^[4], here we have used an assumption that the interaction between the particles is short-range, so that the expansion of the energy is local. The first three terms in the curly brackets of (2.1) are identical with the corresponding expansion in the Landau theory. \mathcal{E}_0 is the energy density of the ground state of the crystal, $\varepsilon(\mathbf{p})$ is the unperturbed quasi-particle energy, and $f(\mathbf{p}, \mathbf{p}')$ is the Landau function defining the interaction between the quasi-particles. The sum of the last two terms of (2.1) is the analog of the energy density in the classical theory of elasticity^[6]. The difference consists in the fact that the tensor ρ_{ik} is proportional to the unit tensor only in the case of crystals of cubic symmetry and in this case, generally speaking, does not reduce to the mass density of the crystal. The structure itself of the last term of the expansion and the property of invariance of the energy with respect to rotations of the system as a whole lead to the usual symmetry conditions for the "elastic tensor" λ_{iklm} : $\lambda_{iklm} = \lambda_{lmik} = \lambda_{kilm}$. The fourth term of the expansion (2.1) defines the interaction of the Fermi excitations with the lattice deformation.

As in the Landau theory^[4], the energy $\varepsilon(\mathbf{p}, x)$ of a quasi-particle with quasi-momentum \mathbf{p} and coordinate x is determined by the variational derivative of the total energy of the system with respect to the distribution function $n(\mathbf{p}, x)$ at fixed lattice deformation:

$$\varepsilon(\mathbf{p}, x) = \varepsilon(\mathbf{p}) + \int \frac{d\mathbf{p}'}{(2\pi)^3} f(\mathbf{p}, \mathbf{p}') \delta n(\mathbf{p}', x) + \xi_i(\mathbf{p}) \frac{\partial u_i(x)}{\partial t} + \zeta_{\alpha}(\mathbf{p}) \frac{\partial u_{\alpha}(x)}{\partial r_i}. \quad (2.2)$$

In writing the expansion (2.1), we have made no assumptions concerning the tensor ρ_{ik} and the functions $f(\mathbf{p}, \mathbf{p}')$ and $\xi_i(\mathbf{p})$. In reality, these are not independent. We shall prove this, by using Galilean invariance.

First, we shall determine the flux density $j_i(x)$ of the quasi-particle number. It is clear that this is equal to

$$j_i(x) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\partial \varepsilon(\mathbf{p}, x)}{\partial p_i} n(\mathbf{p}, x). \quad (2.3)$$

Since, in the ground state in the coordinate system in which the crystal as a whole is at rest, $j_i = 0$ and there is no deformation, on the basis of the expression for $\varepsilon(\mathbf{p}, x)$ (2.2), we obtain

$$j_i(x) = \int \frac{d\mathbf{p}}{(2\pi)^3} \left[v_i + \int dS' f(\mathbf{p}, \mathbf{p}') v_i' \right] \delta n(\mathbf{p}, x) \\ + \int dS v_i \left(\xi_i(\mathbf{p}) \frac{\partial u_i(x)}{\partial t} + \zeta_{\alpha}(\mathbf{p}) \frac{\partial u_{\alpha}(x)}{\partial r_i} \right). \quad (2.4)$$

Here, $v_i \equiv v_i(\mathbf{p}) = \partial \varepsilon(\mathbf{p}) / \partial p_i$ is the unperturbed velocity of a quasi-particle with quasi-momentum \mathbf{p} ,

$$\int dS = \int \frac{d\mathbf{p}}{(2\pi)^3} \delta(\varepsilon(\mathbf{p}) - \mu),$$

and μ is the chemical potential. In deriving (2.4), we

have used the fact that the equilibrium quasi-particle distribution function is of the Fermi type, $n_0(\mathbf{p}) = n_F(\varepsilon(\mathbf{p}))$, and at zero temperature, $\partial n_F(\varepsilon) / \partial \varepsilon = -\delta(\varepsilon - \mu)$. We note that the equation $\varepsilon(\mathbf{p}) = \mu$ defines the Fermi surface of the quasi-particles.

We shall now find an expression for the momentum density $\mathcal{P}_i(x)$ of the lattice. It is the momentum canonically conjugate to the generalized coordinate $u_i(x)$. Obviously, for $\delta n(\mathbf{p}, x) = 0$,

$$\mathcal{P}_i(x) = \rho_{ik} \partial u_k(x) / \partial t.$$

Since

$$\mathcal{P}_i(x) = \frac{\delta \mathcal{L}}{\delta \partial u_i(x) / \partial t}$$

(\mathcal{L} is the Lagrangian function of the system), and the correction to \mathcal{L} linear in δn differs only in sign from the corresponding term in (2.1), we have

$$\mathcal{P}_i(x) = \rho_{ik} \frac{\partial u_k(x)}{\partial t} - \int \frac{d\mathbf{p}}{(2\pi)^3} \xi_i(\mathbf{p}) \delta n(\mathbf{p}, x). \quad (2.5)$$

We shall obtain the first relation between the parameters of the theory by considering the change of energy of the quasi-particle subsystem due to the uniform motion of the lattice sites with constant velocity \mathbf{c} . In the reference system in which the lattice is at rest, the subsystem of quasi-particles moves with velocity $-\mathbf{c}$. The corresponding correction to the energy of the quasi-particles in this coordinate system is equal to

$$\delta E_c^F = -mc_i \int d\mathbf{r} j_i(x) \\ = -mc_i \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[v_i + \int dS' f(\mathbf{p}, \mathbf{p}') v_i' \right] \delta n(\mathbf{p}, x). \quad (2.6)$$

In making the inverse coordinate transformation to the original reference system, it is necessary to add to the energy of the subsystem of quasi-particles the expression

$$c_i \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi)^3} P_i(\mathbf{p}) \delta n(\mathbf{p}, x)$$

($P_i(\mathbf{p})$ is the average value of the momentum of a quasi-particle in a state with quasi-momentum \mathbf{p}) and, thus, the total change of energy of the quasi-particle subsystem produced by the motion of lattice sites with velocity $\partial u_i(x) / \partial t = c_i$ is equal to

$$\delta E^F = \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[P_i(\mathbf{p}) - m(v_i + \int dS' f(\mathbf{p}, \mathbf{p}') v_i') \right] \delta n(\mathbf{p}, x) c_i.$$

Comparing this expression with (2.1), we obtain

$$\xi_i(\mathbf{p}) = P_i(\mathbf{p}) - m \left(v_i + \int dS' f(\mathbf{p}, \mathbf{p}') v_i' \right). \quad (2.7)$$

We note that in the case of the isotropic Fermi liquid, $P_i(\mathbf{p}) = p_i$ and, after equating $\xi_i(\mathbf{p})$ to zero, we obtain the usual Landau-theory relation for the effective mass^[4].

To find the second relation, we note that the mass flux density of the particles of the crystal is equal to the sum of the flux density $m j_i(x)$ of the mass carried by the quasi-particles and the mass flux density associated with the lattice,

$$J_i(x) = m j_i(x) + \mathcal{P}_i(x). \quad (2.8)$$

On transforming to the coordinate system moving with velocity \mathbf{c} , the mass flux density of the crystal changes by an amount $\delta J_i = -N_0 m c_i$ (N_0 is the parti-

cle-number density of the system, and m is the mass of an individual particle). On the other hand, this same change in the mass flux density can be determined from the formulas (2.8), (2.4) and (2.5), if we take into account the law of transformation of the quantities $n(\mathbf{p}, \mathbf{x})$ and $\partial u_i(\mathbf{x})/\partial t$:

$$n(\mathbf{p}, \mathbf{x}) \rightarrow n(\mathbf{p} + m\mathbf{c}, \mathbf{x}), \quad \frac{\partial u_i(\mathbf{x})}{\partial t} \rightarrow \frac{\partial u_i(\mathbf{x})}{\partial t} - c_i.$$

Equating the two expressions for $\delta J_1(\mathbf{x})$, we obtain a relation for the tensor ρ_{ijk} :

$$\rho_{ik} = N_0 m \delta_{ik} - m^2 \int dS v_i \left(v_k + \int dS' f(\mathbf{p}, \mathbf{p}') v_k' \right), \quad (2.9)$$

In essence, the tensor ρ_{ijk} defines that part of the crystal density which is associated with the lattice.

For the isotropic Fermi liquid, taking into account the relation for the effective mass, we obtain the natural result: $\rho_{ijk} = 0$. But in the absence of a Fermi surface, ρ_{ijk} , as in a classical crystal, determines the density of the crystal: $\rho_{ijk} = N_0 m \delta_{ijk}$.

3. We turn to the derivation of the equations of motion describing the weakly nonequilibrium properties of the crystal. One of these, obviously, is the kinetic equation for the quasi-particle distribution function:

$$\frac{\partial n(\mathbf{p}, \mathbf{x})}{\partial t} + \frac{\partial n(\mathbf{p}, \mathbf{x})}{\partial r_i} \frac{\partial \varepsilon(\mathbf{p}, \mathbf{x})}{\partial p_i} - \frac{\partial n(\mathbf{p}, \mathbf{x})}{\partial p_i} \frac{\partial \varepsilon(\mathbf{p}, \mathbf{x})}{\partial r_i} = I(n), \quad (3.1)$$

where $I(n)$ is the collision integral.

For the second equation of motion, we take Hamilton's equation for the lattice:

$$\frac{\partial \mathcal{P}_i(\mathbf{x})}{\partial t} = - \frac{\delta E}{\delta u_i(\mathbf{x})}. \quad (3.2)$$

After substituting (2.2) into (3.1) and (2.5) into (3.2), using the expansion (2.1) and linearizing Eq. (3.1), we obtain a system of equations of motion for the quantum crystal:

$$\begin{aligned} \frac{\partial}{\partial t} \delta n(\mathbf{p}, \mathbf{x}) + v_i \frac{\partial}{\partial r_i} \left\{ \delta n(\mathbf{p}, \mathbf{x}) - \frac{\partial n_0(\varepsilon)}{\partial \varepsilon} \left[\int \frac{d\mathbf{p}'}{(2\pi)^3} f(\mathbf{p}, \mathbf{p}') \delta n(\mathbf{p}', \mathbf{x}) \right. \right. \\ \left. \left. + \xi_i(\mathbf{p}) \frac{\partial u_i(\mathbf{x})}{\partial t} + \zeta_{im}(\mathbf{p}) \frac{\partial u_m(\mathbf{x})}{\partial r_i} \right] \right\} = I(n), \quad (3.3) \\ \rho_{ik} \frac{\partial^2 u_k(\mathbf{x})}{\partial t^2} = \lambda_{ikm} \frac{\partial^2 u_m(\mathbf{x})}{\partial r_k \partial r_i} + \\ + \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\xi_i(\mathbf{p}) \frac{\partial}{\partial t} \delta n(\mathbf{p}, \mathbf{x}) + \zeta_{ik}(\mathbf{p}) \frac{\partial}{\partial r_k} \delta n(\mathbf{p}, \mathbf{x}) \right]. \quad (3.4) \end{aligned}$$

We shall study the determination of the vibrational eigenfrequencies of a quantum crystal. For this, we shall seek a solution of the system of equations (3.3), (3.4) in the form $\delta n(\mathbf{p}, \mathbf{x}) = \delta n(\mathbf{p}, k) e^{i\mathbf{k}\mathbf{x}}$, $u_i(\mathbf{x}) = u_i(k) e^{i\mathbf{k}\mathbf{x}}$ ($\mathbf{k}\mathbf{x} = \mathbf{k} \cdot \mathbf{r} - \omega t$). We are interested in frequencies ω that are high compared with the characteristic inverse collision time of the quasi-particles, so that the collision integral in the right-hand side of Eq. (3.3) can be neglected. As follows from (3.3), the quantity $\delta n(\mathbf{p}, k)$ can be represented in the form

$$\delta n(\mathbf{p}, k) = - \frac{\partial n_0}{\partial \varepsilon} v(\mathbf{p}, k). \quad (3.5)$$

After this, the system of equations for $\nu(\mathbf{p}, k)$ and $u_i(k)$ takes the form

$$\begin{aligned} (\omega - v\mathbf{k}) v(\mathbf{p}, k) = v\mathbf{k} \left[\int dS' f(\mathbf{p}, \mathbf{p}') v(\mathbf{p}', k) \right. \\ \left. + i(-\omega \xi_i(\mathbf{p}) u_i(k) + k_i \zeta_{ik}(\mathbf{p}) u_k(k)) \right], \quad (3.6) \end{aligned}$$

$$\begin{aligned} (\rho_{ik} \omega^2 - \lambda_{ikm} k_i k_m) u_k(k) \\ = - i \int dS (-\omega \xi_i(\mathbf{p}) + k_i \zeta_{ii}(\mathbf{p})) v(\mathbf{p}, k). \quad (3.7) \end{aligned}$$

Eliminating $u_i(k)$, we obtain an equation for $\nu(\mathbf{p}, k)$:

$$(\omega - v\mathbf{k}) v(\mathbf{p}, k) = v\mathbf{k} \int dS' F(\mathbf{p}, \mathbf{p}'; k) v(\mathbf{p}, k), \quad (3.8)$$

where

$$\begin{aligned} F(\mathbf{p}, \mathbf{p}'; k) = f(\mathbf{p}, \mathbf{p}') + (\omega \xi_i(\mathbf{p}) - k_i \zeta_{ii}(\mathbf{p})) D_{ik}^0(k) \\ \times (\omega \xi_k(\mathbf{p}') - k_m \zeta_{mk}(\mathbf{p}')), \quad (3.9) \end{aligned}$$

and $D_{ik}^0(k)$ is the solution of the equation

$$(3.10)$$

Equation (3.8) has a form that is usual for the theory of Fermi liquids^[7]. However, the function $F(\mathbf{p}, \mathbf{p}'; k)$, which plays the role of the Landau function, depends on the wave vector and the vibration frequency. This means that, along with the zero-sound solutions, the system (3.8) can possess an additional family of solutions.

One can convince oneself of this by eliminating the variable $\nu(\mathbf{p}, k)$ from the system (3.6), (3.7), which leads to the following equation for $u_i(k)$:

$$[\rho_{ik} \omega^2 - T_{ik}(k)] u_k(k) = 0, \quad (3.11)$$

where

$$\begin{aligned} T_{ik}(k) = \lambda_{ikm} k_i k_m \\ + \int dS (\omega \xi_i(\mathbf{p}) - k_i \zeta_{ii}(\mathbf{p})) \frac{v\mathbf{k}}{\omega - v\mathbf{k}} (\omega \xi_k(\mathbf{p}, k) - k_m \zeta_{mk}(\mathbf{p}, k)); \quad (3.12) \end{aligned}$$

$\xi_i(\mathbf{p}, k)$ and $\xi_{ik}(\mathbf{p}, k)$ are determined from the equations

$$\xi_i(\mathbf{p}, k) = \xi_i(\mathbf{p}) + \int dS' f(\mathbf{p}, \mathbf{p}') \frac{v\mathbf{k}}{\omega - v\mathbf{k}} \xi_i(\mathbf{p}', k), \quad (3.13)$$

$$\xi_{ik}(\mathbf{p}, k) = \zeta_{ik}(\mathbf{p}) + \int dS' f(\mathbf{p}, \mathbf{p}') \frac{v\mathbf{k}}{\omega - v\mathbf{k}} \zeta_{ik}(\mathbf{p}', k). \quad (3.14)$$

Equation (3.11) has the form of the dynamical equation determining the spectrum of the lattice vibrations. The quantity $T_{ik}(k)$ appearing in it, which as the sense of the dynamical matrix, possesses frequency dispersion. This points to the fact that, apart from the three solutions corresponding to acoustic phonon modes, the vector equation (3.11) has an additional class of solutions. Thus, Eqs. (3.8) and (3.11) determine the spectrum of coupled vibrations of the zero-sound and phonon types. It is not difficult to see that the dispersion law corresponding to these is linear.

4. We proceed to examine the thermodynamic properties of the quantum crystal. In the first place, we shall express the compressibility of the crystal in terms of the quantities appearing in the theory. For this, we make use of the relation

$$\frac{\partial P}{\partial \rho} = \frac{N_0}{m} \left(\frac{\partial N_0}{\partial \mu} \right)^{-1} \quad (4.1)$$

(P is the pressure, and $\rho = N_0 m$ is the density of the crystal) and of the fact that, in the presence of a weakly nonuniform external scalar field $\varphi(\mathbf{r})$ that does not depend on time, the equilibrium condition $\mu + \varphi(\mathbf{r}) = \text{const}$ is fulfilled, whence follows

$$\frac{\partial N}{\partial \mu} = - \frac{\delta N}{\delta \varphi(\mathbf{r})} \Big|_{\varphi(\mathbf{r}) \rightarrow \text{const}} \quad (4.2)$$

In accordance with the basic assumption of our theory, the change of the particle-number density is

equal, obviously, to the sum of the change of the quasi-particle number density and the change of the lattice density.

The most general expression for $\delta N(\mathbf{x})$ in the approximation linear in the deformation has the form

$$\delta N(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \delta n(\mathbf{p}, \mathbf{x}) + \eta_{ik} \frac{\partial u_k(\mathbf{x})}{\partial r_i} \quad (4.3)$$

where η_{ik} is a second-rank symmetric tensor (proportional to the unit tensor for crystals of cubic symmetry). In the presence of the field $\varphi(\mathbf{r})$, to the energy (2.1) we must add the interaction energy

$$\int d\mathbf{r} N(\mathbf{x}) \varphi(\mathbf{x}) = \int d\mathbf{r} \left[N_0 + \int \frac{d\mathbf{p}}{(2\pi)^3} \delta n(\mathbf{p}, \mathbf{x}) + \eta_{ik} \frac{\partial u_k(\mathbf{x})}{\partial r_i} \right] \varphi(\mathbf{x}).$$

The expression for the energy of the quasi-particles takes the form

$$\begin{aligned} \varepsilon(\mathbf{p}, \mathbf{x}) &= \varepsilon(\mathbf{p}) + \varphi(\mathbf{x}) + \int \frac{d\mathbf{p}'}{(2\pi)^3} f(\mathbf{p}, \mathbf{p}') \delta n(\mathbf{p}', \mathbf{x}) \\ &+ \xi_i(\mathbf{p}) \frac{\partial u_i(\mathbf{x})}{\partial r_i} + \zeta_{ik}(\mathbf{p}) \frac{\partial u_k(\mathbf{x})}{\partial r_i}. \end{aligned} \quad (4.4)$$

In the limit where the external field $\varphi(\mathbf{r})$ is independent of time and tends to a constant, the crystal goes over into a new equilibrium state, to which correspond a certain uniform deformation of the lattice and a new quasi-particle distribution, differing from the original one by the replacement of $\varepsilon(\mathbf{p})$ by $\varepsilon(\mathbf{p}) + \delta\varepsilon(\mathbf{p})$. The connection between $\delta\varepsilon(\mathbf{p})$, $\partial u_k/\partial r_i$ and φ will be determined from the relation (4.4)

$$\delta\varepsilon(\mathbf{p}) = \varphi - \int dS' f(\mathbf{p}, \mathbf{p}') \delta\varepsilon(\mathbf{p}') + \zeta_{ik}(\mathbf{p}) \frac{\partial u_k}{\partial r_i} \quad (4.5)$$

(we have taken into account that $\delta n(\mathbf{p}) = n_0(\varepsilon(\mathbf{p}) + \delta\varepsilon(\mathbf{p})) - n_0(\varepsilon(\mathbf{p})) = \delta\varepsilon(\mathbf{p}) (\partial n_0/\partial \varepsilon)$).

As the second equation for $\delta\varepsilon(\mathbf{p})$ and $\partial u_k/\partial r_i$, we take the condition that the energy be a minimum with respect to an infinitesimal change of $\partial u_k/\partial r_i$ in the presence of the field φ :

$$\lambda_{iklm} \frac{\partial u_m}{\partial r_l} - \int dS \zeta_{ik}(\mathbf{p}) \delta\varepsilon(\mathbf{p}) + \eta_{ik} \varphi = 0. \quad (4.6)$$

We shall express the change of energy $\delta\varepsilon(\mathbf{p})$ from Eq. (4.5):

$$\delta\varepsilon(\mathbf{p}) = X(\mathbf{p}) \varphi + \xi_{ik}(\mathbf{p}) \frac{\partial u_k}{\partial r_i}. \quad (4.7)$$

Here $X(\mathbf{p})$ satisfies the equation

$$X(\mathbf{p}) + \int dS' f(\mathbf{p}, \mathbf{p}') X(\mathbf{p}') = 1, \quad (4.8)$$

and the function $\bar{\xi}_{ik}(\mathbf{p})$ is the solution of the equation

$$\xi_{ik}(\mathbf{p}) + \int dS' f(\mathbf{p}, \mathbf{p}') \xi_{ik}(\mathbf{p}') = \zeta_{ik}(\mathbf{p}). \quad (4.9)$$

which, as can be seen from (3.14), coincides with the \mathbf{k} -limit ($\omega = 0, \mathbf{k} \rightarrow 0$) of the function $\xi_{ik}(\mathbf{p}, \mathbf{k})$:

$$\xi_{ik}(\mathbf{p}) = \zeta_{ik}^h(\mathbf{p}) \equiv \lim_{\omega \rightarrow 0, \mathbf{k} \rightarrow 0} \xi_{ik}(\mathbf{p}, \mathbf{k}).$$

By substituting (4.7) into (4.6), we obtain

$$\lambda_{iklm} \frac{\partial u_m}{\partial r_l} = -\bar{\eta}_{ik} \varphi, \quad (4.10)$$

where

$$\lambda_{iklm} = \lambda_{iklm} - \int dS \zeta_{ik}(\mathbf{p}) \xi_{lm}(\mathbf{p}), \quad (4.11)$$

$$\bar{\eta}_{ik} = \eta_{ik} - \int dS \xi_{ik}(\mathbf{p}). \quad (4.12)$$

In deriving (4.10), we have made use of the relation

$$\int dS \zeta_{ik}(\mathbf{p}) X(\mathbf{p}) = \int dS \xi_{ik}(\mathbf{p}), \quad (4.13)$$

which stems from (4.8), (4.9) and (3.14).

Defining the tensor Λ_{iklm} by

$$\Lambda_{iklm} \bar{\lambda}_{lmns} = \delta_{in} \delta_{ks}, \quad (4.14)$$

we obtain from (4.7) and (4.10)

$$\delta\varepsilon(\mathbf{p}) = [X(\mathbf{p}) - \bar{\xi}_{ik}(\mathbf{p}) \Lambda_{iklm} \bar{\eta}_{lm}] \varphi, \quad (4.15)$$

$$\partial u_k / \partial r_i = -\Lambda_{iklm} \bar{\eta}_{lm} \varphi. \quad (4.16)$$

Substituting (4.15) and (4.16) into (4.3), taking account of the relation between $\delta n(\mathbf{p})$ and $\delta\varepsilon(\mathbf{p})$ and the formula (4.2), we shall have

$$\frac{\partial N}{\partial \mu} = \left(\frac{\partial N}{\partial \mu} \right)_F + \bar{\eta}_{ik} \Lambda_{iklm} \bar{\eta}_{lm}, \quad (4.17)$$

where we have introduced the notation

$$\left(\frac{\partial N}{\partial \mu} \right)_F = \int dS X(\mathbf{p}). \quad (4.18)$$

Substituting (4.17) into (4.1), we obtain an expression for the compressibility

$$\frac{\partial P}{\partial \rho} = \frac{N/m}{\left(\frac{\partial N}{\partial \mu} \right)_F + \bar{\eta}_{ik} \Lambda_{iklm} \bar{\eta}_{lm}}. \quad (4.19)$$

In the absence of the subsystem of quasi-particles (which may be regarded as the limit of zero volume of the Fermi surface), $(\partial N/\partial \mu)_F = 0$ and, as we shall see below, $\bar{\eta}_{ik} = N_0 \delta_{ik}$. Therefore, the expression (4.19) goes over into the usual formula of the classical theory of elasticity. The quantity Λ_{iklm} has, in this case, the meaning of the inverse tensor of the elastic moduli.

The tensor η_{ik} that we have introduced above is not, in reality, independent. It can be expressed in terms of other parameters of the theory, on the basis of the following arguments. When an external scalar field with potential $\varphi e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}$ is switched on with the condition that $|\mathbf{k}|/\omega \rightarrow 0, |\mathbf{k}|L \ll 1$ (L characterizes the dimensions of the system), the crystal will perform periodic oscillations as a whole, since the above-mentioned field corresponds to an electric field of frequency ω , uniform over the dimensions of the system; the product of the amplitude of the field intensity with the charge of a particle is equal to $\mathbf{k}\varphi$. According to classical mechanics, the time-averaged correction, associated with the field, to the energy density is equal to

$$\delta\mathcal{E} = \frac{N_c k^2}{m\omega^2} \varphi^2. \quad (4.20)$$

At the same time, $\delta\mathcal{E}$ can be determined by using the equations of motion for $\delta n(\mathbf{p}, \mathbf{x})$ and $u_i(\mathbf{x})$ in an external field. By means of simple but cumbersome calculations, which we shall not give here, one can obtain

$$\begin{aligned} \delta\mathcal{E} &= \left\{ \int dS v_i \left(v_k + \int dS' f(\mathbf{p}, \mathbf{p}') v_k' \right) + \left(\eta_{ii} + \int dS v_i \xi_i(\mathbf{p}) \right) \rho_{im}^{-1} \right. \\ &\quad \left. \times \left(\eta_{mk} + \int dS v_k \xi_m(\mathbf{p}) \right) \right\} \frac{k_i k_k}{\omega^2} \varphi^2. \end{aligned} \quad (4.21)$$

After equating (4.20) and (4.21), taking (3.4) and (3.6) into account, we obtain

$$\eta_{ik} = N_0 \delta_{ik} - \int dS P_i(\mathbf{p}) v_k. \quad (4.22)$$

One further relation between the parameters of the

theory follows from Eq. (4.15):

$$\frac{\partial \varepsilon(\mathbf{p})}{\partial \mu} - X(\mathbf{p}) = \bar{\xi}_{ik}(\mathbf{p}) \Lambda_{iklm} \bar{\eta}_{lm}. \quad (4.23)$$

In the case of an isotropic Fermi liquid, from (4.22), taking into account that $P_i(\mathbf{p}) = p_i$, we obtain $\eta_{ik} = 0$, by virtue of which (4.23) goes over to the relation for $\partial p_0 / \partial \mu$ (p_0 is the Fermi momentum) which arises from the Landau theory^[4]. As the volume of the Fermi surface tends to zero, $\eta_{ik} \rightarrow N_0 \delta_{ik}$; then $\bar{\eta}_{ik} \rightarrow \eta_{ik}$, and from (4.23) it follows that $\bar{\xi}_{ik}(\mathbf{p}) \rightarrow 0$.

We proceed to the determination of the specific heat. According to (2.1), in lowest order in the temperature,

$$C_V = \left(\frac{\partial \mathcal{E}}{\partial T} \right)_{N,V} = \int \frac{d\mathbf{p}}{(2\pi)^3} \varepsilon(\mathbf{p}) \left(\frac{\partial n}{\partial T} \right)_N = \frac{\pi^2}{3} T \int dS. \quad (4.24)$$

The integral

$$\int dS = \int \frac{d\mathbf{p}}{(2\pi)^3} \delta(\varepsilon(\mathbf{p}) - \mu)$$

determines the density of states at the Fermi surface.

It is obvious that a term ($\sim T^3$), corresponding to the contribution of the Bose branches of the excitations to the specific heat of the crystal, must be added to the expression (4.24). The correct allowance for these lies beyond the limits of accuracy of the phenomenological theory, and this limits the applicability of formula (4.24) to the region of extremely low temperatures:

$$T^2 \ll \frac{N_F T_D^3}{N_0 T_F}, \quad T \ll T_F, \quad (4.25)$$

where T_F is the Fermi degeneracy temperatures, N_F is the number of quasi-particles (expressible by the volume of the Fermi surface) and T_D is the temperature determined by the limiting frequency of the phonon excitations.

Up to this point, we have omitted for brevity the symbols for the spin indices of the quantities appearing in the theory. If the spin is taken into account, all the results remain valid, if along with the integrations over $d\mathbf{p}$ and dS it is implied that the trace is to be taken over the spin indices of the integrand. In particular, for example, the integral

$$\int dS = \int \frac{d\mathbf{p}}{(2\pi)^3} \delta(\varepsilon(\mathbf{p}) - \mu)$$

is the total density of states of quasi-particles with both spin directions (we are considering a crystal consisting of particles with spin $1/2$) at the Fermi surface.

If the interaction of the spins has an exchange origin, the Landau function, as in the isotropic case^[4], has the following structure:

$$f_{\alpha\beta\gamma\delta}(\mathbf{p}, \mathbf{p}') = f_0(\mathbf{p}, \mathbf{p}') \delta_{\alpha\gamma} \delta_{\beta\delta} + f_s(\mathbf{p}, \mathbf{p}') \sigma_{\alpha\gamma} \sigma_{\beta\delta}. \quad (4.26)$$

Here σ are the Pauli matrices. In accordance with what we have stated above, the quantity $f_0(\mathbf{p}, \mathbf{p}')$ occurs in the expressions (2.7) and (2.9) for $\xi_i(\mathbf{p})$ and ρ_{ik} . It is obvious also that $\xi_i^{\alpha\beta}(\mathbf{p})$ and $\xi_{ik}^{\alpha\beta}(\mathbf{p})$ are proportional to $\delta_{\alpha\beta}$.

Finally, we shall give an expression for the magnetic spin susceptibility χ of the crystal, which can be obtained by taking into account the structure of the Landau function and the remarks made above:

$$\chi = \mu_0^2 \int dS Y(\mathbf{p}), \quad (4.27)$$

where $Y(\mathbf{p})$ satisfies the equation

$$Y(\mathbf{p}) + \int dS' f_s(\mathbf{p}, \mathbf{p}') Y(\mathbf{p}') = 1, \quad (4.28)$$

and μ_0 is the magnetic moment of an individual particle.

As can be seen from (4.27) and (4.28), the spin susceptibility is completely determined by the contribution of the Fermi excitations. This fact is a consequence of the assumption that the microscopic magnetic-moment density of the crystal is equal to zero in the ground state at $T = 0$. Therefore, excitation of Fermi quasi-particles is the only possibility of formation of a magnetic moment in the presence of a magnetic field.

In conclusion, we shall discuss the connection between the theory developed and the experimental data on solid He³. Sample and Swenson^[8], and also Pandorf and Edwards^[9], performed calorimetric measurements of the specific heat of bcc He³ and, assuming that this is determined by the contribution of the phonons, interpreted their data by a cubic temperature dependence, $C_V \sim (T/T_D)^3$ (T_D is the Debye temperature). It was ascertained that T_D has an appreciable dependence on temperature; on decrease of the temperature, $T_D(T)$ does not tend to a constant, but continues to fall after passing through a maximum ($T_{\max} \sim 10^{-2} T_D$). It is characteristic that, in experiments with solid He⁴ with the same apparatus, it was found that $T_D(\text{He}^4)$ becomes a constant. This gives grounds for supposing that the anomaly found in He³, is not a consequence of spurious effects of the measurements.

The anomalous behavior of the specific heat of bcc He³ was also confirmed by measurement of the thermal conductivity^[10] under conditions in which the mean free path of the phonons does not depend on the temperature or on the thermal coefficient of the pressure $(\partial P / \partial T)_V$ ^[11].

Greywall and Munarin^[12] measured the velocity of sound in bcc He³ and showed that the Debye temperature determined from these data differs from the T_D associated with the specific heat.

In our opinion, the experimentally established anomaly in the specific heat of bcc He³ is a consequence of the contribution of Fermi excitations; in other words, the fall of T_D with decreasing temperature is the result of a transition of the specific heat from a cubic dependence on T to the linear dependence determined by formula (4.24). Unfortunately, for the present there is no possibility of a quantitative comparison of the theory with experiment. For this, it is necessary to extend the specific-heat measurements into the region of lower temperatures, where the contribution of the Fermi quasi-particles is dominant.

¹W. Brenig, Z. Phys. **171**, 60 (1963).

²P. Choquard, The Anharmonic Crystal, W. A. Benjamin, Inc., N.Y., 1967.

³A. F. Andreev and I. M. Lifshits, Zh. Eksp. Teor. Fiz. **56**, 2056 (1969) [Sov. Phys.-JETP **29**, 1107 (1969)].

⁴L. D. Landau, Zh. Eksp. Teor. Fiz. **30**, 1058 (1956) [Sov. Phys.-JETP **3**, 920 (1956)].

⁵J. M. Luttinger, Phys. Rev. **119**, 1153 (1960).

⁶L. D. Landau and E. M. Lifshits, Teoriya uprugosti (Theory of

Elasticity) Fizmatgiz, M., 1965 [English transl. of first edition published by Pergamon Press, London, 1959].

⁷L. D. Landau, Zh. Eksp. Teor. Fiz. **32**, 59 (1957) [Sov. Phys.-JETP **5**, 101 (1957)].

⁸H. H. Sample and C. A. Swenson, Phys. Rev. **158**, 188 (1967).

⁹R. C. Pandorf and D. O. Edwards, Phys. Rev. **169**, 222 (1968).

¹⁰W. C. Thomlinson, Phys. Rev. Lett. **23**, 1330 (1969).

¹¹P. N. Henriksen, M. F. Panczyk, S. B. Trickey, and E. D. Adams, Phys. Rev. Lett. **23**, 518 (1969).

¹²D. S. Greywall and J. A. Munarin, Phys. Rev. Lett. **24**, 1282 (1970); Phys. Rev. Lett. **25**, 261 (1970).

Translated by P. J. Shepherd
180