SOVIET PHYSICS

JETP

A Translation of Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki

Vol. 35, No. 4, pp. 643-830 (Russian Original Vol. 62, No. 4, pp. 1217-1588) October 1972

The Effect of Energy Release on the Emission Spectrum in a Hot Universe

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Submitted October 19, 1971
Zh. Eksp. Teor. Fiz. **62**, 1217-1227 (April, 1972)

An isotropic radiation field, whose spectrum is represented by a linear superposition of Planckian spectra of different temperatures, is considered. It is shown that many mechanisms of energy release at the early stages of the expansion of the universe, e.g., energy dissipation in the adiabatic perturbations of matter density, in primary turbulence, and in the annihilation of matter and antimatter, lead to precisely this type of distortion of the relict black-body radiation spectrum. The same type of spectrum can, in particular, be the result of the "comptonization" of the Planck spectrum after multiple Compton scattering of low-temperature black-body radiation by "hot" electrons, $T_e \gg T_r$. The equilibrium temperature of electrons in such a radiation field, which, naturally, exceeds the initial (prior to heating) radiation temperature, as well as the stationary brightness radiation temperature in the Rayleigh-Jeans region of the spectrum after heating, is calculated. The brightness temperature of the bremsstrahlung of an ionized gas which is optically thick in the decimeter wave band is equal to the electron temperature, i.e., exceeds the brightness temperature in the centimeter range. The absence of an appreciable difference between the relict radiation temperatures in these wave bands allows us to infer that the energy evolved in the universe during the period corresponding to the red shifts, $1.5 \times 10^3 < z < 10^4$, did not exceed 5% of the energy of the radiation.

1. INTRODUCTION

THE universe is filled with relict radiation of temperature $T_r = 2.7°K$[1]. In the course of the expansion of the universe, the radiation temperature decreases adiabatically; earlier (at larger red shifts z) it exceeded $T_r = 2.7(1 + z)°K$. The existence of this radiation, its isotropy and spectrum, which approaches that of black-body radiation, alone gave invaluable information about the past of the universe. In particular, it has been noted[2,3] that the energy released during the early stages of the expansion should lead to deviations of the relict-radiation spectrum from that of black-body radiation. The absence of appreciable deviations imposes limitations on the possible energy released. The aim of this paper is to further investigate the nature of the possible distortions of the relict-radiation spectrum and to obtain the limitations on the energy released in the universe during the period preceding the recombination of hydrogen, $1500 < z < 10^4$.

In our previous paper[2] we considered the distortions of the relict-radiation spectrum resulting from "comptonization"—multiple scattering of the photons by "hot" electrons with $kT_r \ll kT_e \ll m_ec^2$. Under these conditions, it could be assumed that in the rest frame of the electron, the scattering takes place without change of frequency (Thomson scattering). The change in the spectrum is due entirely to the Doppler effect connected with the twofold transition from the laboratory system of coordinates to the rest frame of the electron and back. Since $T_r \ll T_e$, induced scattering does not play any role and the problem is linear. As is well known, the radiation spectrum of a moving black body remains Planckian, but with a changed (and θ-dependent, where θ is the angle) temperature $T_r = T_0(1 + v\cos\theta/c)$. Therefore, it is natural to assume that the spectrum resulting from scattering under the conditions described above can be represented as a superposition of Planckian spectra. As applied to the problem of radiation emission in a hot universe, this implies that the distortions of the spectrum due to scattering by high-temperature electrons, $T_e \gg T_r$, which are on the average at rest, do not differ from the distortions resulting from the presence of relative macroscopic motion of the plasma volumes exchanging photons. It is assumed here that the radiation in each of the plasma volumes is a black-body radiation and is rigidly bound to the matter. The total radiation of a set of sources with black-body-radiation spectrum, but of different temperatures, should have the same type of spectrum.

The shape of the distorted spectrum arising from the comptonization of the black-body radiation has been investigated by us in detail and analytic formulas have been obtained[2]. The proof of the proposition that the superposition of Planckian spectra is similar to the spectrum arising from the comptonization of black-body radiation will therefore allow us to use the existing formulas to describe a wide class of phenomena which, physically, have little in common with comptonization. This proposition will be formally verified below, where a convenient equation and relations for the superposition function will be obtained.

2. REPRESENTATION OF THE SPECTRUM ARISING FROM COMPTONIZATION IN THE FORM OF SUPERPOSITION OF PLANCKIAN SPECTRA

Let us assume that

$$n(\nu,t)=\int R(T,t)\,p(\nu,T)\,dT, \tag{1}$$

where $n=\epsilon_\nu/8\pi\nu^3$ is the occupation number in the sought spectrum of the isotropic (no dependence on angle) nonstationary (t is the time) problem; ϵ_ν is the spectral energy density of the radiation. The function

$$p(\nu,T)=[\exp(\nu/T)-1]^{-1} \tag{2}$$

is the equilibrium Plank function. Here and in the following sections, as well as in the Appendices, we use a system of units in which $h=k=c=1$. Finally, $R(T,t)$ is the temperature distribution function we want to find. The condition $n=\int RpdT$ can be met for any n if we regard it as an integral equation for R for a given p. However, it is semantically expedient to define as a superposition of Planckian spectra only that n for which R is everywhere positive and $\int RdT=1$. Only such a superposition will be obtained in a volume surrounded by black surfaces of different temperatures after averaging over the angles. Under the indicated limitations on R, the assertion that n is a superposition of Planckian spectra becomes meaningful[1].

Let us show that a Planckian radiation spectrum is converted, after scattering by hot electrons, into one which can be represented in the form of such a superposition. The kinetic equation of the interaction of the radiation with electrons for $\nu \ll T_e \ll m_e$ has the form[4]

$$\frac{\partial n}{\partial y}=\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4\frac{\partial n}{\partial\nu}, \tag{3}$$

where the dimensionless parameter

$$y=\sigma_T N_e T_e t/m_e, \tag{4}$$

has been introduced in place of the time, t, and $\sigma_T=8\pi e^4/3m_e^2$ is the Thomson-scattering cross section. Substituting n in the form (1) into (3), we seek the function $R(T,t)$ which satisfies Eq. (3). It is shown in Appendix A that (3) reduces to the equation

$$\frac{\partial R}{\partial y}=T^2\frac{\partial^2 R}{\partial T^2}+6T\frac{\partial R}{\partial T}+4R, \tag{5}$$

the solution of which is

$$R(y,T)=\frac{1}{(4\pi y)^{1/2}}\int R(0,\tau)\exp\left[4y-\frac{1}{4y}\left(\ln\frac{T}{\tau}+5y\right)^2\right]\frac{d\tau}{\tau}.$$

It can be seen from this that R is always positive when $R_0=\delta(T-T_0)$—the case of an initial Planckian spectrum—when

$$R(y,T)=\frac{1}{(4\pi y)^{1/2}}\exp\left[4y-\frac{1}{4y}\left(\ln\frac{T}{T_0}+5y\right)^2\right], \tag{6}$$

as well as for an arbitrary positive initial R.

It is also shown in Appendix A that the solution of Eq. (5) possesses the following property:

$$\frac{\partial}{\partial y}\int RT^k\,dT=k(k-3)\int RT^k\,dT. \tag{7}$$

This result includes: a) the superposition condition—for $k=0$ we obtain $\int RdT=\text{const}$, so that if $\int RdT=1$ at the beginning, then this normalization is conserved; b) the conservation of the number of quanta—for $k=3$ we find $N\sim\int RT^3dT=\text{const}$; c) the law of the growth of the total energy—for $k=4$

$$d\varepsilon/dy=4\varepsilon,\quad \varepsilon\sim e^{4y}; \tag{8}$$

d) the laws of decrease of the brightness temperature in the Rayleigh-Jeans region of the spectrum with time:

$$T_{RJ}=\int TR\,dT\Big/\int R\,dT=\int TR\,dT \tag{9}$$

i.e., $k=1$, and

$$dT_{RJ}/dy=-2T_{RJ},\quad T_{RJ}\sim e^{-2y}. \tag{10}$$

The last three results have been known for a long time[2]; they were obtained directly from Eq. (3) for n. The result (a) is not so trivial: in the frequency representation, i.e., for the function $n(\nu,t)$, it corresponds to the conservation of the quantity

$$\int\frac{n}{\nu}d\nu.$$

However, this quantity diverges and the conservation law loses its meaning. Evidently, the principal result that the comptonized spectrum can be represented in the form of a superposition of Planckian spectra with a positive weight and a conserved normalization ($R\ge 0$, $\int RdT=1$) can be generalized in several directions. First, it can be generalized to the case of scattering by relativistic electrons with an arbitrary spectrum. In this case it is necessary that the scattering function should depend nontrivially only on the ratio ν'/ν, and this imposes the condition $\nu \ll m_e$ in the rest frame of the electron, i.e., $\nu E_e^2 \ll m_e^3$, where E_e is the energy of the electron, including its rest mass. Second, it can be generalized to anisotropic spectra, which depend also on the angle, and in which the angular dependence of the scattering is taken into account. In this case however, it is necessary that the brightness temperature should remain finite and not exceed m_e in any section of the spectrum; in particular, $\nu n\neq\infty$ as $\nu\to 0$, so that the role of the induced scattering could be neglected.

3. SMALL DEVIATIONS FROM THE EQUILIBRIUM SPECTRUM

In the cosmological situation the initial spectrum is an equilibrium spectrum, i.e., $t\to 0$, $R\to\delta(T-T_0)$, and $T_0\to\infty$. The strong interaction of the radiation with the electrons prevents rapid motion on the scale of the mean free path and a rise in the temperature of the electrons. On the other hand, in the period close to modern times, when the existence of high-temperature electrons ($T_e \gg T_r = 2.7(1+z)^\circ$K) is possible,

[1]Many assertions and, in particular, Eq. (5) are true of an expansion in terms of a set of any functions of the ratio ν/T; $n=\int Q(T)f(\nu/T)dT$, where T is a parameter. The choice of the Plank function is connected with the fact that at the early stages in cosmology there is equilibrium, $n=\rho(\nu/T_0)$, $R=\delta(T-T_0)$.

the probability of their interaction with the radiation is small. On the basis of these reasons, we should expect small deviations of the spectrum from the Planck spectrum, i.e., the function R is nearly a δ-function. For R approaching a δ-function, the properties of the spectrum are given by the two moments $\int RTdT$ and $\int RT^2dT$. The first moment is the Rayleigh-Jeans temperature $T_{RJ} \equiv \int RTdT$, since on it depends the behavior of $n(\nu)$ as $\nu \to 0$. It is convenient to define in place of the second moment the dimensionless quantity $u(t)$ that characterizes the width of T and vanishes when R is a δ-function:

$$u(t) = \frac{1}{2T_{RJ}^2} \int (T - T_{RJ})^2 R(T, t)\, dT. \tag{11}$$

In the case when the distortion of the spectrum is due to scattering by hot Maxwellian electrons, the quantity u is exactly equal to the parameter

$$y = \int \frac{kT_e}{m_e c^2} \sigma_T N_e c\, dt.$$

The resulting spectrum (see Appendix B) is of the form

$$n(\nu, t) = p(\nu, T_{RJ})\,[1 + uf(\nu / T_{RJ})], \tag{12}$$

where

$$f(x) = \frac{xe^x}{e^x - 1}\left(\frac{x}{\operatorname{th} x/2} - 2\right).$$

The principal characteristics of the resulting radiation are linear functions of the parameters u: the energy density of the radiation is

$$\varepsilon = \varepsilon_{RJ}(1 + 12u) \tag{13}$$

and the number of quanta in a unit volume

$$N = N_{RJ}(1 + 6u), \tag{14}$$

where ϵ_{RJ} and N_{RJ} correspond to black-body radiation of temperature T_{RJ}. If the distortion of the spectrum is the result of an energy-release process in which the number of quanta is conserved (see the discussion of the astrophysical situation below), then we can (just as in[2]) determine from (14) the initial temperature of the undistorted Planck radiation, $T_0 = T_{RJ}(1 + 2u)$. The parameter u is then determined by the total energy released:

$$u = (\varepsilon - \varepsilon_0) / 4\varepsilon_0 = \Delta\varepsilon / 4\varepsilon. \tag{15}$$

When the released energy goes directly to heat the electrons, so that $T_e \gg T_r$ and small distortions of the spectrum develop from the Compton interaction of the radiation with the electrons,

$$u = (T_e - T_r) N_e \sigma_T t / m_e = \Delta\varepsilon / 4\varepsilon. \tag{16}$$

The theory expounded above, which does not take into account induced effects, is not applicable to multiple ($y > 1$) scattering by electrons with $T_e - T_r < T_r$: we cannot replace the complete Kompaneets equation by Eq. (3). In particular, a typical distortion of the spectrum, corresponding to a photon chemical potential $\mu \neq 0$[3], is not described by a superposition of Planckian spectra.

Formula (12) is, as in the case of comptonization, valid only if $(\nu/T_{RJ})^2 u \ll 1$. At the frequencies $\nu > T_{RJ} u^{-1/2}$ the resulting spectrum is, even for small $u \ll 1$, similar to the spectrum arising from the comptonization of black-body radiation by Maxwellian electrons only when $R(T)$ is a Gaussian function of the temperature. Thus, for example, for a small-scaled random motion of plasma volumes with black-body radiation in each of them and with a Maxwellian velocity distribution, we have

$$R(T, t) = [4\pi u(t)]^{-1/2} \exp\left[-\frac{(T/T_e - 1)^2}{4u(t)}\right], \tag{17}$$

where T_e is the local radiation temperature, and the parameter u is determined by the mean square velocity of the motion, $u = \overline{v}^2/6c^2$. The expression (17) is a consequence of the Doppler formula $T = T_e(1 + vc^{-1}\cos\theta)$. It follows from (6) and (17) that $T_0 = T_e$. Comparing (6) and (13), we easily see that they are identical in the limit of small $y \ll 1$ and $u \ll 1$.

The electron temperature in the field of non-equilibrium radiation. The equilibrium electron temperature in an isotropic radiation field of arbitrary spectrum[5] is equal to

$$T_e = \frac{1}{4} \int n(1 + n) \nu^4 d\nu \Big/ \int n\nu^3 d\nu. \tag{18}$$

We show in Appendix B that for a spectrum of the form (12), i.e., for small deviations of the spectrum from a Planckian spectrum and for $u \ll 1$, the electron temperature is linearly related to the radiation temperature in the Rayleigh-Jeans section of the spectrum:

$$T_e = T_{RJ}(1 + 7{,}4\, u). \tag{19}$$

Evidently, at very low frequencies the bremsstrahlung processes establish a radiation brightness temperature equal to the electron temperature, i.e., exceeding T_{RJ} and T_0.

4. LIMITATIONS ON THE ENERGY RELEASED IN THE UNIVERSE IN THE PRE-RECOMBINATION PERIOD

An obvious example of the application of the formulas obtained above to cosmology is the model of the universe with a developed small-scale turbulence. Prior to the recombination of hydrogen during the red shift $z_{rec} \sim 1500$[6,7], only scales negligible as compared with the mass contained within the horizon were transparent to the radiation; on larger scales the radiation was rigidly bound to the matter and moved with the same velocity as the matter. In the course of the recombination, the optical thickness with respect to Thomson scattering quickly decreased, larger and larger scales became transparent, and the radiation contained in them became intermixed—the resulting spectrum became a superposition of Planckian spectra with different temperatures. If, simultaneously with the energy-containing scale, the whole universe were to become transparent, then the principal effect would be not a deviation of the spectrum from the Planck spectrum, but an angular anisotropy of the radiation[8,9]. Indeed, recombination takes place comparatively slowly and when scales of the order of galaxies and clusters of galaxies become transparent, the optical thickness of the universe with respect to scattering is still very large, and this leads to a strong washing out of the small-scale angular fluctuations of the radiation[10]. The spectral effects, however, remain.

If the whirl theory of the formation of the galaxies is valid, then the turbulence developed after recombination becomes supersonic, generates shock waves, and leads to the heating up of the electrons to high temperatures, comptonization, and additional distortions of the spectrum of the radiation, How are the two enumerated causes of distortion of the spectrum related? The first effect—the intermixing of the Planckian spectra with different temperatures—carries information only about the kinetic energy of motion of the radiation, whereas the second effect carries information about the kinetic energy of the matter. Energetically, their ratio at the instant of recombination is ρ_r/ρ_m, the ratio of the energy density of the radiation to the rest mass of the matter. However, shock waves may be generated some time after the recombination, and this should, in the case of adiabatic expansion, lead to a decrease in the kinetic energy of motion of the matter and of the relative role of comptonization in the distortion of the relict spectrum.

Less obvious, but more important for cosmology, is the existence of an upper bound on the energy release through practically any mechanism during the stage of the expansion of the universe preceding the recombination, $1500 < z < 10^4\Omega^{-1/2}$. The choice of this range of the red shifts is determined by the fact that at $z < 1500$ matter is neutral and its interaction with radiation is weak, while in the earlier energy release, $z < 10^4\Omega^{-1/2}$, Compton scattering has enough time to establish a Bose-Einstein distribution for the quanta, which allows more severe restrictions on the energy release in that period. Here and below, $\Omega = \rho/\rho_{crit} = 8\pi G\rho/3H_0^2$ is the dimensionless density of matter in the universe and $H_0 = 100$ km/sec-Mpsec is the Hubble constant. The elucidation of the primary spectrum of the adiabatic and entropy density perturbations, as well as of the energy content and the spectrum of the initial turbulence, are of great importance for the theory of the formation of galaxies. At the pre-recombination stage, the small-scale density perturbations as well as the turbulent motions damp out because of the presence of radiative viscosity and heat conduction[11-14], processes during which mixing of the Planckian spectra occurs. The annihilation of antimatter and the shock waves lead to an increase in the electron temperature and to comptonization of the radiation. The experimental restrictions on the distortion of the relict spectrum, and consequently on the energy release, enable us to establish the upper bounds of the amplitude of the adiabatic density perturbations with $M < 10^{11}\,M_\oplus$, of the amplitude of the entropy perturbations of the density with $M < 10\,M_\oplus$, of the energy content of the primary turbulence, of the amount of antimatter in the universe, etc (see[15]).

Measurements of the spectrum of the relict radiation in the Wien region, where the distortions predicted by (12) are maximal, are incomplete and are grossly inaccurate (see, however,[16]). Current data are not at variance with $u \leq 0.15$[2]. At the same time, in the centimeter and decimeter wave bands the accuracy of the measurements is considerably higher and the errors in the determination of the brightness temperature of the background do not, apparently, exceed $\Delta T/T \lesssim 10\%$ for $1\text{ cm} < \lambda < 30$ cm and $\Delta T/T \lesssim 30\%$ for $30\text{ cm} < \lambda < 75$ cm[17-19,2]. We recall that it is precisely in the long-wavelength region, where the optical thickness of the universe with respect to bremsstrahlung absorption[20] exceeds unity,

$$\tau_{ff} = 1{,}8\cdot 10^{-2}\int \frac{N_e^2(t)\,g(\nu, T)}{\nu^2(t)\,T_e^{3/2}(t)}\,c\,dt$$

$$= 6\cdot 10^{-23}\frac{\lambda_0^2 g(\lambda, T)}{H_0 T_0^{3/2}}\,\Omega^{3/2}(z_1 - z_{rec}) \approx 1,$$

$$\lambda_0 \approx \frac{500}{\Omega^{3/4}}\,\frac{1}{(z_1 - z_{rec})^{1/2}\,g^{1/2}(\lambda_0, T_0)}\ \text{(см)} \qquad (20)$$

that the brightness temperature of the radiation is equal to the electron temperature and, according to (19), exceeds the radiation temperature in the shorter-wavelength region (see the figure). The establishment time for the electron temperature[21]

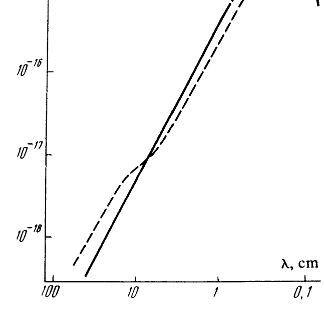
$$t \approx m_e c/4\sigma_T\varepsilon_r$$

is much smaller than the cosmological time in that period. In (20)

$$g = \frac{\sqrt{3}}{\pi}\ln\frac{2{,}35kT_0}{h\nu_0}$$

is the Gaunt factor; λ_0, $T_0 = 2.7$°K, and H_0 correspond to the current values of the wavelength, radiation temperature and the Hubble constant, z_1 is the characteristic energy-equation time. For $\Omega \geq 0.17$ and $z_1 = 2z_{rec}$, we find from (20) that $\lambda \leq 30$ cm (when $\Omega = 1$ and $\lambda_0 \approx 9.3$ cm). It follows from the limitation $\Delta T/T \lesssim 10\%$ and from (19) that $u < 1/74$ and the energy released during the stage $1500 < z < 10^4\Omega^{-1/2}$ did not exceed $\Delta\epsilon/\epsilon = 4u \lesssim 5.4\%$. Remembering that $\Delta T/T < 30\%$ in the region $30\text{ cm} < \lambda_0 < 75$ cm, we easily find from (2) the limitation on $\Delta\epsilon/\epsilon$ for any combination of Ω and z_1.

As an example, let us consider the theory of the primary turbulence[22-24]. Chibisov[14] has shown that this theory is applicable only when $\Omega > 0.3$, otherwise the motions in the energy-containing scale are dissipated during the process of recombination. Estimates from formulas cited in[23] show that for $\Omega > 0.5$ the energy released exceeds the 5% limit obtained above, i.e., the theory under discussion is not at variance with existing observations and is inherently not inconsistent only for $0.3 < \Omega < 0.5$. If the theory is true, then the mean density of matter in the universe lies in this interval and the brightness temperature of the radiation should exhibit a jump of magnitude $\Delta T/T \approx 0.8\,\Omega^{17/6}$ in the spectrum of the relict radiation in the wavelength region $\lambda \approx 3\Omega^{-5/4}$ cm.



APPENDIX A

Let us show that a Planckian radiation spectrum is transformed after scattering by "hot" electrons with $T_e \gg T_r$ into a spectrum that can be represented by a linear superposition of Planckian spectra

$$n(\nu,t)=\int R(T,t)p(\nu,T)dT.$$

The kinetic equation of the interaction of the radiation with the electrons for $\nu \ll T_e \ll m_e$ is of the form[4]

$$\frac{\partial n}{\partial y}=\frac{1}{\nu^2}\frac{\partial}{\partial\nu}\nu^4\frac{\partial n}{\partial\nu}, \qquad \text{(A.1)}$$

where $y = T_e N_e \sigma_T t/m_e$. Substituting $n = \int RpdT$, we seek the function $R(T, t)$ satisfying this equation:

$$\int\frac{\partial R}{\partial y}p\,dT=\nu^{-2}\frac{\partial}{\partial\nu}\nu^4\int R\frac{\partial p}{\partial\nu}dT=\int R\left(\nu^2\frac{\partial^2 p}{\partial\nu^2}+4\nu\frac{\partial p}{\partial\nu}\right)dT. \qquad \text{(A.2)}$$

Since $P \equiv [\exp(\nu/T) - 1]^{-1}$, it follows that

$$\nu^2\frac{\partial^2 p}{\partial\nu^2}+4\nu\frac{\partial p}{\partial\nu}=T^2\frac{\partial^2 p}{\partial T^2}-2T\frac{\partial p}{\partial T}$$

and (A.2) can be represented in the form

$$\int\frac{\partial R}{\partial y}p\,dT=\int R\left(T^2\frac{\partial^2 p}{\partial T^2}-2T\frac{\partial p}{\partial T}\right)dT. \qquad \text{(A.3)}$$

Integrating the right hand side of (A.3) by parts, we obtain Eq. (5) of the text, which is a diffusion-type equation. Equation (5) has the characteristic property (7); let us represent it for convenience in the form

$$\frac{\partial R}{\partial y}=\frac{1}{T^3}\frac{\partial}{\partial T}T^4\frac{\partial TR}{\partial T},$$

from which it follows that

$$\frac{d}{dy}\int RT^k\,dT=\int T^{k-3}\frac{d}{dT}T^4\frac{dTR}{dT}dT$$
$$=-\int T^4(k-3)\frac{dTR}{dT}T^{k-4}\,dT=k(k-3)\int RT^k\,dT.$$

APPENDIX B

Let us find for an arbitrary $R(T)$ an approximate form of the radiation spectrum $n = \int RpdT$, assuming its deviation from the Planckian form to be small. It is useful here to introduce the concept of a Rayleigh-Jeans temperature of the distorted spectrum: at sufficiently low frequencies the Planck function has the form $p = T/\nu$ and we have for the resulting spectrum

$$n=\int Rp\,dT=\frac{1}{\nu}\int RT\,dT=T_{RJ}/\nu,$$

where $T_{RJ} = \int RTdT$ is obviously the brightness temperature in the Rayleigh-Jean region of the spectrum. Expanding the function p in a series about the temperature T_{RJ} up to terms of the second order in $(T - T_{RJ})$:

$$n=\int Rp\,dT=\int R\left[p(T_{RJ})+(T-T_{RJ})\frac{\partial p(T_{RJ})}{\partial T}+\frac{(T-T_{RJ})^2}{2}\frac{\partial^2 p(T_{RJ})}{\partial T^2}\right]dT=p(T_{RJ})\int R\,dT$$
$$+\frac{\partial p(T_{RJ})}{\partial T}\left(\int RT\,dT-T_{RJ}\int R\,dT\right)+\frac{1}{2}\frac{\partial^2 p(T_{RJ})}{\partial T^2}\int(T-T_{RJ})^2R\,dT$$

and taking into account the normalization $\int RdT = 1$ and the definition of the Rayleigh-Jeans temperature $T_{RJ} = \int TRdT$, we find

$$n=p(T_{RJ})+uT_{RJ}^2\frac{\partial^2 p(T_{RJ})}{\partial T^2}, \qquad \text{(B.1)}$$

where u is given by formula (11). For a wide class of functions $R(T)$, which allow a rapid convergence of the series, such an expansion describes the required spectrum sufficiently well. We note that the expressions (B.1) are valid only for $(\nu/T_{RJ})^2u < 1$, when we can neglect the subsequent terms of the Taylor-series expansion of n.

Remembering that $p = [\exp(\nu/T - 1)]^{-1}$, we can also represent Eq. (B.1) in the form

$$n=p(T_{RJ})+u\left[\nu^2\frac{\partial^2 p(T_{RJ})}{\partial\nu^2}+2\nu\frac{\partial p(T_{RJ})}{\partial\nu}\right], \qquad \text{(B.2)}$$

from which we obtain formula (12).

Let us now find the energy density $\epsilon = \int n\nu^3 d\nu$ and the density $N = \int n\nu^2 d\nu$ of the photons of the radiation whose spectrum is described by the formulas (B.1), (B.2), and (12). Integrating (B.2) by parts, we find

$$\varepsilon=\int n\nu^3\,d\nu=\int p(T_{RJ})\nu^3\,d\nu+u\int\left(\nu^2\frac{\partial^2 p}{\partial\nu^2}+2\nu\frac{\partial p}{\partial\nu}\right)\nu^3\,d\nu$$
$$=\int p(T_{RJ})\nu^3\,d\nu+12u\int p(T_{RJ})\nu^3\,d\nu=\varepsilon_{RJ}(1+12u) \qquad \text{(B.3)}$$

and similarly

$$N=\int n\nu^2\,d\nu=N_{RJ}(1+6u), \qquad \text{(B.4)}$$

where ϵ_{RJ} and N_{RJ} are the energy and number of the photons in black-body radiation with temperature T_{RJ}. The formulas (12), (B.3), and (B.4) are identical with the corresponding formulas for small distortions of the spectrum arising from its comptonization[2]. However, as in the case of comptonization, they are applicable only when $(\nu/T)^2u \ll 1$, i.e., they describe distortions in the comparatively low-frequency region of the spectrum. It is clear that a complete similarity of the discussed spectrum distortions in the Wien region with the distortions arising from comptonization is possible only if the function $R(T)$ is Gaussian. This is precisely the case realized when the velocity distribution of Planck-radiation sources of equal temperature is Maxwellian.

APPENDIX C

The equilibrium electron temperature in an isotropic radiation field with an arbitrary spectrum is given by formula (18)[5]. We shall seek this temperature, assuming that the spectrum of the radiation $n = \int RpdT$ is described by the formulas (B.1) and that $u \ll 1$. Using (B.2) and (B.3), we find that

$$\int n\nu^3\,d\nu=(1+12u)\int p(T_{RJ})\nu^3\,d\nu=(1+12u)\varepsilon_{RJ}.$$

Using (B.2), we seek in a first approximation in $u \ll 1$

$$\int n(1+n)\nu^4\,d\nu=\int p(1+p)\nu^4\,d\nu+u\int(2p+1)\left(\nu^2\frac{\partial^2 p}{\partial\nu^2}+2\nu\frac{\partial p}{\partial\nu}\right)\nu^4 d\nu. \qquad \text{(C.1)}$$

The integral in (C.1) is equal to

$$\int p(1+p)\nu^4\,d\nu=-T_{RJ}\int\frac{\partial p}{\partial\nu}\nu^4\,d\nu=4T_{RJ}\int p\nu^3\,d\nu=4T_{RJ}\,\varepsilon_{RJ}.$$

It can be shown that for $p \equiv [\exp(\nu/T_{RJ}) - 1]^{-1}$ we have

$$(2p+1)\frac{\partial p}{\partial\nu}=-\frac{\partial^2 p}{\partial\nu^2}T_{RJ},$$

$$(2p+1)\frac{\partial^2 p}{\partial\nu^2}=-\frac{2}{3}\frac{\partial^3 p}{\partial\nu^3}T_{RJ}-\frac{1}{3T_{RJ}}\frac{\partial p}{\partial\nu}$$

and therefore the second integral in (C.1) is easily evaluated by parts:

$$\int (2p+1)\left(\nu^2\frac{\partial^2 p}{\partial\nu^2}+2\nu\frac{\partial p}{\partial\nu}\right)\nu^4\,d\nu = -\frac{2}{3}T_{RJ}\int\frac{\partial^3 p}{\partial\nu^3}\nu^6\,d\nu$$
$$-\frac{1}{3T_{RJ}}\int\frac{\partial p}{\partial\nu}\nu^6\,d\nu - 2T_{RJ}\int\frac{\partial^2 p}{\partial\nu^2}\nu^5\,d\nu = 40T_{RJ}\int p\nu^3\,d\nu$$
$$+\frac{2}{T_{RJ}}\int p\nu^5\,d\nu = 4T_{RJ}\int p\nu^3\,d\nu\left[10+\int p\nu^5\,d\nu/2T_{RJ}^2\int p\nu^3\,d\nu\right].$$

The ratio

$$\int p\nu^5\,d\nu\Big/\int p\nu^3\,d\nu = T_{RJ}^2\cdot 40\pi^2/21$$

is well known (see[25]). We obtain for the electron temperature the simple formula:

$$\theta_e \approx T_{RJ}[1+u({}^{20}/{}_{21}\pi^2-2)] \approx T_{RJ}(1+7{,}4\ u). \qquad \text{(C.2)}$$

[1]A. A. Penzias and R. W. Wilson, Astrophys. J. **142**, 419 (1965).
[2]Ya. B. Zeldovich and R. A. Sunyaev, Astrophys. Space Sci. **4**, 301 (1969).
[3]R. A. Sunyaev and Ya. B. Zeldovich, Astrophys. Space Sci. **7**, 20 (1970).
[4]A. S. Kompaneets, Zh. Eksp. Teor. Fiz. **31**, 876 (1956) [Sov. Phys.-JETP **4**, 730 (1957)].
[5]Ya. B. Zel'dovich and E. V. Levich, Zh. Eksp. Teor. Fiz. Pis'ma Red. **11**, 57 (1970) [JETP Lett. **11**, 35 (1970)].
[6]Ya. B. Zel'dovich, V. G. Kurt, and R. A. Syunyaev, Zh. Eksp. Teor. Fiz. **55**, 278 (1968) [Sov. Phys.-JETP **28**, 146 (1969)].
[7]P. J. E. Peebles, Astrophys. J. **153**, 1 (1968).
[8]J. Silk, Nature (Lond.) **216**, 453 (1968).
[9]G. V. Chibisov and L. M. Ozernoy, Astrophys. Lett. **3**, 189 (1969).
[10]Ya. B. Zeldovich and R. A. Sunyaev, Astrophys. Space Sci. **7**, 201 (1970).
[11]J. Silk, Astrophys. J. **151**, 459 (1967).
[12]P. J. E. Peebles and J. T. Yu, Astrophys. J. **162**, 815 (1970).
[13]S. Weinberg, Astrophys. J. **168**, 175 (1971).
[14]G. V. Chibisov, Astron. Zh. **49** (1), (1972) [Sov. Astron.-AJ,(1972)].
[15]Ya. B. Zeldovich and R. A. Sunyaev, Astrophys. Space Sci. **9**, 353 (1970).
[16]A. G. Blair, J. G. Beery, F. Edeskuty, R. D. Hiebert, J. P. Shipley, and K. D. Williamson, Jr., Phys. Rev. Lett. **27**, 1154 (1971).
[17]R. A. Stokes, R. B. Partridge, and D. T. Wilkinson, Phys. Rev. Lett. **19**, 1199 (1967).
[18]T. F. Howell and J. P. Shakeshaft, Nature (Lond.) **216**, 753 (1967).
[19]S. A. Pelyushenko and K. S. Stankevich, Astron. Zh. **46**, 283 (1969) [Sov. Astron.-AJ **13**, 223 (1969)].
[20]S. A. Kaplan and S. B. Pikel'ner, Mezhzvezdnaya sreda (The Interstellar Medium), Fizmatgiz, 1963 [Eng. Transl., Harvard University Press, Cambridge, Mass., 1970].
[21]R. Weymann, Astrophys. J. **145**, 560 (1966).
[22]L. M. Ozernoĭ and A. D. Chernin, Astron. Zh. **44**, 1131 (1967) [Sov. Astron.-AJ **11**, 907 (1968)].
[23]L. M. Ozernoĭ and G. V. Chibisov, Astron. Zh. **47**, 769 (1970) [Sov. Astron.-AJ **14**, 615 (1971)].
[24]T. Matsuda, H. Sato, and H. Takeda, Prog. Theor. Phys. **46**, 416 (1971).
[25]L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Nauka, 1964 [Eng. Transl., Addison-Wesley, Reading, Mass., 1969].

Translated by A. K. Agyei
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