SOVIET PHYSICS JETP VOLUME 35, NUMBER 3 SEPTEMBER, 1972

Hydrodynamics of Photons in an Inhomogeneous Plasma

B. A. Trubnikov

I. V. Kurchatov Atomic Energy Institute

Submitted October 7, 1971

Zh. Eksp. Teor. Fiz. **62**, 971–979 (March, 1972)

Quasihydrodynamic equations for transverse waves in an inhomogeneous plasma are obtained in the geometric optics approximation. Examples of bounded equilibrium radiation blobs in which the light pressure is balanced by the plasma pressure are considered. The spectrum of radiation balancing a plasma of prescribed profile is calculated.

1. The propagation of low-amplitude electromagnetic waves in an inhomogeneous plasma has been considered in many papers (see, for example, the monograph[1], and the literature cited therein). A number of papers[3-5] have considered monochromatic waves of intensities so high that the radiation pressure becomes comparable with the plasma pressure. In this paper we solve, in the geometrical optics approximation, certain problems involving the interaction of intense nonmonochromatic radiation bunches with a plasma.

2. In the geometrical-optics approximation, the motion of transverse quanta in an inhomogeneous plasma is described by the equations

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{\partial\omega_\mathbf{k}}{\partial\mathbf{k}},\quad \frac{d\mathbf{k}}{dt}=-\frac{\partial\omega_\mathbf{k}}{\partial\mathbf{r}};\quad \omega_\mathbf{k}=\sqrt{\omega_0^2(\mathbf{r})+c^2k^2}. \tag{1}$$

Here $\mathbf{v}$ is the group velocity of the waves, $\mathbf{k}$ is the wave vector, and $\omega_0(\mathbf{r}) = [4\pi e^2 n_e(\mathbf{r})/m]^{1/2}$ is the electron plasma frequency. If we introduce the momentum $\mathbf{p} = \hbar\mathbf{k}$ and the "mass" $\mu_\mathbf{k} = \hbar\omega_\mathbf{k}/c^2$ of the quantum, then Eqs. (1) can be rewritten in the form

$$\mathbf{p}=\mu\mathbf{v},\quad \frac{d\mathbf{p}}{dt}=-\frac{\alpha}{\omega_\mathbf{k}}\nabla n_e(\mathbf{r});\quad \alpha=2\pi\hbar\frac{e^2}{m}, \tag{2}$$

so that the force exerted on the quantum by the external medium is proportional to the gradient of the electron density $n_e(\mathbf{r})$. If we introduce the function $N_\mathbf{k}(\mathbf{r}, t)$, representing the number of quanta in the state $\mathbf{k}$, and normalized in such a way that the quantity $N(\mathbf{r}) = 2\int N_\mathbf{k}d\mathbf{k}/8\pi^3$ is equal to the number of quanta with two possible polarizations in a unit volume, then all these quanta will be acted upon by the summary force (see (2))

$$\mathbf{f}_n=-U\nabla n_e,\quad U=\alpha\int\frac{N_\mathbf{k}}{\omega_\mathbf{k}}2\frac{d\mathbf{k}}{8\pi^3}. \tag{3}$$

The quantity U has the dimension of energy, and we shall show below that it is equal to the average kinetic energy of an electron oscillating in the rapidly alternating field of the waves.

3. In the absence of absorption and scattering of the quanta, the function $N_\mathbf{k}(\mathbf{r}, t)$ satisfies the kinetic equation

$$\frac{\partial}{\partial t}N_\mathbf{k}+\operatorname{div}(\mathbf{v}N_\mathbf{k})+\frac{\partial}{\partial\mathbf{k}}(\dot{\mathbf{k}}N_\mathbf{k})=0, \tag{4}$$

which when multiplied respectively by 1, $\hbar\mathbf{k}$, and $\hbar\omega_\mathbf{k}$ and upon integration over the number of states $2d\mathbf{k}/8\pi^3$ (with allowance for polarization) yields the quasihydrodynamic equations for the "gas" of quanta:

$$\frac{\partial}{\partial t}N+\operatorname{div}(N\langle\mathbf{v}\rangle)=0,\ \frac{\partial}{\partial t}\mathbf{P}=-\nabla\cdot\hat{\pi}+$$
$$\frac{\partial}{\partial t}w+c^2\operatorname{div}\mathbf{P}-U\frac{\partial}{\partial t}n_e=0. \tag{5'}$$

(A similar system was previously obtained for longitudinal waves in[6].) We have introduced here the obvious notation

$$N=\int\Phi_\mathbf{k}d\mathbf{k},\quad \langle\mathbf{v}\rangle=\frac{1}{N}\int\mathbf{v}\Phi_\mathbf{k}\,d\mathbf{k},\quad \mathbf{P}=\int\hbar\mathbf{k}\Phi_\mathbf{k}\,d\mathbf{k},$$
$$\pi_{\alpha\beta}=\int\mu v_\alpha v_\beta\Phi_\mathbf{k}d\mathbf{k},\ w=\int\hbar\omega_\mathbf{k}\Phi_\mathbf{k}d\mathbf{k},\ \Phi_\mathbf{k}=N_\mathbf{k}/4\pi^3. \tag{6}$$

Contraction of the tensor $\hat{\pi}$ yields the relation

$$\operatorname{Sp}\hat{\pi}=\pi_{\alpha\alpha}=\int\hbar\frac{c^2k^2+\omega_0^2-\omega_0^2}{\omega_\mathbf{k}}\Phi_\mathbf{k}d\mathbf{k}=w-2Un_e. \tag{7}$$

We note that the system (5)--(5′) is not closed. In the stationary case we have the equation

$$\nabla\cdot\hat{\pi}=\mathbf{f}_n=-U\nabla n_e, \tag{8}$$

which describes the equilibrium of the plasma with the radiation.

4. It is useful to trace a more detailed derivation of (5′) directly from Maxwell's equations

$$\frac{\partial}{\partial t}\mathbf{B}=-c\operatorname{rot}\mathbf{E},\ \frac{\partial}{\partial t}\mathbf{E}=c\operatorname{rot}\mathbf{B}-4\pi\mathbf{j}, \tag{9}$$

where $\mathbf{j} = en\mathbf{v}$ and $m\dot{\mathbf{v}} = e(\mathbf{E} + c^{-1}\mathbf{v}\times\mathbf{B})$. Writing down the relation

$$\frac{\partial}{\partial t}\frac{E^2+B^2}{8\pi}=-\operatorname{div}\frac{c}{4\pi}[\mathbf{EB}]-\mathbf{jE}, \tag{10*}$$

we expand the field in plane waves

$$\mathbf{E}(\mathbf{r},t)=\int\mathbf{E}_\mathbf{k}e^{i(\mathbf{kr}-\omega_\mathbf{k}t)}\,d\mathbf{k}, \tag{11}$$

which can propagate in the plasma. Here $\mathbf{E}_\mathbf{k}$ is assumed to be a function that varies slowly with the time and with the coordinates. Assuming the existence of many harmonics with random phases, we introduce the correlation function

$$\overline{(\mathbf{E}_\mathbf{k}^*\mathbf{E}_{\mathbf{k}'})}=K_\mathbf{k}(\mathbf{r},t)\delta(\mathbf{k}-\mathbf{k}'). \tag{12}$$

The bar denotes averaging over the phases. Recognizing that $\mathbf{B}_\mathbf{k} = c\mathbf{k}\times\mathbf{E}_\mathbf{k}/\omega_\mathbf{k}$ and $\mathbf{E}_\mathbf{k}\perp\mathbf{k}$ (transversality), we obtain the averaged Poynting vector

$$\overline{\mathbf{S}}=\frac{c}{4\pi}\overline{[\mathbf{E},\mathbf{B}]}=\frac{c}{4\pi}\int d\mathbf{k}\int d\mathbf{k}'\frac{c\mathbf{k}'}{\omega_{\mathbf{k}'}}\overline{(\mathbf{E}_\mathbf{k}^*\mathbf{E}_{\mathbf{k}'})}=\frac{c^2}{4\pi}\int\frac{\mathbf{k}}{\omega_\mathbf{k}}K_\mathbf{k}d\mathbf{k}. \tag{13}$$

If we define the number of quanta $N_\mathbf{k}$ by the relation

*$[\mathbf{EB}]\equiv\mathbf{E}\times\mathbf{B}$.

$$N_{\mathbf{k}} = \pi^2 K_{\mathbf{k}} / \hbar\omega_{\mathbf{k}}, \tag{14}$$

then, taking the definitions (6) into account, we easily verify that

$$\overline{\mathbf{S}} = c^2\mathbf{P}, \frac{1}{8\pi}\overline{E^2} = \frac{1}{2}w, \frac{1}{8\pi}\overline{B^2} = \frac{1}{2}w - Un_e. \tag{15}$$

and thus the energy of the quantum per unit volume is

$$w = \int \hbar\omega_{\mathbf{k}} N_{\mathbf{k}} 2 \frac{d\mathbf{k}}{8\pi^3} = \frac{1}{8\pi}(\overline{E^2 + B^2}) + Un_e \tag{16}$$

and includes, besides the field energy, also the kinetic energy Un_e of the electrons oscillating in the field[7].

Substituting the obtained values of $\overline{E^2}$, $\overline{B^2}$, and $\overline{S}$ in the average equation (10)

$$\frac{\partial}{\partial t}(\overline{E^2 + B^2})/8\pi + \operatorname{div}\overline{\mathbf{S}} = -\overline{\mathbf{jE}} = -en_e(\overline{\mathbf{vE}}), \tag{17}$$

we obtain

$$\frac{\partial}{\partial t}w + c^2 \operatorname{div}\mathbf{P} - U\frac{\partial}{\partial t}n_e = n_e\left[\frac{\partial}{\partial t}U - e(\overline{\mathbf{vE}})\right], \tag{18}$$

and to obtain equality with (5′) it is necessary that the right-hand side vanish.

5. To prove the validity of the last relation

$$e(\overline{\mathbf{vE}}) = \frac{\partial}{\partial t}U, \tag{19}$$

we represent the position of the electron in the form $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$, where $\rho \ll R$. The quantity ρ describes the rapid oscillations of the charge about the average position $\mathbf{R}$. We represent the equation of motion of the electrons in the form (we expand the fields in terms of ρ)

$$m(\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}) = e\{\mathbf{E}(\mathbf{R}) + (\boldsymbol{\rho}\nabla)\mathbf{E}(\mathbf{R}) + c^{-1}[\dot{\boldsymbol{\rho}}\mathbf{B}(\mathbf{R})]\}, \tag{20}$$

and for fast oscillations we have

$$m\ddot{\boldsymbol{\rho}} = m\dot{\mathbf{v}} = e\mathbf{E}(\mathbf{R}, t),\ \mathbf{v} = \frac{e}{m}\int_{-\infty}^{t}\mathbf{E}(t')\,dt'. \tag{21}$$

This oscillation velocity ν must be substituted in (19), from which $\mathbf{R}$ drops out.

$$e(\overline{\mathbf{vE}}) = \frac{\partial}{\partial t}\int_{-\infty}^{t} dt' e(\overline{\mathbf{E v}})_{t'} = \frac{e^2}{m}\frac{\partial}{\partial t}\int_{-\infty}^{t} dt' \mathbf{E}^*(t')\int_{-\infty}^{t'}\mathbf{E}(t'')\,dt''$$

$$= \frac{e^2}{m}\frac{\partial}{\partial t}\int d\mathbf{k}\,d\mathbf{k}'\frac{\overline{(\mathbf{E}_{\mathbf{k}}^*\mathbf{E}_{\mathbf{k}'})}}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})\omega_{\mathbf{k}'}}. \tag{22}$$

The fields were represented here in the form of the expansions (11). Symmetrization under the integral signs in (22)

$$\frac{1}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})\omega_{\mathbf{k}'}} \to \frac{1}{2}\left[\frac{1}{(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})\omega_{\mathbf{k}'}} + \frac{1}{(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})\omega_{\mathbf{k}}}\right] = \frac{1}{2\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}$$

leads to the result

$$e(\overline{\mathbf{vE}}) = \frac{e^2}{2m}\frac{\partial}{\partial t}\int\frac{K_{\mathbf{k}}}{\omega_{\mathbf{k}}^2}d\mathbf{k} = \frac{\partial}{\partial t}U, \tag{23}$$

which proves (19), as well as (5′) by virtue of (18).

6. Averaging the equation of motion (20) over the fast oscillations and recognizing that

$$\overline{[\dot{\boldsymbol{\rho}}\mathbf{B}]} = -\left[\overline{\boldsymbol{\rho}\frac{\partial}{\partial t}\mathbf{B}}\right] = c\overline{[\boldsymbol{\rho}\,\operatorname{rot}\mathbf{E}]},$$

we obtain

$$m\ddot{\mathbf{R}} = \mathbf{F}_1 = e\{\overline{(\boldsymbol{\rho}\nabla)\mathbf{E}} + \overline{[\boldsymbol{\rho}[\nabla\mathbf{E}]]}\} = e\nabla\overline{(\boldsymbol{\rho}\check{\mathbf{E}})}. \tag{24}$$

The caret over $\mathbf{E}$ indicates the action of the operator ∇, but since $\mathbf{E} = m\dot{\nu}/e$, we have $e(\overline{\rho\check{\mathbf{E}}}) = m(\overline{\rho\dot{\check{\nu}}}) = -m(\overline{\nu\nu})$, and consequently $\mathbf{F}_1 = -\frac{1}{2}m\nabla(\overline{\nu^2})$.

Using expression (21) for ν and the expansion of the field (11) we obtain, taking (12) and (14) into acoount

$$\frac{m\overline{\mathbf{v}^2}}{2} = \frac{e^2}{2m}\int d\mathbf{k}\,d\mathbf{k}'\overline{(\mathbf{E}_{\mathbf{k}}^*\mathbf{E}_{\mathbf{k}'})}/\omega_k\omega_{k'} = U, \tag{25}$$

so that the quantity U introduced in (3) is the average kinetic energy of the electron oscillating in the field.

On the other hand, substituting (25) in the force $\mathbf{F}_1$, we get from (24)

$$m\ddot{\mathbf{R}} = \mathbf{F}_1 = -\nabla U \tag{26}$$

and consequently, the quantity U plays the role of the potential energy at slow electron displacements.

We note that an analogous situation is encountered in thermodynamics, where the quantity $U_{gas} = c_V T(\frac{3}{2}kTN$ at $\gamma = \frac{5}{3})$ is the kinetic energy of the fast thermal motions of the molecules and plays at the same time the role of the potential energy $U_{gas} = \int_V^\infty p\,dV = pV/(\gamma - 1)$ in the slow process of adiabatic $(pV^\gamma = \text{const})$ expansion of the gas.

7. Multiplying $\mathbf{F}_1$ by n_e, we obtain the force acting on all the electrons in a unit volume, and therefore the hydrodynamic equation describing the motion of the plasma in the field of the waves can be written in the

$$\rho\dot{\mathbf{V}} = -\nabla p_{\text{gas}} + \mathbf{f}_E; \quad \mathbf{f}_E = n_e\mathbf{F}_1 = -n_e\nabla U. \tag{27}$$

Formulas (26) and (27) generalize the results obtained earlier[3-9] for monochromatic fields to include the case of a nonmonochromatic set of waves with random phases.

In the case of stationary equilibrium between the plasma and the radiation, we obtain from (8) and (27) the self-consistent equations

$$\nabla\cdot\hat{\pi} = \mathbf{f}_n = -U\nabla n_e, \quad \nabla p_{\text{gas}} = \mathbf{f}_E = -n_e\nabla U. \tag{28}$$

Assuming quasineutrality $n_e = n_i = n(\mathbf{r})$ and, in the simplest case, constancy of the total temperature $T_+ = T_i + T_i$ of the electrons and ions in space we obtain from the second equation

$$p_{\text{gas}} = nT_+, \quad n(\mathbf{r}) = n(\infty)e^{-U/T_+} \ (\text{at}\ \ U(\infty) = 0). \tag{29}$$

This Boltzmann distribution is analogous to that obtained in[3,4] for monochromatic waves. If we assume not the field but the plasma density profile to be specified, then we obtain from (29) the equilibrium value of the quantity

$$U(\mathbf{r}) = T_+ \ln g^{-1}, \quad g = n(\mathbf{r})/n(\infty). \tag{30}$$

The first equation of (28) then takes the form

$$\nabla\cdot\hat{\pi} = n(\infty)T_+ \ln g\nabla g = -\nabla(p_{\text{gas}} + nU) \tag{31}$$

and in a number of cases it is possible to obtain from it the distribution of the field that balances a specified plasma profile.

8. Let us consider a concrete example of a similar self-consistent problem for a profile in the form (see Fig. 1)

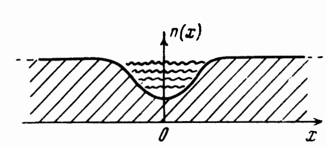

FIG. 1. Density well in a plasma filled with radiation.

$$n(x)=n(\infty)-\frac{n(\infty)-n(0)}{\mathrm{ch}^2(x/a)}=n(\infty)g,\qquad g(x)=1-\frac{q}{\mathrm{ch}^2(x/a)},\quad (32)$$

where $q = 1 - (n(0)/n(\infty)) < 1$. In the case $n(\infty) = 0$, such a profile would correspond to a flat layer of a plasma with thickness on the order a, with a density that drops off towards the periphery. As first shown by Epstein[10], the wave problem of reflection from such a layer can be solved exactly. In our case $n(\infty) > n(0)$, corresponding to a "small well," and the problem consists of finding the radiation that is strapped in it and has a pressure capable of balancing the indicating profile.

Let us consider the case when the radiation consists of waves propagating only along the x axis. The tensor $\pi_{\alpha\beta}$ (see (6)) has in this case only the component π_{xx}, which obviously coincides with the contraction (7), $\pi_{xx} = w - 2nU$, and since all the quantities depend only on x, by integrating (31) and assuming that there is no radiation at infinity ($x = \pm\infty$), we obtain

$$w-nU+p_{\mathrm{gas}}=\mathrm{const}=p_{\mathrm{gas}}(\infty).\quad (33)$$

Taking (30) into account, we obtain the energy density

$$w(x)=\overline{E^2}/4\pi=\rho_{\mathrm{gas}}(\infty)\,(1-g+g\ln g^{-1}).\quad (34)$$

This expression is valid for any flat layer. For a concrete layer, say of the form (32), we can obtain from this the total energy of the trapped radiation (at $n(0) = 0$, $q = 1$, $\xi = x/a$)

$$W=\int_{-\infty}^{+\infty}w(x)\,dx=ap_{\mathrm{gas}}(\infty)\int_{-\infty}^{+\infty}(\mathrm{ch}^{-2}\xi-\mathrm{th}^2\xi\ln\mathrm{th}^2\xi)\,d\xi$$
$$=\left(\frac{\pi^2}{2}-2\right)ap_{\mathrm{gas}}(\infty),\quad (35)$$

per column of unit cross section $dydz = 1$.

9. The radiation considered by us is not in thermodynamic equilibrium, and therefore cannot be completely described by specifying only the quantities w and U in Eqs. (28) which do not even form a closed system. Their solution is therefore not unique.

Moreover, it is clear that the specified plasma profile can be balanced by the radiation pressure only at a definite spectral composition of the latter. In essence, therefore, it is necessary to know the distribution function $N_{\mathbf{k}}(\mathbf{r}, t)$ itself, and not only its integral characteristics

$$U=2\pi\hbar\frac{e^2}{m}\int\frac{N_{\mathbf{k}}}{\omega_{\mathbf{k}}}2\frac{d\mathbf{k}}{8\pi^3}\ \text{and}\ w=\int\hbar\omega_{\mathbf{k}}N_{\mathbf{k}}2\frac{d\mathbf{k}}{8\pi^3}.\quad (36)$$

In the case of a flat layer with trapped waves propagating only along the x axis (anisotropic one-dimensional radiation), this problem admits of a unique solution in the case of a monotonic density profile $g(x) = n(x)/n(\infty)$, of the type shown in Fig. 1.

10. In fact, the general solution of (4) is the function $N_{\mathbf{k}}$, which depends in arbitrary fashion on the integrals of motion of the quantum. In the one-dimensional case $k_y = k_z = 0$, and therefore the only integral of motion is the frequency $\omega_{\mathbf{k}}$ itself, so that here $N_{\mathbf{k}} = f(\omega_{\mathbf{k}})\delta(k_y)\delta(k_z)$. Taking this unique dependence into account, we write down the quantity $U(x)$ (36) in the form ($d\mathbf{k} = dk_x dk_y dk_z$)

$$U(x)=\int_{-|k_m|}^{+|k_m|}u_1(\omega_{\mathbf{k}})\,dk_x;\qquad u_1(\omega_{\mathbf{k}})=2\pi\hbar\frac{e^2}{m}\iint\frac{N_{\mathbf{k}}}{\omega_{\mathbf{k}}}2\frac{dk_y\,dk_z}{8\pi^3}.\quad (37)$$

Here $|k_m|$ is the maximum value of $|k_x|$. Recognizing that $\omega_{\mathbf{k}}$ and hence also u_1 are even functions of k_x, and changing over to a new variable $\omega_{\mathbf{k}}$, we have

$$U=2\int_0^{k_m}u_1\,dk_x=2\int u_1\frac{d\omega_{\mathbf{k}}}{\partial\omega_{\mathbf{k}}/\partial k_x}=\frac{1}{c}\int u_1(\omega_{\mathbf{k}})\frac{d\omega_{\mathbf{k}}^2}{\sqrt{\omega_{\mathbf{k}}^2-\omega_0^2(x)}},\quad (38)$$

where the integration with respect to ω is carried out, using as limits the local plasma frequency $\omega_0(x)$ and the maximum frequency trapped in the well $\omega_0(x\to\infty) = \omega_0(\infty) = \sqrt{4\pi e^2 n(\infty)/m}$. It is convenient to introduce a new integration variable $s = \omega_{\mathbf{k}}^2/\omega_0^2(\infty)$ and the relative plasma profile in the well $g(x) = n(x)/n(\infty) \le 1$. Then (38) takes the form

$$U(x)=\frac{\omega_0(\infty)}{c}\int_{g(x)}^{1}u_1(s)\frac{ds}{\sqrt{s-g(x)}},\quad (39)$$

and $u(x)$ depends in turn, for the Boltzmann distribution (29), on the profile $g(x)$ in accordance with the formula (30), so that if we put $f(s) = u_1\omega_0(\infty)/cT_+$, then by equating (30) and (39) we obtain the Abel integral equation

$$\ln\frac{1}{g}=\int_g^1\frac{f(s)\,ds}{\sqrt{s-g}}\qquad(0\le g\le 1),\quad (40)$$

a solution of which is the function

$$f(s)=\frac{2}{\pi\sqrt{s}}\,\mathrm{arc\,tg}\sqrt{\frac{1-s}{s}}.$$

11. Gathering the results, we obtain the distribution function of the quanta trapped in a well with arbitrary plasma-wall profile

$$N_{\mathbf{k}}=\delta(k_y)\delta(k_z)4\pi\frac{mc}{\hbar e^2}T_+\,\mathrm{arc\,tg}\sqrt{\left(\frac{\omega_0(\infty)}{\omega_{\mathbf{k}}}\right)^2-1},$$

where $\omega_{\mathbf{k}} = \sqrt{\omega_0^2(x) + c^2k^2}$. The dependence on the concrete profile $n(x)$ enters in $N_{\mathbf{k}}$ only via the frequency $\omega_{\mathbf{k}}$, so that the energy density can be represented in the form

$$w(x)=\int\hbar\omega_{\mathbf{k}}N_{\mathbf{k}}2\frac{d\mathbf{k}}{8\pi^3}=\int_{\omega_0(x)}^{\omega_0(\infty)}w_\omega(x)\,d\omega,\quad (41)$$

where (here $\nu = \sqrt{s} = \omega/\omega_0(\infty)$)

$$w_\omega(x)=2\iint\hbar\omega_{\mathbf{k}}N_{\mathbf{k}}2\frac{dk_y\,dk_z}{8\pi^3}\frac{\partial k_x}{\partial\omega_{\mathbf{k}}}$$
$$=\frac{8}{\pi}\frac{p_{\mathrm{gas}}(\infty)}{\omega_0(\infty)}\frac{\nu^2}{\sqrt{\nu^2-g(x)}}\,\mathrm{arctg}\frac{\sqrt{1-\nu^2}}{\nu}$$

is the local energy density per unit frequency interval.

Integrating this expression with respect to x, we obtain the spectral distribution of the total energy

$$W_\omega=\int w_\omega(x)\,dx=\frac{8}{\pi}p_{\mathrm{gas}}(\infty)\frac{s}{\omega_0(\infty)}\,\mathrm{arctg}\sqrt{\frac{1-s}{s}}\int_{-|x_t|}^{+|x_t|}\frac{dx}{\sqrt{s-g(x)}}.\quad (42)$$

The last integral must be taken over the region where $s - g(x) > 0$. For the concrete profile (32) considered by us earlier, it is equal to

$$\int[s-1+q\,\mathrm{ch}^{-2}(x/a)]^{-1/2}\,dx=\pi a/\sqrt{1-s},\quad (43)$$

so that in this case formula (42) becomes

$$dW=W_\omega d\omega=8ap_{\mathrm{gas}}(\infty)F(\nu)\,d\nu,\quad (44)$$

where the function $F(\nu)$ is

$$F(\nu)=\frac{\nu^2}{\sqrt{1-\nu^2}}\,\mathrm{arctg}\frac{\sqrt{1-\nu^2}}{\nu}=\begin{cases}{}^1/_2\pi\nu^2 & \text{for}\quad \nu\ll 1,\\ 1 & \text{for}\quad \nu\to 1.\end{cases}$$

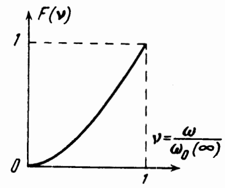

FIG. 2. Spectrum $F(\nu)$ of equilibrium radiation in a well with density profile (32).

The spectrum $F(\nu)$ is shown in Fig. 2. It is easy to verify that the integral of (44) over the frequencies gives the value of the total energy ($\nu_{min} = 0$ if $q = 0$

$$W = 8ap_{gas}(\infty)\int_{\nu_{min}=0}^{1} F(\nu)\,d\nu = \left(\frac{\pi^2}{2} - 2\right) a p_{gas}(\infty),$$

obtained earlier in (35). At given W and $P_{gas}(\infty)$, this relation determines the dimension a of the well.

12. It is easy to consider also the inverse problem of finding the equilibrium plasma profile, given the spectrum of the total energy W_ω. Then, in accordance with (42), one should regard as known the function

$$J(s) = \int_{-|x_s|}^{+|x_s|} \frac{dx}{\sqrt{s - g(x)}} = 2\int_{g(0)}^{s} \frac{dx(g)}{dg}\frac{dg}{\sqrt{s-g}} = \frac{\pi\omega_0(\infty)}{8p_{gas}(\infty)}\,\frac{W_\omega}{s\,\mathrm{arctg}\sqrt{s^{-1}-1}}. \tag{45}$$

This equation can be solved only if the spectrum W_ω vanishes at $\nu^2 \le g(0) < 1$ and extends up to $\nu = 1$. Then, solving the Abel equation (45), we obtain for the profile $g(x)$ the inverse relation

$$x = x(g) = \frac{1}{2\pi}\int_{g(0)}^{g} \frac{J(s)\,ds}{\sqrt{g-s}}.$$

For example, substituting here $J(s) = \pi a/\sqrt{1-s}$ from (43)), we get $x(g) = a\cosh^{-1}[(1 - g(0))/(1 - g)]$, which corresponds to the profile (32).

13. Let us consider finally a spherically symmetrical well in a plasma and assume for simplicity that the radiation consists entirely of quanta moving only on circular orbits, so that $k_r = 0$. Putting $k_\perp = (k_\theta^2 + k_\varphi^2)^{1/2}$ and equating the centrifugal and radiant forces in (2), we have

$$\mu\frac{v_\perp^2}{r} = \frac{a}{\omega_k}\frac{\partial n}{\partial r}, \quad \text{meaning} \quad c^2k_\perp^2 = \frac{\omega_0^2(\infty)}{2}\,r\frac{\partial g(r)}{\partial r}.$$

Therefore on a circle of a given radius r there can move only quanta with frequency

$$\omega_k = \sqrt{\omega_0^2(r) + c^2k_\perp^2} = \omega_0(\infty)\sqrt{g + \frac{r}{2}\frac{\partial g}{\partial r}}, \qquad g(r) = \frac{n(r)}{n(\infty)} \tag{46}$$

as a result of which the quantities U and w (see (32)) turn out to be connected by the relation

$$w = \int \hbar\omega_k^2 \frac{N_k}{\omega_k}\, 2\frac{dk}{8\pi^3} = \left(2n + r\frac{\partial n}{\partial r}\right)U,$$

and the tensor $\hat{\pi}$ has obviously the form (see (7))

$$\hat{\pi} = \left(\hat{\delta} - \frac{\hat{rr}}{r^2}\right)\left(\frac{w}{2} - Un\right) = \left(\hat{\delta} - \frac{\hat{rr}}{r^2}\right)U\frac{r}{2}\frac{\partial n}{\partial r}$$

The first equilibrium equation in (28) is satisfied automatically, and from the second we have $U = T_* \ln(1/g)$, so that

$$w(r) = p_{gas}(\infty)\left(2g + r\frac{\partial}{\partial r}g\right)\ln(1/g).$$

For example, for a spherical well with density profile $g(r) = 1 - \cosh^{-2}[(r/a)^3]$ (see (32)) we have

$$w(r) = 2p_{gas}(\infty)\left(\mathrm{th}^2\xi + 3\xi\frac{\mathrm{th}\,\xi}{\mathrm{ch}^2\xi}\right)\ln\frac{1}{\mathrm{th}^2\xi}; \qquad \xi = \left(\frac{r}{a}\right)^3,$$

and then the total energy in the well is

$$W = \int_0^\infty w(r)\,4\pi r^2\,dr = \frac{4}{3}\pi a^3 p_{gas}(\infty)\varkappa,$$

where (see (35))

$$\varkappa = \int_0^\infty (3\,\mathrm{ch}^{-2}\xi + \mathrm{th}^2\xi\ln\mathrm{th}^2\xi)\,d\xi = 5 - \frac{\pi^2}{4}.$$

In view of the one-to-one correspondence of the radius and the frequency, which is given by (46), it is easy to write for this case also the distribution of the total energy with respect to the frequencies, $dW = W_\omega d\omega$, where

$$W_\omega = w(r)\,4\pi r^2\frac{dr}{d\omega} = \frac{4}{3}\pi a^3 w(r)\frac{d\xi}{d\omega} = \frac{W}{\omega_0(\infty)}\varphi(\nu),$$

and the function $\varphi(\nu)$, normalized by the condition $\int_0^1 \varphi(\nu)d\nu = 1$, is determined from the parametric relation

$$\varphi(\nu) = \frac{4}{\varkappa}\,\frac{(\mathrm{sh}^2\xi + 3\xi\,\mathrm{th}\,\xi)^{3/2}}{5\,\mathrm{sh}\,\xi + 3\xi\,\mathrm{ch}\,\xi(1 - 3\,\mathrm{th}^2\xi)}\ln\frac{1}{\mathrm{th}^2\xi},$$

$$\nu = \sqrt{\mathrm{th}^2\xi + 3\xi\frac{\mathrm{th}\,\xi}{\mathrm{ch}^2\xi}}.$$

For small $\nu \ll 1$ we have, in particular, $\varphi(\nu) = (\nu^2/\kappa)\ln(2/\nu)^2$.

In analogy with a spherical well, it is easy to construct also a cylindrical well with quanta moving in circles, we shall not stop to discuss this case.

In conclusion we note that in the concentric model with circular orbits of the transverse quanta, which was considered above, four-wave processes of induced scattering do not change the distribution function N_k, so that the evolution of trapped radiation is possible only as a result of electron collisions and a three-wave process in which one longitudinal and two transverse quanta take part. These questions, however, deserve a separate analysis.

The author is most grateful to M. A. Leontovich, T. F. Volkov, and L. I. Rudakov for critical remarks and for a discussion of the results.

[1]V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (The Diffusion of Electromagnetic Waves in Plasma), Fizmatgiz, 1960.

[2]L. M. Brekhovskikh, Volny i sloistykh sredakh (Waves in Layered Media), AN SSSR, 1957.

[3]T. F. Volkov, in: Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problem of Controlled Thermonuclear Reactions), Vol. 3, p. 336, AN SSSR, 1958.

[4]G. A. Askar'yan, Zh. Eksp. Teor. Fiz. **42**, 1567 (1962) [Sov. Phys.-JETP **15**, 1088 (1962)].

[5]V. P. Silin, Zh. Eksp. Teor. Fiz. **53**, 1662 (1967) [Sov. Phys.-JETP **26**, 955 (1968)].

[6]A. A. Vedenov, A. V. Gordeev, and L. I. Rudakov, Plasma Phys. **9**, 719 (1967).

[7]V. N. Tsytovich, Nelineĭnye éffekty v plazme (Nonlinear Effects in Plasma), Nauka, 1967.

[8]A. V. Gaponov and M. A. Miller, Zh. Eksp. Teor. Fiz. **34**, 242 (1958) [Sov. Phys.-JETP **7**, 168 (1958)].

[9]L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **39**, 1450 (1960) [Sov. Phys.-JETP **12**, 1008 (1961)].

[10]P. Epstein, Reflection of Waves in an Inhomogeneous Absorbing Medium; Proc. Natl. Acad. Sci. USA **16**, 627 (1930).

Translated by J. G. Adashko
115