SOVIET PHYSICS JETP VOLUME 35, NUMBER 4 OCTOBER, 1972

Theory of a Surface "Mixed" State in Type I Superconductors

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Submitted November 9, 1971

Zh. Eksp. Teor. Fiz. 62, 1540–1547 (April, 1972)

The structure of a "superconducting" layer covering the inner surface of a hollow cylindrical superconductor with a current is elucidated. The electromagnetic impedance of the inner surface is calculated. The behavior of the system in a transverse external magnetic field is investigated.

A singly-connected type I superconductor, in which an electric current is flowing, can exist in three possible states, depending on the value of the current: superconducting, normal, and intermediate. The intermediate state is[1-3] a set of macroscopic (i.e., large in comparison with the superconducting coherence length) regions occupied alternately by normal and superconducting phases. The situation changes markedly if we are dealing with a multiply-connected superconductor with a current. In this case, in addition to the intermediate state, there also ought to exist the so-called surface "mixed" state. The conditions for the emergence of a "mixed" state are realized, for example, in a hollow superconducting cylinder with a current flowing along its axis. As was noted by L. D. Landau,[4] the intermediate state vanishes in a hollow cylinder if the current exceeds $I_i = I_c(r_1^2 + r_2^2)/2r_1r_2$, where $I_c = cH_cr_2/2$ is the critical field, and r_1 and r_2 are the respective radii of the inner and outer surfaces of the specimen. However, for $I > I_i$, the purely normal state also cannot exist. Actually, for an arbitrary value of the current, the magnetic field inside the opening is equal to zero and is therefore small close to the inner surface. It is then clear that even at $I \gg I_i$ the normal state is unstable against the onset of superconductivity close to the internal surface.

It could be thought that a macroscopic superconducting layer would be formed on the inner surface of the specimen, but this is impossible, since the separation boundary of the normal and superconducting phases would be in this case in a state of neutral equilibrium and the presence of an electric field of even small magnitude in the normal phase would lead to continuous motion of the boundary in the direction of the inner surface of the sample. In this connection, L. D. Landau[4] assumed that the metal undergoes near the inner surface a transition into some "mixed" state, in which superconductivity and an electric field exist simultaneously. The existence of this state was demonstrated experimentally by I. Landau and Sharvin.[5]

The presence of the "mixed" state plays an important role also in the current range $I_c < I < I_i$, when the sample is found in the intermediate state. That is, normal layers, which appear on the inner surface, should be covered by a layer of the "mixed" state. In the work of one of the authors,[6] the structure of the intermediate state which appears under these conditions was elucidated. The problem of the structure of the mixed state itself was not studied in that case. The goal of the present research is the clarification of the structure of the surface "mixed" state. We shall consider the range of currents $I > I_i$, where the intermediate state is absent and the properties of the sample as a whole are determined by the presence of the mixed state.

1. THE STRUCTURE OF THE SURFACE LAYER

We shall consider a massive hollow cylindrical sample of a pure type-I superconductor, i.e., we shall assume that the characteristic radius of the sample R and the free path length of the electrons l are sufficiently large. The range of values of the total current I begins with $I \sim I_i \sim cH_cR$ and extends to $I \sim I_m \sim cH_cR^2/\xi$, where I_m is the value of the current above which the surface mixed state disappears, as we shall see; $\xi = \xi(T)$ is the coherence length. The indicated values of the current are not small in the sense of the magnetic field produced by them. This magnetic field of the current exceeds the characteristic value of H_c for the superconductor. It is essential, however, that the electric field $E \sim I/R^2\sigma$ (σ is the conductivity of the normal phase) is proportional to $1/Rl$ for $I \sim I_i$, and is proportional to $1/l$ for $I \sim I_m$. Expressed in terms of units that are natural for the superconductor, the electric field is therefore small. We should thus consider the problem in which the electric field is finite but the electric field tends to zero. In the case of a superconductor, the problem of the effect of an infinitesimally small electric field is not trivial—an infinitesimally small static field can in principle produce finite changes in the properties of the system. A similar situation arises in the flow of a superfluid under the action of a pressure gradient. In the latter case, the general picture of the flow is well known. Vortex filaments are formed in the liquid, the interaction of which with the walls of the vessel balances the pressure gradient. If the pressure gradient tends to zero, then the number of vortex filaments also tends to zero, and therefore the properties of the liquid in a particular part of the volume are identical with the properties of a liquid in thermodynamic equilibrium. We shall assume that a similar situation applies in our case, and shall neglect the electric field everywhere, and consider a system that is in equilibrium for a given value of the current.

As shown above, the purely normal state of a hollow cylinder is unstable against the onset of superconductivity near the inner surface of the sample. To ascertain the structure of the resultant surface of the "superconducting" layer, we write out the Ginzburg-Landau equation with account of the normal current

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$$-\xi^2\frac{d^2\psi}{dx^2}+(A^2-1)\psi+\psi^3=0,$$
$$\lambda^2\frac{d^2A}{dx^2}=A\psi^2-\lambda^2\xi\frac{8\pi e}{\hbar c^2}j_n, \tag{1}$$

where ψ is the ratio of the ordering parameter to its equilibrium value in a bulk superconductor, A is the vector potential of the magnetic field, expressed in units of $2^{1/2}H_c\lambda$, λ the penetration depth of the magnetic field in the superconductor, e the charge of the electron, and j_n the density of the normal current. The coordinate x is measured from the inner surface of the cylinder into the interior of the sample. The solution of interest to us is shown qualitatively in Fig. 1. The layer thickness d, as we see, is always greater than or of the order of ξ. Inasmuch as the parameter κ of the Ginzburg-Landau theory is usually small in pure type I superconductors, the thickness d is large in comparison with the penetration depth λ. The magnetic field near x = d changes from zero to some value H_0.

Since $\lambda \ll d$, then the value of the potential of the magnetic field A in the basic part of the volume of the layer is determined by the condition of the vanishing of the right-hand side of the second equation of (1). Recognizing that $j_n \sim I/R^2$, $I \lesssim cH_cR^2/\xi$ and, as we see, $\psi \sim 1$, we find

$$A\sim\frac{\lambda_2\xi}{\psi^2}\frac{ej_n}{\hbar c^2}\lesssim\frac{\lambda^2H_ce}{\hbar c}\sim\frac{\lambda}{\xi}\sim\varkappa.$$

This means that the effect of the normal current can be neglected in the case under consideration when $\kappa \ll 1$, and we can solve the ordinary equations of Ginzburg-Landau, i.e., Eq. (1) with $j_n = 0$. The vector potential A for $x < d$ is then virtually equal to zero, and therefore the function ψ is determined by the equation $-\xi^2\psi'' - \psi + \psi^3 = 0$, with the boundary conditions $\psi'(0) = 0$, $\psi(d) = 0$. The first of these is the ordinary Ginzburg-Landau equation on the boundary of the superconductor-vacuum, the meaning of the second is clear from Eq. (1). The solution is expressed in terms of the elliptical sine:

$$\psi=\frac{2^{1/2}k}{(1+k^2)^{1/2}}\,\mathrm{sn}\left(\frac{d-x}{\xi(1+k^2)^{1/2}},k\right), \tag{2}$$

and the parameter k entering into (2) ($0 < k < 1$) is connected with the layer thickness by the relation

$$d/\xi(1+k^2)^{1/2}=K(k),$$

where K(k) is the complete elliptical integral of the first kind.

The magnetic field H_0 is most simply determined if we use the well-known first integral of the Ginzburg-Landau equation:

$$(1-A^2)\psi^2-{}^1/_2\psi^4+\lambda^2A'^2+\xi^2\psi'^2=\text{const.} \tag{3}$$

As $x \to \infty$, we have $\psi = 0$, $\psi' = 0$, $A' = H_0/2^{1/2}H_c\lambda$; if also x = 0, then $A' = A = 0$, $\psi' = 0$. Equating the values of the first integral at z = 0 and $x = \infty$, we find

$$H_0^2=2H_c^2[\psi^2(0)-{}^1/_2\psi^4(0)].$$

These formulas allow us to express all the characteristics of the "superconducting" layer in terms of the single parameter k:

$$d=\xi(1+k^2)^{1/2}K(k),\qquad \psi(0)=\frac{2^{1/2}k}{(1+k^2)^{1/2}},\qquad \frac{H_0}{H_c}=\frac{2k}{1+k^2}. \tag{4}$$

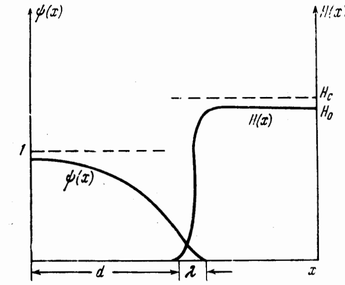

FIG. 1

The same parameter k remains arbitrary in this case. The fact that the solution of the Ginzburg-Landau equations contains an arbitrary parameter in the given case becomes clear if we note that the solution we have obtained describes, in particular, the macroscopic superconducting layer, whose thickness can of course be arbitrary. The value of the parameter k will be determined below from the condition that the thermodynamic potential of the system be minimal.

2. DEPENDENCE OF THE LAYER PARAMETERS ON THE CURRENT

For a given value of the total current through the sample, the thermodynamic potential $\mathcal{F}$ should tend toward its minimal value (see[7], where this potential is denoted by $\tilde{\mathcal{F}}$), the density of which in the normal phase can be assumed to be equal to $(H_c^2 - H^2)/8\pi$, if we take the initial measuring point of the energy such that in the superconducting phase the value of $\mathcal{F}$ is equal to zero.

We use a cylindrical set of coordinates (r, φ, z) with an axis coinciding with the axis of the sample. By virtue of the symmetry of the problem, the only magnetic field component differing from zero is $H_\varphi \equiv H$. In the normal metal, this satisfies the equation

$$\frac{d^2H}{dr^2}+\frac{1}{r}\frac{dH}{dr}-\frac{H}{r^2}=0,$$

whence it follows that

$$H=a/r+br, \tag{5}$$

where a and b are arbitrary constants. One condition for their determination is obtained from the connection between the magnetic field on the external surface of the sample and the total current I: $H(r_2) = 2I/cr_2$. The other follows from the properties, set forth in the preceding section, of the "superconducting" layer that covers the inner surface of the sample: $H(r_1 + d) = H_0$. As a result, we find

$$a=-\frac{2I}{c}\frac{(r_1+d)^2}{r_2^2-(r_1+d)^2}+H_0\frac{(r_1+d)r_2^2}{r_2^2-(r_1+d)^2},$$
$$b=\frac{2I}{c}\frac{1}{r_2^2-(r_1+d)^2}-H_0\frac{r_1+d}{r_2^2-(r_1+d)^2}. \tag{6}$$

The total thermodynamic potential per unit length of the sample is equal to the sum $\mathcal{F} = \mathcal{F}_n + \mathcal{F}_s$, where

$$\mathcal{F}_n=\int_{r_1+d}^{r_2}\frac{H_c^2-H^2}{8\pi}2\pi r\,dr$$

is the contribution of the normal region $r_2 > r > r_1 + d$, and

$$\mathcal{F}_s=r_1\frac{H_c^2}{2}\int_0^d dx\left\{\frac{1}{2}-\psi^2+\frac{\psi^4}{2}+\psi'^2\right\} \tag{7}$$

is the contribution of the superconducting region. The appearance of the term $\frac{1}{2}$ in the curly brackets of this latter equation is connected with the fact that we reckon the potential from the value in the superconducting phase.

We shall carry out further calculations separately for two overlapping regions of currents. First, let the current satisfy the inequality $I \ll I_c(R/\xi)$. In this case, as we shall see below, the parameter k is close to unity, and the field H_0 is close to H_c. Let $\mathcal{F}_{n0}$ be the value of the potential $\mathcal{F}_n$ for $d = 0$ and $H = H_c$. The difference $\mathcal{F}_n - \mathcal{F}_{n0}$ can be expanded in powers of the small quantities $(H_c - H_0)/H_c$ and d/r_1 and limited to the first nonvanishing terms. As a result, we get

$$\mathcal{F}_n - \mathcal{F}_{n0} = \frac{1}{2}\frac{r_1 r_2^2}{r_2^2 - r_1^2}\left(a_0 \ln\frac{r_2}{r_1} + b_0\frac{r_2^2 - r_1^2}{2} - \frac{a_0}{2}\frac{r_2^2 - r_1^2}{r_2^2} - \frac{b_0}{4}\frac{r_2^4 - r_1^4}{r_2^2}\right)\left\{\left(b_0 - \frac{a_0}{r_1^2}\right)\frac{d}{\xi} + H_c - H_0\right\}, \quad (8)$$

where the quantities a_0 and b_0 are obtained from a and b if we set $d = 0$ and $H_0 = H_c$. For the potential $\mathcal{F}_s$, we find from Eq. (7)

$$\mathcal{F}_s = \frac{1}{2} r_1 \xi H_c^2 \left\{\frac{d}{\xi}\left[\frac{1}{2} - \frac{2k^2}{(1+k^2)^2}\right] + \frac{4k^2}{(1+k^2)^{3/2}}\int_0^1 [(1-w^2)(1-k^2w^2)]^{1/2} dw\right\}; \quad (9)$$

the ratio of the potential $\mathcal{F}_s$ to the difference $\mathcal{F}_n - T_{n0}$ is of the order of $\xi/r_1 \ll 1$; therefore, to determine the equilibrium value of the parameter k, it suffices to minimize the expression in the curly brackets of Eq. (8). Using the asymptotic formula $K(k) = \frac{1}{2}\ln 8/(1 - K)$, which is valid for $k \to 1$, we find

$$k = 1 - \left\{\frac{\sqrt{2} r_2 \xi}{r_2^2 - r_1^2}\frac{I - I_i}{I_c}\right\}^{1/2}. \quad (10)$$

The thickness of the layer in the region of currents under consideration is large in comparison with the coherence length

$$d = \frac{\xi}{2\sqrt{2}}\ln\left\{\frac{32\sqrt{2}(r_2^2 - r_1^2)}{r_2 \xi}\frac{I_c}{I - I_i}\right\}$$

in connection with which the energy gap Δ on the inner surface $(r = r_1)$ is close to the value Δ_0 in the bulk superconductor, and the field H_0 is close to H_c:

$$\frac{\Delta_0 - \Delta}{\Delta_0} = \left\{\frac{r_2\xi(I - I_i)}{2\sqrt{2}(r_2^2 - r_1^2)I_c}\right\}^{1/2}, \qquad \frac{H_c - H_0}{H_c} = 2\left(\frac{\Delta_0 - \Delta}{\Delta_0}\right)^2.$$

In the case of large currents $I \gg I_c$, it is convenient to calculate the difference between the potential $\mathcal{F}_n$ and the potential $\mathcal{F}_N$ of a sample located in a purely normal state. The latter is obtained from the formula

$$\mathcal{F}_N = \frac{1}{4}\int_{r_1}^{r_2} r\,dr(H_c^2 - H^2),$$

where the field H has the previous value (5) but the constants a and b are equal to

$$a_N = -\frac{2I}{c}\frac{r_1^2}{r_2^2 - r_1^2}, \qquad b_N = \frac{2I}{c}\frac{1}{r_2^2 - r_1^2}.$$

We have

$$\mathcal{F}_n - \mathcal{F}_N = \frac{I}{4c}\frac{r_2^2 r_1^3}{(r_2^2 - r_1^2)^2}\left(\frac{r_2^4 - r_1^4}{r_1^2 r_2^2} - 4\ln\frac{r_2}{r_1}\right)\left\{\frac{4I}{c}\frac{d}{r_2^2 - r_1^2} - H_0\right\}. \quad (11)$$

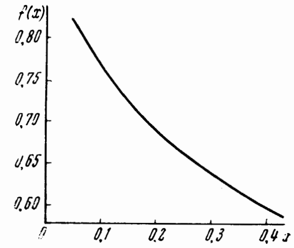

FIG. 2

The potential $\mathcal{F}_s$ can again be neglected, and the condition for equilibrium written as the condition of the minimum of the expression in the curly brackets of Eq. (11), i.e.,

$$\frac{2I}{I_c}\frac{r_2\xi}{r_2^2 - r_1^2}(1+k^2)^{1/2}K(k) - \frac{2k}{1+k^2} = \min. \quad (12)$$

Then the parameter k is determined as a function of the current:

$$k = f\left(\frac{2I}{I_c}\frac{r_2\xi}{r_2^2 - r_1^2}\right). \quad (13)$$

A plot of the function $f(x)$, obtained as a result of numerical calculation, is shown in Fig. 2. We note that the result obtained from (12) for $I\xi/I_cR \ll 1$ agrees with Eq. (10), as was to be expected. The relations (13) and (4) completely determine the parameters of the surface layer in the region of large currents.

The current $I = I_m$ above which the surface mixed state vanishes and the sample undergoes a transition to the purely normal state is determined from the condition of the vanishing of the difference $\mathcal{F}_n - \mathcal{F}_N$ or, what is the same thing, from the condition of the vanishing of the minimum of the expression (12). The corresponding limiting value of the parameter k is a root of the equation

$$(1+k^2)E(k) - (2 - 3k^2 + k^4)K(k) = 0,$$

as is easily seen. Here $E(k)$ is a complete elliptic integral of the second kind. Numerical calculation yields $k = 0.58$. The ratio of the current I_m of transition to the normal state to the critical current is expressed in terms of the limiting value of k by the formula

$$\frac{I_m}{I_c} = \frac{r_2^2 - r_1^2}{r_2\xi}\frac{k}{(1+k^2)^{3/2}}K^{-1}(k) \approx 0{,}21\frac{r_2^2 - r_1^2}{r_2\xi}. \quad (14)$$

We note that I_m greatly exceeds the previously calculated value[8] I_{c2}, at which the fluctuations of the superconducting ordering parameter increase when approaches from above in the normal state. From the point of view of the thermodynamic theory developed here, the current I_{c2} determines the limit below which the normal state is absolutely unstable against the superconducting fluctuations. Actually, even at $I = I_m$ a first-order transition into the mixed state should take place.

In concluding this section, we write down the formulas obtained for the electromagnetic impedance Z of the inner surface of the sample. If the frequency of the alternating field is not too large, the surface impedance of the superconductor is uniquely determined (see[7]) by the penetration depth λ. Since the thickness of the "superconducting" layer is large in comparison with

λ, the difference between the field penetration depth in the sample and the value that is characteristic for the infinite superconductor is wholly due to the difference between the energy gap Δ on the inner surface and Δ_0. Recognizing that the surface impedance is proportional to the penetration depth, which is in turn is inversely proportional to the energy gap, we find

$$\frac{Z}{Z_s}=\frac{\Delta_0}{\Delta}=\left(\frac{1+k^2}{2k^2}\right)^{1/2},$$

where Z_S is the surface impedance of the bulk superconductor. If the current is weak, i.e., $I \ll I_c(R/\xi)$, then the impedance Z is close to Z_S:

$$\frac{Z-Z_s}{Z_s}=\left\{\frac{r_2\xi}{2\sqrt{2}(r_2^2-r_1^2)}\frac{I-I_l}{I_c}\right\}^{1/2}.$$

The quantity Z increases with increasing current, reaching 1.4 Z_S at $I = I_m$, after which it changes jumpwise to the value characteristic for the metal in the normal state.

3. THE EFFECT OF NONCOAXIALITY OF THE SURFACES OF THE SAMPLE

The behavior of the system in a region of large currents is shown to be very sensitive to a displacement δ of the axes of the inner and outer cylindrical surfaces relative to one another. For the validity of the formulas of the previous section, therefore, it is necessary that the very rigorous inequality $\delta \ll \xi$ be satisfied. In this section, we shall consider the case in which $\delta \gg \xi$ but still $\delta \ll r_1$. The problem under study is also interesting for another reason. The fact is that, in the presence of the noncoaxiality δ in the normal metal, a homogeneous magnetic field

$$H_\delta = 2I\delta/c(r_2^2-r_1^2)$$

appears on the inner surface. It then follows that the behavior of the noncoaxial sample is equivalent to the behavior of a sample with $\delta = 0$, placed in an external magnetic field H_δ perpendicular to the axis.

We have seen above that if the current is not too great, i.e., $I \ll I_m$, then the thickess of the "superconducting" layer that covers the inner surface is large in comparison with ξ. Since the magnetic field on the outer surface of such a layer should be tangential everywhere and equal in magnitude to H_c, then it is clear that this surface will have a cylindrical shape and will be coaxial with the outer surface of the sample. Moreover, if we note that it is energywise advantageous to choose the radius of the surface of the layer to be as small as possible, we arrive at the configuration shown in Fig. 3. As is clear from the drawing, the distribution of the magnetic field in the normal phase in this case is identical with the distribution arising in the coaxial sample, whose inner surface is covered by a superconducting layer, with the parameters $d = \delta$ and $H_0 = H_c$. It is therefore easy to write down the difference $\mathcal{F} - \mathcal{F}'_N$, where $\mathcal{F}'_N$ is the potential of the coaxial normal sample for $I \gg L_c$, using Eq. (11). Noting that the difference of the potentials for the noncoaxial and coaxial normal samples is proportional to $(\delta/R)^2$ and is therefore negligibly small, we have

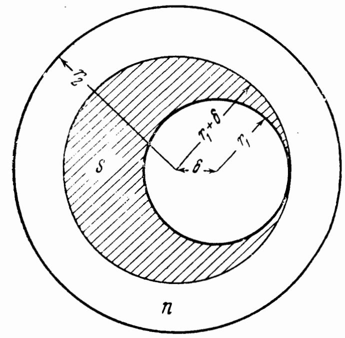

FIG. 3

$$\mathcal{F}-\mathcal{F}_N=\frac{I}{4c}\frac{r_2^2r_1^3}{(r_2^2-r_1^2)^2}\left(\frac{r_2^2-r_1^2}{r_1^2r_2^2}-4\ln\frac{r_2}{r_1}\right)\left\{\frac{4I}{c}\frac{\delta}{r_2^2-r_1^2}-H_c\right\}.$$

If $I < I_1 \equiv cH_c(r_2^2 - r_1^2)/4\pi$, then the given expression is negative, i.e., the formation of the layer is energywise advantageous. For $I = I_1$, the potential $\mathcal{F}$ is comparable with $\mathcal{F}_N$; however, if the sample were to undergo here a transition to the normal state, the magnetic field inside the opening would be equal to $H_\delta = H_c/2$, i.e., significantly less than H_c. The normal state for $I_1 < I < 2I_1$ is more advantageous than the state with a homogeneous "superconducting" layer, but not stable relative to the appearance of superconductivity close to the inner surface of the sample. We therefore reach the conclusion that the surface layer cannot be homogeneous in the interval of currents from some value up to $2I_1$, and should represent a system of alternating regions of superconducting and normal phase. For $I > 2I_1$ the superconducting regions close to the surface disappear and the metal goes over into the purely normal state. We emphasize that, since $\delta \gg \xi$, this takes place for currents significantly smaller than I_m.

In conclusion, we express our gratitude to I. L. Landau and Yu. V. Sharvin for stimulating comments and advice, and to L. P. Gor'kov and I. M. Lifshitz for attention to the work and useful discussion.

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Translated by R. T. Beyer
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