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# Nonlinear Theory of Interaction Between a Bounded Relativistic Beam and a Plasma

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A nonlinear theory of interaction between bounded relativistic beams and a plasma is developed which permits one to elucidate the main dynamic regularities of the beam particles during development of plasma-beam instability and also to determine the field amplitudes and beam radii. Limiting cases of monovelocity and quasilinear beams are considered.

## 1. INTRODUCTION

AT the present time, owing to the rapid development of accelerator technology, it is possible to perform experiments on the interaction of relativistic beams with a plasma (see, for example[1,2]). It is therefore of interest to investigate theoretically the instabilities of relativistic beams in a plasma[1-7]. The linear theory of this effect was developed in[3-5], and a quasilinear analysis for beams of unlimited radius was carried out in[1,6] in the approximation of a strong longitudinal magnetic field and in[1,2] in the absence of a magnetic field.

In the present paper we consider the excitation of electrostatic oscillations by relativistic beams with limited radii. This makes it possible to analyze the dynamics of the beam particles in both the longitudinal and transverse directions and to determine the maximum amplitudes of the fields and the variation of the beam radius at the instant of nonlinear stabilization of the instabilities.

The nonlinear effects that occur when a single-velocity beam interacts with a plasma are taken into account within the framework of a model in which the beam is specified by a sequence of charged bunches (thin charged disks) moving through the plasma and separated by a distance equal to the wavelength. For a sufficiently broad beam $a \gg ƛ\gamma_0^2$ ($a$ is the initial radius of the beam, $ƛ = v_0/\omega_2$, $v_0$ is the initial beam velocity, and $\omega_2$ is the plasma frequency), of electrons moving on parallel paths, the dynamics of the instability does not differ in principle from the case of an unbounded beam in a strong longitudinal magnetic field[1,6]. The field energy density reaches a value $E^2 = 8\pi n_1 m v_0^2 \gamma_0^2 (\nu/2)^{1/3}$ ($\nu = n_1/n_2$, $n_1$ and $n_2$ are the beam and plasma densities) within a time $\tau_{\parallel} \sim (\gamma_0/\omega_2)\nu^{-1/3}$. For a beam of smaller radius $a \ll ƛ\gamma_0^2$, the presence of considerable radial fields leads to a compression of the "unstable" electron bunches in the transverse direction, to a radius $R \sim (c/\omega_2)(\gamma_0/\nu)^{1/3}$ within a time $\tau_\perp \sim \omega_2^{-1}(\gamma_0/\nu)^{1/3}$ (without a significant change in the longitudinal beam velocity) and to a defocusing of the "passive" particles of the beam, which fall at the initial instant of time into the accelerating phases of the field[1)].

[1)]The feasibility, in principle, of radial focusing of a beam of charged particles under conditions of beam-plasma instability is pointed out in[8,9], where nonrelativistic beams are considered.

The concluding part of the paper deals with the quasilinear theory of the interaction of a bounded relativistic beam with a plasma. After determining the energy density of the field excited by the beam in the plasma, an estimate is obtained of the beam radius $R \sim c\gamma_0/\nu\omega_2$ established at the end of the quasilinear stage of instability development, i.e., after a time $\tau_{qu} \sim \gamma_0/\nu\omega_2$. Just as in the case of a "cold" beam (relativistic with respect to the transverse velocities), the radius is determined by the ratio of the speed of light to the instability increment and is determined by the distance traversed by an electron situated on the beam axis within the time of instability development, i.e., by the instant when the gradient of the thermal pressure in the beam becomes balanced by the focusing field of the instability. The difference lies in the fact that in the quasilinear case the focusing of each beam particle occurs under the action of a resonant harmonic of the field, the energy density of which $\nu n_1 m v_0^2$ is small compared with the analogous quantity in the hydrodynamic case, when there is one wave. This reduces the focusing efficiency and leads to an increase of the beam radius in the kinetic case for the same beam and plasma parameters. At the same time, however, the presence of a large number of waves leads to uniform focusing of the kinetic beam over the entire length and makes it possible to avoid the loss of some of the particles.

The method considered here for focusing the beam in the plasma can be used for controlled thermonuclear fusion[10,11], where a strong-current relativistic beam produces a sufficiently dense plasma on the path to the target. Thus, a beam producing a plasma with density $n_2 \sim n_1$ can be focused down to $R < 1$ cm if its parameters satisfy the inequality $\sqrt{n_1} > 5 \cdot 10^5 \gamma_0$. At an initial beam radius $a = 10$ cm and $\gamma_0 = 10$ this corresponds to a current $3 \times 10^7$ A. The corresponding focusing time is of the order of $3 \times 10^{-10}$ sec.

## 2. LINEAR THEORY

Let an electron beam with density $n_1$ move along the axis of a cylindrical conducting tube filled with a plasma of density $n_2$. The electrostatic field curl $\mathbf{E} = 0$ and $\mathbf{E} = -\text{grad}\,\Phi$ excited in the plasma by a beam of low density $n_1 \ll n_2$, is described by the linearized system of equations of motion of the plasma electrons, which must be considered in conjunction with the Poisson equation

$$\left(\frac{\partial^2}{\partial t^2}+\omega_2{}^2\right)\Delta\Phi=4\pi e\frac{\partial^2 n_1}{\partial t^2},\qquad \omega_2{}^2=\frac{4\pi e^2 n_{02}}{m} \tag{1}$$

($n_{02}$ is the density of the plasma ions).

We transform Eq. (1) by expanding the potential $\Phi$ in a system of orthogonal waveguide functions

$$\Phi_s=J(\lambda_\nu r/b)\exp(-ik_\parallel z) \tag{2}$$

($b$ is the plasma radius, $k_\parallel$ the longitudinal wave number, $J_0(\lambda_p)=0$, $k=(\lambda_p/b,\ k_\parallel)$), and by averaging both parts of (1) with respect to the variables $r$ and $z$:

$$\left(\frac{d^2}{dt^2}+\omega_2{}^2\right)\Phi(\mathbf{k})=-\frac{4\pi e}{k^2}\frac{2}{b^2J_1{}^2(\lambda_p)}\int_0^b r\,dr\,J_0\left(\lambda_p\frac{r}{b}\right)\times\frac{k_\parallel}{2\pi}\int_0^{2\pi/k_\parallel}\frac{\partial^2 n_1}{\partial t^2}\exp(ik_\parallel z)\,dz, \tag{3}$$

where $k^2=\lambda_p^2/b^2+k_\parallel^2$.

In the linear approximation, expressing the ac component of the beam density $\tilde{n}_1$ in terms of the potential from the linearized hydrodynamic equations of motion of the beam

$$\tilde{n}_1(\mathbf{k})=-\frac{e}{m\gamma_0}\frac{1}{(\omega-k_\parallel v_0)^2}\sum_{p=1}^{\infty}\frac{1}{k^2}\left(\frac{k_\parallel{}^2}{\gamma_0{}^2}+\frac{\lambda_p{}^2}{b^2}\right)\cdot J_0\left(\lambda_p\frac{r}{b}\right)\Phi(\mathbf{k}) \tag{4}$$

($v_0$ is the longitudinal velocity and $\gamma_0=(1-v_0^2/c^2)^{-1/2}$) and substituting it into Eq. (3), we arrive at a dispersion equation for the harmonic with wave vector $\mathbf{k}$

$$1-\frac{\omega_2{}^2}{\omega^2}-\left(\frac{k_\parallel{}^2}{k^2}\frac{1}{\gamma_0{}^2}+\frac{\lambda_p{}^2}{k^2b^2}\right)\frac{\omega_1{}^2}{\gamma_0(\omega-k_\parallel v_0)}=0,\qquad \omega_1{}^2=\frac{4\pi e^2 n_1}{m}. \tag{5}$$

In the derivation of (5) it was assumed that the beam electrons are uniformly distributed over the waveguide cross section.

Solving (5), we obtain the increment for the most unstable harmonic $\omega_2=k_\parallel v_0$:

$$\mu=\frac{\sqrt{3}}{2^{4/3}}\left(\frac{k_\parallel{}^2}{k^2}\frac{1}{\gamma_0{}^2}+\frac{\lambda_p{}^2}{k^2b^2}\right)^{1/3}\left(\frac{\nu}{\gamma_0}\right)^{1/3}\omega_2,\qquad \omega_2-\omega=\mu/\sqrt{3}. \tag{6}$$

It follows from (16) that owing to the anisotropy of the longitudinal and transverse masses the oscillations that develop predominantly are the transverse ones ($\lambda_p/b \gg k_\parallel$), for which the increment is maximal and equal to

$$\mu_\perp\approx\frac{\sqrt{3}}{2^{4/3}}\left(\frac{\nu}{\gamma_0}\right)^{1/3}\omega_2. \tag{7}$$

This result was obtained for the unbounded case $b\to\infty$ in[1].

## 3. NONLINEAR BOUNDED MONOENERGETIC BEAM

Since the relativistic electron beams obtained in accelerators are made up of sequences of bunches, it is of interest to estimate the energy transferred by such a strongly modulated beam to a plasma. We shall therefore consider the excitation of electrostatic oscillations by a sequence of infinitesimally thin charged disks.

The density of such a beam can be represented in the form

$$n_1=\sum_{s=-\infty}^{\infty}\sigma_s(t,r)\,\delta[z-sl-z_s(t,r)], \tag{8}$$

where $\sigma_s(t,r)$ is the electron surface density in each layer, $l$ is the distance between layers, and $z_s(t,r)$ is the coordinate of each layer. Substituting (8) into the equation for the potential (1), we seek solutions of this equation in the form

$$\Phi(t,\mathbf{r})=\varphi(t,r)\exp(i\omega_M t-ik_\parallel z), \tag{9}$$

where $\omega_M=2\pi v_0/l$ is the beam modulation frequency, which is assumed to be close to the plasma frequency $\omega_2$: $|\omega_M-\omega_2|\ll\omega_2$, and $\varphi$ is a slowly varying function: $|\dot{\varphi}|\ll\omega_2\varphi$. Then Eq. (3) can be represented in the form

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-k_\parallel{}^2\right)\left(\frac{\partial\varphi}{\partial t}+\frac{i}{2}\varepsilon\omega_M\varphi\right)=2\pi ie\omega_M\frac{\sigma}{l}e^{i\psi}. \tag{10}$$

In the derivation of (10) we have averaged over the longitudinal period of the beam and introduced the notation $\psi=k_\parallel z-\omega_M t$ and $\epsilon=1-\omega_2^2/\omega^2$.

With the same notation, the equations of motion of the electron bunches become

$$\frac{\partial}{\partial t}\left(\gamma\frac{\partial z}{\partial t}\right)=-\frac{e}{m}\mathrm{Re}[ik_\parallel\varphi e^{-i\psi}], \tag{11}$$

$$\frac{\partial}{\partial t}(\gamma v_r)+v_r\frac{\partial}{\partial r}(\gamma v_r)=\frac{e}{m}\mathrm{Re}\left[\frac{\partial\varphi}{\partial r}e^{-i\psi}\right], \tag{12}$$

$$\frac{\partial\sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rv_r\sigma)=0. \tag{13}$$

We assume that the plasma radius greatly exceeds the wavelength $k_\parallel b\gg 1$, and that the beam radius $b\gg a$, so that the field geometry is determined by the ratio of the beam radius to the wavelength and we consider first a sufficiently broad beam $k_\parallel a\gg 1$. The terms proportional to the derivatives with respect to $r$ in the left-hand side of (10) can then be neglected and the equation for $\varphi$ can be represented in the form

$$\dot{\varphi}+\frac{i}{2}\omega_M\varepsilon\varphi=-\frac{2\pi ie\omega_M n_1}{k_\parallel{}^2}e^{i\psi},\qquad n_1=\frac{\sigma}{l}. \tag{14}$$

We assume that the following inequality is satisfied

$$k_\parallel a\gg\gamma_0{}^2, \tag{15}$$

so that the effective radial force exerted on the bunch by the field is small compared with the longitudinal force, and in addition, $v_r(t=0)=0$. Under these assumptions we can neglect the transverse displacement of the electrons and assume, when integrating (11) and (14), that the electron density $n_1$ is constant at each point $r$.

The system of equations (11) and (14) coincides with the analogous equations obtained in[7] for an unbounded beam, and can be integrated. Introducing the amplitude and phase of the field $ik_\parallel\varphi=Ee^{i\vartheta}$, we represent the closed system of equations in the form

$$\frac{d}{dt}(\gamma\dot{z})=-\frac{e}{m}E\cos(\psi-\vartheta),\qquad \dot{E}=\frac{2\pi e\omega_M n_1}{k_\parallel}\cos(\psi-\vartheta),\qquad \dot{\vartheta}=-\frac{1}{2}\varepsilon\omega_M+\frac{2\pi e\omega_M n_1}{k_\parallel E}\sin(\psi-\vartheta). \tag{16}$$

From the first and second equations of (16) we get the momentum conservation law

$$n_1 m\dot{z}\gamma+E^2/4\pi v_0=n_1 m v_0\gamma_0. \tag{17}$$

Expressing $\dot{\psi}$ in terms of $\dot{z}$ from (17) and introducing the variable $\eta = \psi - \vartheta$, we lower the order of the system of equations (16):

$$\dot{E} = 2\pi e n_1 v_0 \cos\eta,$$
$$\dot{\eta} = -\frac{2\pi e n_1 v_0}{E}\sin\eta + \omega_2\left\{1 - \frac{\gamma_0 - w^2}{[1 + v_0^2 c^{-2}(\gamma_0 - w^2)^2]^{1/2}}\right\} \quad (18)$$

(we have put $\epsilon = 0$ and $w^2 = E^2/4\pi n_1 m v_0^2$). Integrating (18) with initial conditions $\eta(0) = E(0) = 0$, we get

$$\frac{\omega_1}{\omega_2} w \sin\eta = w^2 + \frac{c^2}{v_0^2}\left\{\left[1 + \frac{v_0^2}{c^2}(\gamma_0 - w^2)^2\right]^{1/2} - \gamma_0\right\}. \quad (19)$$

The foregoing analysis shows that the electron bunches situated at the initial instant of time at the phase $\eta = 0$ corresponding to the maximum of the interaction of particles with the field subsequently lag the wave and shift into the region of phases $\eta = \pi/2$, where the force acting on the bunch is equal to zero. Putting in (19) $\sin\eta = 1$, we obtain an equation for the maximum field amplitude, which has an approximate solution in the form

$$(E^2/8\pi)_{max} \approx n_1 m v_0^2 \gamma_0^2 (\nu/2)^{1/3}, \qquad \gamma_0 \nu^{1/3} \lesssim 1. \quad (20)$$

The characteristic growth time of the field amplitude turns out to be of the order of

$$\tau_\parallel \sim \gamma_0 \omega_2^{-1/3} \omega_1^{-2/3}.$$

In the case when inequality (15) is not satisfied, the transverse motion of the electrons become significant. For a beam with parameters

$$1 \ll k_\parallel a \ll \gamma_0^2 \quad (21)$$

the longitudinal displacement of the bunches can be neglected, and the equations for the field (14) must be considered in conjunction with Eqs. (12) and (13). Making the substitution $\varphi = \varphi_0 e^{i\vartheta}$, we obtain

$$\frac{\partial\varphi_0}{\partial t} = \frac{2\pi e \omega_M n_1}{k_\parallel^2}\sin(\psi - \vartheta), \quad (22)$$

$$\frac{\partial\vartheta}{\partial t} = -\frac{1}{2}\varepsilon\omega_M - \frac{2\pi e\omega_M n_1}{k_\parallel^2 \varphi_0}\cos(\psi - \vartheta), \quad (23)$$

$$\frac{\partial}{\partial t}(\gamma v_r) + v_r\frac{\partial}{\partial r}(\gamma v_r) = \frac{e}{m}\frac{\partial}{\partial r}[\varphi_0\cos(\psi - \vartheta)]. \quad (24)$$

The effective beam density $n_1 = \sigma/l$, which enters in (22), satisfies the continuity equation (13).

Since no exact solution of the system (22)–(24) could be obtained we present below a qualitative analysis. We note first that since the longitudinal beam velocity does not change, we can put $\psi = 0$ without loss of generality. Further, at $\vartheta_0 = \pi/2$, at the start of the process, the amplitude of the potential increases linearly with the time: $\varphi_0 = -2\pi e n_1 \omega_M t/k_\parallel^2$. The presence of a frequency deviation ($\epsilon \neq 0$) leads to a drift of the field phase and to a detuning of the resonance between the beam and the wave when the phase reaches a value $\vartheta \sim 1$. Since the law governing the time variation of the phase is close to linear, the characteristic time of development of the transverse instability turns out to be of the order of $\tau_\perp \sim 1/\omega_M\epsilon$, and the amplitude of the potential reaches a value

$$\varphi_{max} = -\frac{4\pi e n_1}{k_\parallel^2 \varepsilon}\operatorname{sign}\varepsilon. \quad (25)$$

When the amplitude and phase of the field increase, a radial force $F_\perp$ appears and acts on the electron bunch. According to (24), the maximum of this quantity corresponds to $\vartheta = 0$ or $\vartheta = \pi$:

$$F_{\perp max} = -\frac{4\pi e^2}{k_\parallel^2 \varepsilon}\frac{\partial n_1}{\partial r}. \quad (26)$$

Since the electron density in the beam reaches a maximum value on the axis and drops off towards the periphery, it follows that $\partial n_1/\partial r < 0$ and the field exerts a focusing action on the beam at $\epsilon < 0$ and defocuses the bunches at $\epsilon > 0$. It follows from (24) and (26) that the radial displacement $\Delta R$ of the electron during the time of development of the instability turns out to be of the order of

$$(\Delta R)^2 \sim \frac{1}{k_\parallel^2}\frac{\omega_1^2}{\omega_2^2}\frac{1}{\gamma_0}\frac{1}{|\varepsilon|^3}. \quad (27)$$

The presence of radial focusing of the beam at a modulation frequency $\omega_M < \omega_2 (\epsilon < 0)$ has the following physical explanation. In the reference frame of the beam, the dielectric constant of the plasma at the modulation frequency $\omega_M = 2\pi v_0/l$ is negative (and it is precisely this frequency which is appreciable near the resonance[9]), so that the Coulomb repulsion force between the electrons $F_C = e^2/r\epsilon$ gives way to attraction and self-contraction of the bunches takes place[9].

With increasing initial beam radius, the dynamics of the instability does not change in principle, but the radial force of the field on the beam decreases. For a beam with radius small compared with the wavelength, $k_\parallel a \ll 1$, we can neglect in (10) the terms proportional to $k_\parallel^2$, and the dependence on the variable $r$ can be eliminated from (10), (12), and (13) by making the change of variables

$$\varphi(t, r) = g(t)\frac{r^2}{2}e^{i\vartheta(t)}, \qquad v_r = \frac{\dot{R}(t)}{R(t)}r. \quad (28)$$

Equations (10) and (12) then take the form

$$\dot{g} = -\pi e\omega_M n_{10}\frac{a^2}{R^2}\sin(\psi - \vartheta);$$
$$\dot{\vartheta} = -\frac{1}{2}\varepsilon\omega_M + \frac{\pi e\omega_M n_{10}}{g}\frac{a^2}{R^2}\cos(\psi - \vartheta);$$
$$\ddot{R} - \frac{e}{m\gamma_0}g\cos(\psi - \vartheta)R = 0. \quad (29)$$

In the derivation of the third equation in (29) we have assumed that the condition $\gamma_0^{-2} \gg v_r^2/c^2$ is satisfied and we have taken the relativistic factor outside the sign of the derivative with respect to time.

We shall show now that the formally introduced function $R(t)$ coincides with the beam radius. To this end, we consider the continuity equation (13):

$$\frac{\partial n_1}{\partial t} + \frac{\dot{R}}{R}\frac{1}{r}\frac{\partial}{\partial r}(r^2 n_1) = 0, \quad (13')$$

which satisfies the following conditions at $t = 0$:

$$n_1(0, r) = \begin{cases} n_{10}, & r \leqslant a \\ 0, & r > a \end{cases}. \quad (30)$$

We take the Laplace transform with respect to the variable $\tau = -\ln(R/a)$:

$$n_1(p, r) = r^{p-2}\left[A - \int_0^r \xi^{1-p} n_1(0, \xi)\, d\xi\right], \quad (31)$$

and determine the integration constant $A$ from the condition $n_1(p, a) = 0$. The inverse transformation leads to the formula

$$n_1(t,r)=\begin{cases} n_{10}a^2/R^2, & r\leqslant R(t), \\ 0, & r>R(t). \end{cases} \tag{32}$$

Relation (32) was used in the derivation of the equations in (29).

Just as in the preceding case (when the inequality (21) is satisfied), the bunches become shifted in phase into the region $\psi - \vartheta = 0$ or $\psi - \vartheta = \pi$, depending on the sign of $\epsilon$, as the instability develops. The radial state of the beam is then described by the equation

$$\ddot{R}-\frac{1}{2}\frac{\omega_1^2}{\gamma_0\varepsilon}\frac{a^2}{R}=0. \tag{33}$$

In the case when $\epsilon < 0$, radial stability is obtained. Multiplying both halves of (33) by $\dot{R}$ and integrating, we obtain

$$\dot{R}^2=-\frac{\omega_1^2a^2}{|\varepsilon|\gamma_0}\ln\frac{R}{a}, \quad \dot{R}_0=0. \tag{34}$$

Since $\dot{R}^2 > 0$, the allowed region of variation $R < a$ corresponds to self-collapse of the electron bunches. We note that as $R \to 0$ and $\dot{R} \to \infty$ the hydrodynamic velocity of the beam remains finite, since $r < R$.

In the modulation-frequency region $\epsilon > 0$, the beam becomes defocused.

At small beam radii, the transverse thermal scatter may become significant. To take this effect into account, we introduce the gradient $-T\partial n_1/\partial r$ of the kinetic pressure in the beam ($T$ is the transverse temperature) into the right-hand side of the radial equation of motion (12) and consider the transverse steady state of the beam $v_r = 0$:

$$e\frac{d\varphi}{dr}+\frac{T}{n_1}\frac{dn_1}{dr}=0, \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)=\frac{4\pi en_1}{|\varepsilon|}. \tag{35}$$

Integrating the first equation, we express the density in terms of the potential

$$n_1(r) = n_1(0)\exp(-e\varphi/T) \tag{36}$$

and eliminate it from the second equation of (35):

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right)-\frac{4}{R^2}\exp(-\Phi)=0; \quad \Phi=\frac{e\varphi}{T}. \tag{37}$$

The solution of (37) is

$$\Phi=2\ln\left(1+\frac{r^2}{R^2}\right), \quad R^2=\frac{T|\varepsilon|}{\pi e^2n_1(0)}. \tag{38}$$

Substituting (38) in (36), we obtain the radial distribution of the density in the beam

$$n_1(r)=\frac{n_1(0)}{(1+r^2/R^2)^2}. \tag{39}$$

It follows from this relation that the electron density remains practically constant at $r \lesssim R$ and decreases rapidly when $r > R$. With increasing electron density in the beam, the effective radius decreases like $1/\sqrt{n_1}$. Integrating both halves of (39) with respect to $rdr$, we arrive at the relation

$$N_c=\int_0^\infty n_1r\,dr=\frac{T}{\pi e^2}|\varepsilon|, \tag{40}$$

which determines the condition for the existence of the stationary regime. At $N > N_C$ radial oscillations arise in the beam, and in the opposite case $N < N_C$ there is irreversible defocusing of the beam under the influence of the kinetic-pressure gradient.

Thus, the foregoing analysis demonstrates the feasibility of radial self-focusing of a strongly modulated beam in a plasma if the beam modulation frequency is close to the plasma frequency: $\omega_2 - \omega_M \ll \omega_2$. In addition, the model considered above enables us to analyze the dynamics of the particles in a beam of continuous density under conditions of beam-plasma instability. According to formula (6), the dielectric constant of the plasma at the self-modulation frequency is negative and equal to $\epsilon = -(\nu/2\gamma_0)^{1/3}$. If the beam is furthermore relativistic with respect to the transverse velocities, $T \sim mc^2\gamma_0$, then the "unstable" electron bunches, which transfer their energy to the wave, compress in accordance with (38) to

$$R \sim (c/\omega_2)(\gamma_0/\nu)^{1/6}. \tag{41}$$

Strictly speaking, formula (38) can be used for a beam whose radius does not exceed the wavelength. However, in view of the physical equivalence of the processes, it is also qualitatively correct for a broader beam whose radius satisfies the inequality (21).

Simultaneously with focusing of the beam particles that fall into the slowing-down phases of the field, the instability field defocuses the particles that are in the accelerating phases.

## 4. QUASILINEAR THEORY OF INSTABILITY OF A BOUNDED BEAM

During the quasilinear stage of development of the instability, when the electron beam excites a broad spectrum of longitudinal wave numbers in the plasma, conditions are produced for the focusing of the beam uniformly over its entire length. This effect is the result of the fact that as their longitudinal velocities diffuse, the beam electrons interact with a large number of field harmonics, spending the greater part of the time in the slowing-down phases, where the particles give up their energy to the field and become focused.

Taking into account the cylindrical symmetry of the problem, we represent the kinetic equation for the distribution function of the beam in the form[8]

$$\frac{\partial f}{\partial t}+\frac{v_r}{r}\frac{\partial}{\partial r}(rf)+v_\parallel\frac{\partial f}{\partial z}+e\mathbf{E}\frac{\partial f}{\partial \mathbf{w}}=0, \tag{42}$$

where $\mathbf{v}$ is the velocity and $\mathbf{w}$ the momentum of the beam particles. Separating the slowly and rapidly varying parts of the distribution function[8], we arrive at the relations

$$\frac{\partial f_0}{\partial t}+\frac{v_r}{r}\frac{\partial}{\partial r}(rf_0)+e\left\langle \mathbf{E}\frac{\partial f_1}{\partial \mathbf{w}}\right\rangle=0,$$

$$rf_k=e\sum_{s=0}i^{s+1}\frac{v_r^s}{(\omega-k_\parallel v_0)^{s+1}}\frac{\partial^s}{\partial r^s}\left(r\mathbf{E}_k\frac{\partial f_0}{\partial \mathbf{w}}\right). \tag{43}$$

When calculating the alternating part of the beam density

$$n_k=\int f_k d\mathbf{w} \tag{44}$$

we shall assume the expansion parameter in the sum (43) to be small:

$$\eta \sim v_r/(\omega-k_\parallel v_0)r \sim v_r/\omega_2 r \ll 1 \tag{45}$$

and retain only the terms with $s = 0$ and $s = 1$:

$$n_k = e\int \frac{\partial}{\partial w_\parallel}\left[iE_{\parallel k} f_0 + \frac{\partial w_\parallel}{\partial v_\parallel}\frac{v_r}{k_\parallel}\frac{1}{r}\frac{\partial}{\partial r}\left(rE_{rk}\frac{\partial f_0}{\partial w_r}\right)\right]\frac{d\mathbf{w}}{\omega - k_\parallel v_\parallel}. \quad (46)$$

We expand the fields $E_{\parallel k}$ and $E_{rk}$ in terms of the eigenfunctions of the waveguide and assume that $f_0$ does not depend on the variable r. Then, substituting (46) into the equation for the potential (3), we arrive at a dispersion equation for the harmonic with wave vector $\mathbf{k} = (\lambda_p/b,\ k_\parallel)$:

$$1 - \frac{\omega_2^2}{\omega^2} + \frac{4\pi e^2 k_\parallel}{k^2}\frac{2}{b^2 J_1^2(\lambda_p)}\int_0^b r\,dr\int h(r,\mathbf{w})\frac{d\mathbf{w}}{\omega - k_\parallel v_\parallel} = 0, \quad (47)$$

where

$$h(r,\mathbf{w}) = \frac{\partial}{\partial w_\parallel}\left[J_0^2\left(\lambda_p\frac{r}{b}\right)f_0 - \frac{\partial w_\parallel}{\partial v_\parallel}\frac{\lambda_p^2}{k_\parallel^2 b^2}J_1^2\left(\lambda_p\frac{r}{b}\right)v_r\frac{\partial f_0}{\partial w_r}\right]. \quad (48)$$

Solving (47), we obtain the quasilinear increment

$$\mu_{\kappa p} = \frac{\pi}{2}\frac{4\pi e^2\omega_2}{k^2}\frac{2}{b^2 J_1^2(\lambda_p)}\int_0^b r\,dr\int\left[\frac{\partial w_\parallel}{\partial v_\parallel}h(r,\ \mathbf{w})\right]_{\overline{w}_\parallel} dw_r. \quad (49)$$

In the derivation of (49) we used the presence of the $\delta$ function and integrated with respect to the variable $w_\parallel$, so that $\overline{w}_\parallel$ is the root of the equation $v_\parallel(w_r,\ w_\parallel) = \omega_2/k_\parallel$:

$$\overline{w}_\parallel^2 = \frac{\beta_{ph}^2}{1-\beta_{ph}^2}(m^2c^2 + w_r^2), \qquad \frac{\partial w_\parallel}{\partial v_\parallel} = \frac{m}{(1-\beta_{ph}^2)^{3/2}}\left(1 + \frac{w_r^2}{m^2c^2}\right)^{1/2}, \quad (50)$$

where $\beta_{ph} = \omega_2/ck_\parallel$.

Taking (49) into account, we represent the equation for the field energy density in the form

$$\frac{\partial E_{kp}^2}{\partial t} = \pi\frac{4\pi e^2\omega_2 m}{k^2}\frac{2}{b^2 J_1^2(\lambda_p)}\frac{E_{kp}^2}{(1-\beta_{ph}^2)^{3/2}}\int_0^b r\,dr\int\left(1 + \frac{w_r^2}{m^2c^2}\right)^{1/2} h(r,\ \mathbf{w})\Big|_{\overline{w}_\parallel} dw_r. \quad (51)$$

To obtain an equation for the background component of the beam distribution function $f_0$, we substitute $f_k$ from the second equation of (43) in the first equation and then average over the variables r and $w_r$. As a result we obtain

$$\frac{\partial}{\partial t}\int_0^b r\,dr\int f_0\,dw_r - \pi e^2\sum_{p=1}^{\infty}\frac{\partial}{\partial w_\parallel}\frac{E_{kp}^2}{v_{ph}}\int_0^b r\,dr\int h(r,\ \mathbf{w})\Big|_{\overline{w}_\parallel} dw_r = 0. \quad (52)$$

The integration with respect to $k_\parallel$ was carried out with the aid of a $\delta$ function.

It is impossible to integrate Eqs. (51) and (52) in general form, since the integrals in the right-hand sides of (51) and (52) are different. To continue the calculations and to obtain an analytic solution, we take the factor $(1 + w_r^2/m^2c^2)^{1/2}$ outside the integration sign at the point $\overline{w}_r$, where the steady-state distribution function reaches a maximum with respect to the transverse momenta. Then, integrating Eqs. (51) and (52) and using the second equation of (50), we obtain

$$\sum_{p=1}^{\infty}\frac{b^2}{2}J_1^2(\lambda_p)\left(1 + \frac{\lambda_p^2}{k_\parallel^2 b^2}\right)E_{kp}^2 = \frac{4\pi v_{ph}^2}{k_\parallel}\frac{dw_\parallel}{dv_{ph}}\Big|_{\overline{w}_r}\int_0^b r\,dr\int(f_\infty - f_M)\,d\mathbf{w}, \quad (53)$$

where $f_\infty$ is the height of the plateau, with respect to the longitudinal momenta, on the distribution function, and $f_M$ is the initial distribution function of the beam.

Expressing $f_\infty$ in terms of the number of particles per unit beam length

$$f_\infty = \frac{2}{b^2}\frac{N}{w_2 - w_1}, \qquad N = \int_0^b r\,dr\int f_0\,d\mathbf{w} \quad (54)$$

and transforming the left-hand side of (53) with the aid of the relation

$$E_{kp}^2\frac{b^2}{2}J_1^2(\lambda_p)\left(1 + \frac{\lambda_p^2}{k_\parallel^2 b^2}\right) = \int_0^b |\mathbf{E}_{kp}(r)|^2 r\,dr, \quad (55)$$

we obtain the formula

$$\sum_{p=1}^{\infty}\int_0^b |\mathbf{E}_{kp}(r)|^2 r\,dr = \frac{4\pi v_{ph}^2 N}{k_\parallel}\frac{dw_\parallel}{dv_{ph}}\Big|_{\overline{w}_r}\frac{w_\parallel - w_1}{w_2 - w_1}, \quad (56)$$

where $w_2 > w_1$ are the limits of the plateau with respect to the longitudinal momenta on the beam distribution function[6].

Integrating both halves of (56) with respect to the phase velocities, in analogy with[6], we determine the energy density of the field excited by a relativistic bounded beam in the plasma

$$\frac{1}{8\pi}\int_0^b |\mathbf{E}|^2 r\,dr = \frac{1}{4}Nmc^2\left(\gamma_0 - \frac{1}{2\beta_0}\ln\frac{1+\beta_0}{1-\beta_0}\right), \qquad \beta_0 = \frac{v_0}{c}. \quad (57)$$

We note that when recalculated in terms of a unit volume of the plasma, formula (57) leads to the same result as the analogous expression in[6], which was derived for the unbounded case.

We analyze the transverse particle motion in the beam on the basis of the equation

$$\frac{\partial f_0}{\partial t} + \frac{v_r}{r}\frac{\partial}{\partial r}(rf_0) - D(r)\frac{\partial^2 f_0}{\partial w_r^2} = 0,$$
$$D(r) = \pi e^2\int |E_{rk}(r)|^2\delta(\omega_2 - k_\parallel v_\parallel)\,dk_\parallel, \quad (58)$$

which is obtained from (43) (by retaining the highest-order term of the expansion in the parameter $\eta$, which vanishes after averaging over $w_r$ in Eqs. (51) and (52)). Simultaneously with the formation of the plateau on the distribution function and with the cessation of the growth of the field amplitude, a stationary transverse state of the beam is established, $\partial f_0/\partial t = 0$. The solution of Eq. (58) was obtained for this case in[8] and makes it possible to express the steady-state distribution function $f_\infty(r,\ w_r)$ in terms of its value on the axis $f(0,\ w_r)$, or equivalently, to express the beam radius R in terms of $\overline{w}_r$. The same result can be obtained by comparing the second and third terms in (58) and using for the estimates $D \sim \pi e^2 n_1 mc^2\omega_2^{-1}$:

$$R \sim \overline{v}_r^3\gamma_0\omega_2/c^2\omega_1^2. \quad (59)$$

The transverse velocity $\overline{v}_r$ can be estimated by noting that the transverse beam temperature increases with increasing field in accordance with the diffusion law. Comparing the first and third terms in (58), we obtain $\overline{v}_r \sim c$, and from (59) it follows that

$$R \sim c\gamma_0/\omega_2\nu. \quad (60)$$

The establishment of a stationary transverse state of the beam can be explained physically in the following manner. The interaction of the electrons with the inhomogeneous random radiation field leads, on the one hand, to radial focusing and diffusion of the electrons towards the beam axis, and on the other hand to transverse heating and thermal expansion of the beam. The equilibrium between these processes is established when the beam radius reaches a value R determined by formula (60).

Using formula (60), we find that the expansion parameter (45) in the kinetic equation turns out to be

$$\eta \sim \nu / \gamma_0 \ll 1. \tag{61}$$

It should be noted that for a beam with a distribution function that is smeared with respect to the velocities we can obtain nonlinear stabilization of the beam instability[12] as a result of transfer of the oscillation energy into the nonresonant part of the spectrum via induced scattering.


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118