Similarity Hypothesis and the Hydrodynamic Description of Turbulence

G. A. KUZ'MIN AND A. Z. PATASHINSKIĬ

Thermophysics Institute, Siberian Division, USSR Academy of Sciences

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An hypothesis of scaling invariance of the local structure of turbulence is formulated. The restrictions which it imposes on the form of the multi-point distribution functions for the relative velocities, momenta and their variational derivatives with respect to external force are considered. The situation which arises in the hydrodynamic description of turbulence if the hypothesis proposed is correct is analyzed by employing quantum field theory methods.

1. The attempt to solve the problem of the hydrodynamic description of fully developed turbulence leads to the difficulties that are well known in different variants of theories with strong interaction. The infinite or very large number of degrees of freedom of the system is such that it is impossible to describe it in terms of individual independent (or weakly dependent) variables and by the same token find the solution of problems even in the form of series in really small parameters.

The present understanding of the properties of a locally homogeneous and isotropic turbulence stems from the ideas of Kolmogorov.[1] The theory of self-similarity of correlations, proposed by Kolmogorov, postulates a multidimensional distribution of the probability of the velocities. Two essential assumptions concerning this distribution made it possible to obtain certain relationships explicitly. First, it is assumed that the probability distribution has the similarity property, in that a simultaneous variation of all the scales in a certain ("inertial") scale interval leaves the picture of the fluctuations invariant apart from the scale transformation. Second, it is assumed that only one parameter determines all the statistical properties of the system in this scale interval and that this parameter is the energy flux from the large to the small scales. Improvements of the Kolmogorov scheme are also known (see the review in [2]).

Similarity properties were recently observed in another object with strongly interacting fluctuations, namely in thermodynamic systems near their critical points.[3] In this case, however, both the first investigation in which the similarity properties were observed[4] and the succeeding more general studies[5,6] have shown how the similarity properties can be reconciled with the microscopic equations of the theory. We do not speak here of the proof of similarity. Similarity, in our opinion, is a rather widespread if not common property of systems with strong interactions. The purpose of the present paper is to investigate, assuming similarity, methods of reconciling the correlations in the exact equations of turbulence theory.

2. We consider first an idealized case of a locally homogeneous and isotropic turbulence.[2] The pulsations are fed from excitation of harmonics with infinitely large wavelength, and the viscosity is infinitesimally small. We consider liquid motions that are described by a system of hydrodynamic equations. A more definite mathematical formulation of the problem is given below (see Sec. 4).

Let us perturb the flux by a small external force $\mathbf{h}(\mathbf{x}, t)$. The simplest characteristics of the reaction of the system to an external action are the variational derivatives, taken at $\mathbf{h} = 0$, of the multidimensional probability distributions of the velocity pulsations in the external force. In the case considered by us, the theory has no characteristic length and time scales. Accordingly, it is natural to assume that the following similarity hypothesis holds:

a) length and time scale transformations of the form

$$x_i' = \lambda x_i, \quad t' = \lambda^\alpha t \tag{2.1}$$

(where α is a certain fixed number and λ is arbitrary) do not change the probability distributions for the relative velocities

$$\begin{gathered} \mathbf{v}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}+\mathbf{x}_0, t+t_0) - \mathbf{u}(\mathbf{x}_0, t_0), \text{ i.e.,} \\ dW_{\mathbf{x}_1, t_1, \ldots, \mathbf{x}_n, t_n}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = dW_{\mathbf{x}_1', t_1', \ldots, \mathbf{x}_n', t_n'}(\mathbf{v}_1', \ldots, \mathbf{v}_n'), \\ \mathbf{v}' = \lambda^{1-\alpha}\mathbf{v}; \end{gathered} \tag{2.2}$$

b) the transformations (2.1) in conjunction with the substitution

$$\delta / \delta\mathbf{h}(\mathbf{x}, t) \to \lambda^{4-\alpha}\delta / \delta\mathbf{h}(\mathbf{x}', t') \tag{2.3}$$

do not change the variational derivatives of the probability distributionswith respect to the external force, taken at $\mathbf{h}(\mathbf{x}, t) = 0$:

$$\frac{\delta^m W_{\mathbf{x}_1, t_1, \ldots, \mathbf{x}_n, t_n}(\mathbf{v}_1, \ldots, \mathbf{v}_n)}{\delta\mathbf{h}(\mathbf{y}_1, \tau_1)\ldots\delta\mathbf{h}(\mathbf{y}_m, \tau_m)} = \lambda^{(4-\alpha)m}\frac{\delta^m W_{\mathbf{x}_1', t_1', \ldots, \mathbf{x}_n', t_n'}(\mathbf{v}_1', \ldots, \mathbf{v}_n')}{\delta\mathbf{h}(\mathbf{y}_1', \tau_1')\ldots\delta\mathbf{h}(\mathbf{y}_m', \tau_m')}. \tag{2.4}$$

In terms of the moments of the relative velocities, this hypothesis can be reformulated in the following form: the substitutions

$$\begin{gathered} \mathbf{x} \to \lambda\mathbf{x}, \quad t \to \lambda^\alpha t, \quad \mathbf{v} \to \lambda^{\alpha-1}\mathbf{v}, \\ \delta / \delta\mathbf{h}(\mathbf{x}, t) \to \lambda^{4-\alpha}\delta / \delta\mathbf{h}(\mathbf{x}', t') \end{gathered} \tag{2.5}$$

do not change the moments of the relative velocities or their variational derivatives as $\mathbf{h} \to 0$.

Let us consider first the limitations imposed by this hypothesis on the quantity $\Delta v(\mathbf{x}, t) = (\langle v^2(\mathbf{x}, t)\rangle)^{1/2}$. From the condition

$$\langle v^2(\mathbf{x}, t)\rangle = \lambda^{2(\alpha-1)}\langle v^2(\lambda\mathbf{x}, \lambda^\alpha t)\rangle \tag{2.6}$$

it follows that

$$\Delta v(\mathbf{x}, t) = x^{1-\alpha}\varphi(ct / x^\alpha), \tag{2.7}$$

where φ is a certain function of its argument, and c is a constant of dimension $[L^\alpha/T]$. We assume in (2.2)

$$\lambda = [1/\Delta v(x_1, t_1)]^{1/(1-\alpha)}. \quad (2.8)$$

It follows from (2.7) and (2.8) that we can write for W

$$dW_{x_1, t_1, \ldots, x_n, t_n}(v_1, \ldots, v_n) = dW'_{t_1/x_1^\alpha, \ldots, t_n/x_n^\alpha, x_2/x_1, \ldots, x_n/x_1}\left(\frac{v_1}{\Delta v_1}, \ldots, \frac{v_n}{\Delta v_1}\right); \quad (2.9)$$

at n = 1 this equation changes into

$$dW_{x,t}(v) = dW'_{t/x^\alpha}(v/\Delta v),$$

which means that the transformation (2.1) does not change the distribution function for the relative velocity measured in units of Δv (compare with [3]). For a moment of arbitrary order we obtain in analogy with (2.7)

$$\langle v_{i_1}(x_1, t_1) \ldots v_{i_n}(x_n, t_n)\rangle = x_1^{n(1-\alpha)} \varphi_{\{i\}}\left(\frac{t_1 c}{x_1^\alpha}, \ldots, \frac{t_n c}{x_n^\alpha}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right) \quad (2.10)$$

(The index $\{i\}$ denotes the aggregate of all the indices $i_1, i_2, \ldots, i_n$). Finally, for the variational derivatives of (2.10) with respect to **h** we have from (2.4)

$$\left\langle \frac{\delta^m v_{i_1}(x_1, t_1) \ldots v_{i_n}(x_n, t_n)}{\delta h_{j_1}(y_1, \tau_1) \ldots \delta h_{j_m}(y_m, \tau_m)} \right\rangle = x_1^{n(1-\alpha)-m(4-\alpha)} \cdot$$

$$\times \varphi_{\{i,j\}}\left(\frac{t_1 c}{x_1^\alpha}, \ldots, \frac{t_n c}{x_n^\alpha}, \frac{\tau_1 c}{y_1^\alpha}, \ldots, \frac{\tau_m c}{y_m^\alpha}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}, \frac{y_1}{x_1}, \ldots, \frac{y_m}{x_1}\right) \quad (2.11)$$

($\{i, j\}$ is the aggregate of the indices $i_1, \ldots, i_n$; $j_1, \ldots, j_m$).

According to (2.10) and (2.11), if we know the degree of homogeneity of any one of the moments (say (2.7)) then we know the degree of homogeneity of all the variational derivatives (2.11) that describe the reaction of the system to a small external perturbation. It has recently become possible to study experimentally the propagation of external actions in a turbulent liquid, i.e., to study the response functions directly (see, for example, [7]). We consider it important to verify the theory not only at the level of the spectral characteristics of the velocity field, but also with respect to quantities of another type, such as the response functions.

Formulas (2.10) and (2.11) can be obtained directly from dimensionality considerations by assuming that the parameter c, with the dimension $[c] = [L^\alpha/T]$, determines the pulsation regime completely. If we take c^3 to be equal to the average rate of energy dissipation $\dot{\epsilon}$, then $\alpha = 2/3$ and formulas (2.10) coincide with those obtained by Kolmogorov.[1]

We consider now the case when the viscosity ν has a finite value. From the parameters c and ν we can construct quantities with the dimensions of length and time:

$$\eta = (\nu/c)^{1/(2-\alpha)}, \quad \tau_\eta = \eta^\alpha/c.$$

Formulas (2.9)–(2.11) will then contain the quantities x_1/η and t_1/τ_η as arguments. We assume that the similarity hypotheses formulated above are valid for spatial intervals much larger than η and for temporal intervals much larger than τ_η. Assume now that the external turbulence scales L and T also have finite values. In this case the arguments of the functions (2.9)–(2.11) can also be t_1/T and x_1/L. This means that all the functions can depend on L/η, i.e., on the Reynolds number. We can expect, however, the existence of a scale interval in which this dependence is weak, so that the main influence of the finite character of the Reynolds number lies in the fact that it limits the distance interval in which the similarity hypothesis is valid.

3. We shall need subsequently a formulation of the similarity hypotheses for the case when the turbulence is homogeneous, isotropic, and stationary. The fields of the velocity and of the random exciting force can be expanded in Fourier-Stieltjes intervals.[2] For the velocity difference we have in this case

$$v_i(x, t) = \int \{\exp[i(\mathbf{k}(\mathbf{x}+\mathbf{x}_0) - \omega(t+t_0))] - \exp[i(\mathbf{k}\mathbf{x}_0 - \omega t_0)]\} dZ(\mathbf{k}, \omega). \quad (3.1)$$

The n-th-order moment of v_i is expressed in terms of the moment of the quantities **dZ**

$$\langle v_{i_1}(x_1, t_1) \ldots v_{i_n}(x_n, t_n)\rangle$$

$$= \int \prod_{j=1}^{n} \{\exp[i(\mathbf{k}_j(\mathbf{x}_j + \mathbf{x}_0) - \omega_j(t_j + t_0))] - \exp[i(\mathbf{k}_j x_0 - \omega_j t_0)]\} \langle dZ_{i_1}(\mathbf{k}_1, \omega_1) \ldots dZ_{i_n}(\mathbf{k}_n, \omega_n)\rangle. \quad (3.2)$$

If the similarity hypothesis of Sec. 2 is valid, then the moments for dZ remain unchanged as

$$\mathbf{k} \to \lambda \mathbf{k}, \quad \omega \to \lambda^\alpha \omega, \quad d\mathbf{Z} \to \lambda^{1-\alpha} d\mathbf{Z}, \quad (3.3)$$

i.e.,

$$\langle dZ_{i_1}(\mathbf{k}_1 \omega_1) \ldots dZ_{i_n}(\mathbf{k}_n, \omega_n)\rangle = \lambda^{n(1-\alpha)} \langle dZ_{i_1}(\lambda \mathbf{k}_1, \lambda^\alpha \omega_1) \ldots dZ_{i_n}(\lambda \mathbf{k}_n, \lambda^\alpha \omega_n)\rangle. \quad (3.4)$$

According to [2]

$$\langle dZ_{i_1}(\mathbf{k}_1, \omega_1) \ldots dZ_{i_n}(\mathbf{k}_n, \omega_n)\rangle$$

$$= F_{\{i\}}(\mathbf{k}_1, \omega_1, \ldots, \mathbf{k}_{n-1}, \omega_{n-1}) \delta\left(\sum_{i=1}^{n} \mathbf{k}_i\right) \cdot \delta\left(\sum_{i=1}^{n} \omega_i\right) d\mathbf{k}_1 d\omega_1 \ldots d\mathbf{k}_n d\omega_n. \quad (3.5)$$

From this and (3.4) we find that F are homogeneous tensors of the form

$$F_{\{i\}}(\mathbf{k}_1, \omega_1, \ldots, \mathbf{k}_{n-1}, \omega_{n-1}) = k_1^{-4n+3+\alpha} \times F'_{\{i\}}\left(\frac{\omega_1}{k_1^\alpha}, \ldots, \frac{\omega_{n-1}}{k_{n-1}^\alpha}, \frac{\mathbf{k}_2}{k_1}, \ldots, \frac{\mathbf{k}_{n-1}}{k_1}\right). \quad (3.6)$$

At n = 2 we have

$$F_{ij}(\mathbf{k}, \omega) = k^{\alpha-5} F'_{ij}(\omega/k^\alpha).$$

The tensor F_{ij} can be expressed in terms of one scalar function F with the aid of the equation[2]

$$F_{ij}(\mathbf{k}, \omega) = \Delta_{ij}(\mathbf{k}) F(\mathbf{k}, \omega), \quad \Delta_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2.$$

Thus

$$F_{ij}(\mathbf{k}, \omega) = k^{\alpha-5} F'(\omega/k^\alpha) \Delta_{ij}(\mathbf{k}). \quad (3.7)$$

We add a small (nonrandom) increment $h_i(\mathbf{x}, t)$ to the external exciting force and assume that it can be expanded in a Fourier integral:

$$h_i(\mathbf{x}, t) = \int \exp[i(\mathbf{k}\mathbf{x} - \omega t)] h_i(\mathbf{k}, \omega) d\mathbf{k}\, d\omega.$$

Using the condition that the turbulence is homogeneous and stationary, and also the similarity hypothesis, we can show that

$$\left\langle \frac{\delta^m dZ_{i_1}(\mathbf{k}_1, \omega_1) \ldots dZ_{i_n}(\mathbf{k}_n, \omega_n)}{\delta h_{j_1}(\mathbf{q}_1, \beta_1) \ldots \delta h_{j_m}(\mathbf{q}_m, \beta_m)} \right\rangle = G_{\{i,j\}}(\mathbf{k}_1, \omega_1, \ldots, \mathbf{k}_{n-1}, \omega_{n-1}, \mathbf{q}_1, \beta_1, \ldots, \mathbf{q}_m, \beta_m) \quad (3.8)$$

$$\times \delta\left(\sum_{i=1}^{n} \mathbf{k}_i - \sum_{j=1}^{m} \mathbf{q}_j\right) \delta\left(\sum_{i=1}^{n} \omega_i - \sum_{j=1}^{m} \beta_j\right) d\mathbf{k}_1 d\omega_1 \ldots d\mathbf{k}_n d\omega_n d\mathbf{q}_1 d\beta_1 \ldots d\mathbf{q}_m d\beta_m,$$

where

$$G_{\{i,j\}}(\mathbf{k}_1, \omega_1, \ldots, \mathbf{k}_{n-1}, \omega_{n-1}, \mathbf{q}_1, \beta_1, \ldots, \mathbf{q}_m, \beta_m) = k_1^{-4n+(1-2\alpha)m+3+\alpha}$$

$$\times g_{\{i,j\}}\left(\frac{\omega_1}{k_1^\alpha}, \ldots, \frac{\omega_{n-1}}{k_{n-1}^\alpha}, \frac{\beta_1}{q_1^\alpha}, \ldots, \frac{\beta_m}{q_m^\alpha}, \frac{\mathbf{k}_2}{k_1}, \ldots, \frac{\mathbf{k}_{n-1}}{k_1}, \frac{\mathbf{q}_1}{k_1}, \ldots, \frac{\mathbf{q}_m}{k_1}\right). \quad (3.9)$$

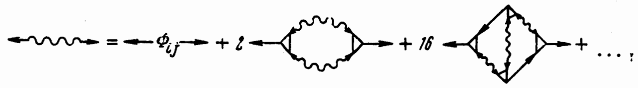

FIG. 1

At m = n = 1 this equation takes the form

$$G_{ij}(\mathbf{k}, \omega) = k^{-\alpha} g(\omega / k^{\alpha}) \Delta_{ij}(\mathbf{k}); \tag{3.10}$$

at m = 2 and n = 1 formula (3.8) yields ($G_{\{i,j\}} \equiv \Lambda_{ijl}$)

$$\Lambda_{ijl}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1) = k^{1-3\alpha} \Lambda'_{ijl}\left(\frac{\omega}{k^{\alpha}}, \frac{\omega_1}{k_1{}^{\alpha}}, \frac{\mathbf{k}_1}{k}\right) \tag{3.11}$$

4. Let us discuss a situation that arises in the dynamic description of turbulence if the similarity hypotheses formulated above are valid. We use for this purpose a diagram technique of the type developed by Wyld.[8] Let us describe the formulation of the problem. For a viscous incompressible liquid the velocity field u(x, t) obeys the Navier-Stokes equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{l} \nabla p + \nu \Delta u_i + f_i, \tag{4.1}$$

where f is the external force and is the source of the motion, while the remaining symbols are standard. The source energy is transferred to the liquid at a rate

$$\dot{\varepsilon} = \int \mathbf{uf}\, d\mathbf{x}$$

and is then dissipated by the viscosity. It is assumed that the random field of the external forces is stationary, homogeneous, isotropic, and has a normal distribution. The latter means that any mean value of the type

$$\langle f_{i_1}(\mathbf{x}_1, t_1) \ldots f_{i_n}(\mathbf{x}_n, t_n) \rangle$$

can be represented by a sum of products of all possible pairwise averages (the analog of the Wick theorem in quantum field theory). We shall assume henceforth that the spectral tensor of the external-force field $\Phi_{ij}(\mathbf{k}, \omega)$ differs from zero only when $k \leqslant L^{-1}$ and $\omega \leqslant T^{-1}$, and that the similarity hypotheses are satisfied for a sufficiently large interval $k \gg L^{-1}$, $\omega \gg T^{-1}$.

Any velocity function can be represented with the aid of (4.1) by a formal functional series in $\mathbf{f}(\mathbf{x}, t)$. The problem of averaging becomes defined if the method of summing the series after term-by-term averaging is specified. We assume that such a method exists. After partial summation in the functional series we can obtain the following system of equations (all the quantities are henceforth considered in the Fourier representation).

The equation for the spectral velocity-field tensor takes the form shown in Fig. 1.[8]
The spectral tensor $F_{ij}(\mathbf{k}, \omega)$ will henceforth be denoted by a wavy line, the vertex $\Gamma_{ijl}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1)$ by a triangle, and the Green's tensor $G_{ij}(\mathbf{k}, \omega)$ by an arrow. The quantity $G_{ij}(-\mathbf{k}, -\omega)$ is denoted by an arrow in the opposite direction. We can write for the vertex an equation of the form shown in Fig. 2:[8]
The point denotes the nonrenormalized vertex

$$P_{ijl}(\mathbf{k}) = -\frac{i}{(2\pi)^4}[k_j \Delta_{il}(\mathbf{k}) + k_l \Delta_{ij}(\mathbf{k})]. \tag{4.2}$$

The Green's tensor G_{ij} can be expressed in terms of the

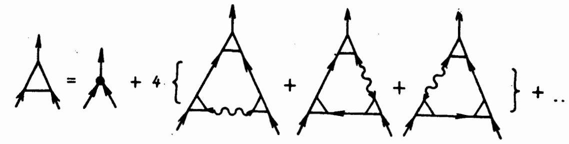

FIG. 2

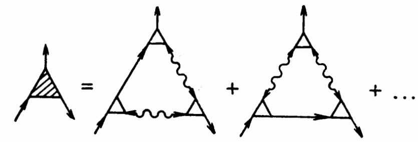

FIG. 3

"mass operator" $\Sigma^{(1)}_{\alpha\beta}$ with the aid of the Dyson equation[8]

$$G_{ij}(\mathbf{k}, \omega) = \frac{\delta_{ij}}{-i\omega + \nu k^2} + \frac{\delta_{i\alpha}}{-i\omega + \nu k^2} \Sigma^{(1)}_{\alpha\beta}(\mathbf{k}, \omega) G_{\beta j}(\mathbf{k}, \omega). \tag{4.3}$$

A diagram that does not split into two parts when any of the lines in it is broken will be called nodal. It is easy to see that $\Sigma^{(1)}_{\alpha\beta}$ is a sum of nodal diagrams with one input and one output, while the vertex Γ is a sum of nodal diagrams with two inputs and one output. We introduce, finally, the vertex Γ, defined as a sum of nodal diagrams with one input and two outputs (see Fig. 3):

$$U_i(\mathbf{k}, \omega) = G_{i\alpha}(\mathbf{k}, \omega) h_{\alpha}(\mathbf{k}, \omega).$$

Then $\Sigma^{(1)}_{\alpha\beta}$ can be expressed in terms of Γ and Γ′ with the aid of the relation

FIG. 4

The system of equations in Figs. 1–4 (4.3) is complete. Substituting dZ in the form of a graphical expansion (see [8]) in the left-hand side of (3.8), we can verify that the tensor $G_{\{i,j\}}(\mathbf{k}_1, \omega_1, \ldots, \mathbf{k}_{n-1}, \omega_{n-1}, \mathbf{q}_1, \beta_1, \ldots, \mathbf{q}_m, \beta_m)$ corresponds to the sum of all possible diagrams with m inputs and n outputs.[9,10] It follows therefore that the Green's tensor $G_{ij}(\mathbf{k}, \omega)$ describes in the linear approximation the reaction of the average velocity field $U(\mathbf{k}, \omega)$ to a small force perturbation $h(\mathbf{k}, \omega)$:

$$U_i(\mathbf{k}, \omega) = G_{i\alpha}(\mathbf{k}, \omega) h_{\alpha}(\mathbf{k}, \omega). \tag{4.4}$$

The vertex Γ is connected with Λ (see (3.11)) by the equation

$$\Lambda_{ijl}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1) = G_{i\alpha}(\mathbf{k}, \omega) G_{j\beta}(\mathbf{k}_1, \omega_1) \cdot \cdot G_{l\gamma}(\mathbf{k} - \mathbf{k}_1, \omega - \omega_1) \Gamma_{\alpha\beta\gamma}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1). \tag{4.5}$$

If we assume that the similarity hypotheses holds true (see Sec. 2), then $\Gamma_{ijl}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1)$ can be represented in accordance with (4.5), (3.10), and (3.11) in the form

$$\Gamma_{ijl}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1) = k g_{ijl}\left(\frac{\omega}{k^{\alpha}}, \frac{\omega_1}{k_1{}^{\alpha}}, \frac{\mathbf{k}_1}{k}\right). \tag{4.6}$$

Let us consider the system of equations in the region where the wave numbers and the frequencies of the quantities in the left-hand sides are large compared with the reciprocals L^{-1} and T^{-1} of the principal length and time scales. We assume that the spectral tensor $\Phi_{ij}(k, \omega)$ of the external forces differs from zero only when $k \leqslant L^{-1}$ and $\omega \leqslant T^{-1}$. Then the first term in the right-hand side of the equation in Fig. 1 drops out. Let us estimate the contribution made to an arbitrary diagram of the left-hand side of the equation of Fig. 1 by the region in which the integration variables are of the order of the wave numbers and frequencies of the external lines. A diagram F_n containing n integrations with respect to $dkd\omega$ contains n + 1 wavy lines, 2n vertices, and 2n tensors G. Hence

$$F_n \sim (G\Gamma)^{2n} F^{n+1} (k^3\omega)^n.$$

Using (3.7), (3.10), and (4.6), we get

$$F_n \sim k^{\alpha-5} \sim F,$$

i.e., F_n is of the same order as the left-hand side. A similar fact can be established also for the equations in Figs. 2 and 3.

Thus, the similarity hypothesis agrees with the system of equations if the region in question makes the main contribution to the integrals. To this end it suffices that the vertex Γ decrease rapidly if any one of its arguments becomes small in comparison with the remaining ones. In the equation of Fig. 2 there is a diagram that does not satisfy this requirement, namely the nonrenormalized vertex. Therefore the diagrams in the right-hand side of the equation should contain a contribution that cancels out $P_{ijl}(\mathbf{k})$ in the region where the arguments of the external lines differ strongly from one another in order of magnitude. The source of such a contribution may be either the remote terms of the series, or the contribution from the integration regions $q_i \ll k$ and $\omega_i \ll \omega$, where k and ω are of the order of the wave numbers and frequencies of the external lines. A similar situation apparently also holds in a Bose liquid.

Let us consider the case when the nonrenormalized vertex in the equation of Fig. 2 cancels out completely. The cancellation of this vertex in the region where its arguments are quantities of different order is equivalent to its complete cancellation, since the difference between these two cases is determined by the behavior of the diagrams when the arguments are of the same order of magnitude. By multiplying the equation of Fig. 2 by G and introducing a new vertex $\Gamma^{(c)}$, connected with Γ by the equation

$$\Gamma_{ijl}^{(c)}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1) = G_{i\alpha}(\mathbf{k}, \omega)\Gamma_{\alpha jl}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1),$$

we eliminate the function G from the equations of Figs. 1 and 2. Analogously, by multiplying the equation of Fig. 3 by $G_{i\alpha}(\mathbf{k}, \omega)$ and $G_{j\beta}(\mathbf{k}_1, \omega_1)$ and introducing the new vertex

$$\Gamma_{ijl}^{(c)\prime} = G_{i\alpha}(\mathbf{k}, \omega) G_{j\beta}(\mathbf{k}_1, \omega_1)\Gamma'_{\alpha\beta l}(\mathbf{k}, \omega, \mathbf{k}_1, \omega_1),$$

we eliminate G from this equation as well. Solving the equations of Figs. 1–3, we can in principle, determine F, $\Gamma^{(c)}$ and $\Gamma^{(c)\prime}$. Then, using Eq. (4.3) and Fig. 4, we can determine G. It will be shown in Sec. 5 that $\Sigma^{(1)}$ determines the action of turbulent viscosity on the pulsations. Accordingly, we shall assume that when $k \ll \eta^{-1}$ and $\omega \ll \tau_\eta^{-1}$ we can neglect the term νk^2 in comparison with $\Sigma^{(1)}$. Then the dimensional constant ν drops out from (4.3). All that matters here is the sign of the viscosity ν, which determines the direction of going around the poles during the integration, and by the same token, the sign of the turbulent viscosity (see Sec. 5). If we put $\nu \equiv 0$, then we obtain a temperature-distribution regime that differs from that considered by us.

5. Let us assume, in accord with [1], that the dimensionality parameter of the average energy dissipation rate determines completely the statistical pulsation regime in the inertial interval. As noted above, this corresponds to the case $\alpha = 2/3$.

We consider Eq. (4.4), which describes the response of the averaged velocity field to a small force perturbation. We multiply it by $U^*(\mathbf{k}, \omega)$ and integrate with respect to ω:

$$\varphi(k) \equiv \int |U(\mathbf{k}, \omega)|^2 d\omega = \int G_{i\alpha}(\mathbf{k}, \omega) h_\alpha(\mathbf{k}, \omega) U_i^*(\mathbf{k}, \omega) d\omega.$$

According to (3.10), in order for the spectral function $\varphi(k)$ to vary like $k^{-11/3}$, it is necessary that the supply power $U_\alpha^* h_\alpha$ vary like k^{-3}. This means that wave packets that occupy a region of the order of k in the wave-number space expend a power that does not depend on the value of k to overcome the turbulent viscosity.

The foregoing suggests the possibility of a somewhat different interpretation of the parameter $\dot{\epsilon}$ used in Kolmogorov's similarity theory. According to [11], the turbulent medium should be regarded as made up of wave packets. These packets, produced by nonlinear energy supply, lose energy as they overcome the turbulent velocity, and are replenished. The case $\alpha = 2/3$ thus corresponds to a situation wherein the wave packets of all scales are constructed in similar fashion and lose equal amounts of power when overcoming the turbulent viscosity.

Let us examine from this point of view the equation for the spectral tensor $F_{ij}(\mathbf{k}, \omega)$, which can be derived by partial summation of the series in analogy with (4.3):

$$F_{ij}(\mathbf{k}, \omega) = \frac{\delta_{ij}}{-i\omega + \nu k^2}[\Sigma_{\alpha\beta}^{(1)}(\mathbf{k}, \omega) F_{\beta j}(\mathbf{k}, \omega) + \Sigma_{\alpha\beta}^{(2)}(\mathbf{k}, \omega) G_{\beta j}(\mathbf{k}, \omega)],$$

where $\Sigma_{\alpha\beta}^{(2)}$ is the sum of nodal diagrams with two outputs. Hence

$$[(-i\omega + \nu k^2)\delta_{i\alpha} - \Sigma_{i\alpha}^{(1)}] F_{\alpha j} = \Sigma_{i\alpha}^{(2)} G_{\alpha j}. \tag{5.1}$$

According to (4.3) and (4.2)

$$[(-i\omega + \nu k^2)\delta_{i\alpha} - \Sigma_{i\alpha}^{(1)}] U_\alpha = h_i.$$

Multiplying this equation by $U_i^*(\mathbf{k}, \omega)$ and using the fact that $\Sigma_{i\alpha}^{(1)} = \Sigma^{(1)}\Delta_{i\alpha}(\mathbf{k})$ by virtue of the isotropy of the turbulence, we obtain

$$(-i\omega + \nu k^2 - \Sigma^{(1)})|U|^2 = hU^*. \tag{5.2}$$

We see therefore that the term with $\Sigma^{(1)}$ describes the action of the "turbulent viscosity" on the introduced perturbation. We shall assume that the term νk^2 can be neglected by comparison with $\Sigma^{(1)}$ in the inertial interval. A comparison of (5.1) with (5.2) shows that the term $\Sigma^{(2)}G$ can be interpreted as the energy supplied by the nonlinear interactions. The term $\Sigma^{(1)}F$ describes the conversion of the energy of the harmonic k, ω into the energy of all other harmonics.

From (4.3) and (3.10) we have

$$\Sigma^{(1)} \sim -i\omega - G^{-1} \sim k^\alpha. \tag{5.3}$$

The condition $\alpha = \frac{2}{3}$ coincides, in accordance with (3.7) and (5.3), with the condition

$$\int_0^k \Sigma^{(1)} F \, d\mathbf{k} \, d\omega \sim \text{const},$$

This can be interpreted as meaning that the power required to overcome the turbulent viscosity is independent of the dimension of the packet.

The physical picture corresponding to this case is as follows. In order to increase the amplitudes of the pulsations $u(\mathbf{k}, \omega)$ forming a wave packet of scale λ by an amount $\Delta u(\mathbf{k}, \omega)$ it is necessary, according to (5.2), to have a power

$$\Delta\dot{\varepsilon} = \int \Delta u(\mathbf{k}, \omega) h(\mathbf{k}, \omega) \, d\mathbf{k} \, d\omega,$$

where the integral is taken over the region $\Delta^3 k \sim 1/\lambda^3$ occupied by the packet in k-space, $h(\mathbf{k}, \omega)$ is the external force, and

$$\Delta\varepsilon \sim \Sigma^{(1)}(\mathbf{k}, \omega) |\Delta u(\mathbf{k}, \omega)|^2 \Delta^3 k_\lambda \omega_\lambda. \tag{5.4}$$

In the state of stationary turbulence, the packet obtains a power $\epsilon(\lambda)$ from the turbulent system and dissipates it through turbulent viscosity (i.e., it gives up the power to the turbulent system). This quantity $\dot{\epsilon}(\lambda)$ can be naturally called the energy-conversion power. Were we to "turn off" the supply, leaving only dissipation, then the power of the external force $h(\mathbf{k}, \omega)$ which must be consumed to maintain the pulsations of the packet at the level $|u(\mathbf{k}, \omega)|^2$ is

$$\dot{\varepsilon}_k \sim \frac{\Delta\varepsilon}{|\Delta u(\mathbf{k}, \omega)|^2} |u(\mathbf{k}, \omega)|^2 \approx \Sigma^{(1)} |u(\mathbf{k}, \omega)|^2 k^3 \omega_k. \tag{5.5}$$

We assume that the energy conversion $\dot{\epsilon}(\lambda) \approx \dot{\epsilon}_k$, $k \approx 1/\lambda$ does not depend on the scale. In this case, using the definition (3.5), we get from (5.5)

$$F \Sigma^{(1)} k^3 \omega_k \sim \varepsilon \sim \text{const},$$

which, as found above, leads to the Kolmogorov law.

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Note added in proof (28 January 1972). According to Sec. 4, the role of the effective expansion parameter is played by the quantity

$$\varkappa = (G\Gamma)^2 F k^3 \omega \sim \text{const}.$$

We introduce the turbulent viscosity $\nu_T = \Sigma_1/k^2$. The quantity $Fk^3\omega$ is the square of the characteristic velocity of scale $\sim 1/k$. Taking (4.6) and (5.3) into account, Eq. (*) can be rewritten in the form

$$\varkappa \sim v_k^2 k^{-2} / \nu_T^2 = Re_k^2 \sim \text{const},$$

i.e., the Reynolds number of pulsations of scale $1/k$, determined from the turbulent viscosity, does not depend on the scale.

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