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Nonlinear Interaction Between Waves in a Plasma with Random Inhomogeneities

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The nonlinear interaction of waves in a plasma containing random inhomogeneities is considered in the quasihydrodynamic approximation. The fundamental equations are obtained for quantities averaged over the inhomogeneity ensemble. The contracted equations for the amplitudes and phases of the interacting (transverse and longitudinal) waves in an isotropic plasma are investigated in the one-dimensional approximation on the basis of the asymptotic method.

THE nonlinear interaction of waves has been studied in sufficient detail in application to a homogeneous plasma (see, for example, [1-4]). There is undoubted interest in the generalization of the known results to the case in which the plasma is inhomogeneous. In particular, studies have been carried out in this direction for a regularly inhomogeneous plasma (see, for example, [5]). The effect of random inhomogeneities on the process of the nonlinear interaction of waves has not appeared in the literature to date, so far as we know. In optics, it is true that a number of questions (frequency doubling, stimulated Mandel'shtam-Brillouin scattering, stimulated Rayleigh wing scattering, etc.) on the statistical theory of nonlinear processes have already had their development.[6,7] In the present work, a systematic derivation is carried out of the equations for inhomogeneities in the electron concentration, averaged over the ensemble of quantities that enter into the quasihydrodynamic equations for the plasma. On the basis of their asymptotic method, one-dimensional contracted equations are obtained for the amplitudes and phases of the interacting waves (transverse and longitudinal). An analysis of the various limiting cases is carried out.

1. DERIVATION OF THE EQUATIONS FOR AVERAGED QUANTITIES

The system of quasihydrodynamic equations describing the isotropic heating of the plasma to the temperature T has the form[1)]

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=\frac{e}{m}\mathbf{E}+\frac{e}{mc}[\mathbf{vH}]-\frac{v_T^2}{N}\operatorname{grad}N+\mathbf{G}, \qquad (1)^*$$

$$\frac{\partial N}{\partial t}+\operatorname{div}N\mathbf{v}=0,$$

$$\operatorname{rot}\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}=0, \quad \operatorname{rot}\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{4\pi e}{c}N\mathbf{v},$$

where **E** and **H** are the electric and magnetic fields, **v** and N are the velocity and concentration of the electrons of the plasma, e and m are their charge and mass, c is the velocity of light, $v_T^2 = \kappa T/m$ (κ, Boltzmann's constant), **G** is the external field that creates an inhomogeneous distribution of the electron concentration in the equilibrium state. This concentration can be written in the form

$$N=N_0+\delta N(x)+\langle\tilde{N}(x,t)\rangle+N'(x,t), \qquad (2)$$

where $N_0 + \delta N(x)$ is the equilibrium value and $\delta N(x)$ the random departure of the electron density from the mean value, which latter is equal to N_0. The quantity $\langle\tilde{N}(x, t)\rangle + N'(x, t)$ is the perturbation under the action of the electromagnetic field, which is represented in the form of the sum of the average over the ensemble $\langle\tilde{N}\rangle$ and the fluctuating part $N'(x, t)$. The brackets in (2) indicate statistical averaging, the value of $\mathbf{G} = (v_T^2/N_0)\operatorname{grad}\delta N(x)$. In what follows, the fluctuations of $\tilde{N}$ are assumed to depend on a single coordinate x (one-dimensional inhomogeneity).

Representing the variables **v**, **E**, **H** in (1) in the form of mean values and their fluctuating deviations, and averaging the system (1), we obtain the equations for the mean quantities

$$\frac{\partial}{\partial t}\langle\mathbf{v}\rangle-\frac{e}{m}\langle\mathbf{E}\rangle+\frac{v_T^2}{N_0}\operatorname{grad}\langle\tilde{N}\rangle-\frac{v_T^2}{N_0^2}\operatorname{grad}\langle N'\delta N\rangle$$
$$=(\langle\mathbf{v}\rangle\nabla)\langle\mathbf{v}\rangle+\frac{e}{mc}[\langle\mathbf{v}\rangle\langle\mathbf{H}\rangle]+\frac{v_T^2}{N_0}\langle\tilde{N}\rangle\operatorname{grad}\langle\tilde{N}\rangle,$$
$$\frac{\partial}{\partial t}\langle\tilde{N}\rangle+N_0\operatorname{div}\langle\mathbf{v}\rangle+\operatorname{div}\langle\delta N\mathbf{v}'\rangle=-\operatorname{div}\langle\tilde{N}\rangle\langle\mathbf{v}\rangle,$$
$$\operatorname{rot}\langle\mathbf{E}\rangle+\frac{1}{c}\frac{\partial}{\partial t}\langle\mathbf{H}\rangle=0,$$
$$\operatorname{rot}\langle\mathbf{H}\rangle-\frac{1}{c}\frac{\partial}{\partial t}\langle\mathbf{E}\rangle-\frac{4\pi e}{c}N_0\langle\mathbf{v}\rangle-\frac{4\pi e}{c}\langle\delta N\mathbf{v}'\rangle=\frac{4\pi e}{c}\langle\tilde{N}\rangle\langle\mathbf{v}\rangle.$$

In the derivation of (3), it is assumed that:

1) the wave perturbations and fluctuations of the electron concentration are small, $|\langle\tilde{N}\rangle|$, $|N'|$, $|\delta N| \ll N_0$;

2) the nonlinearities are so weak that the complex amplitudes of the interacting waves are slowly changing (in the scale of wavelengths and periods of oscillation) functions of the coordinates and the time.

The conditions 1) and 2) permit us to discard in the set (3) the nonlinear terms of the form $\langle(\mathbf{v}'\nabla)\mathbf{v}'\rangle$, $(e/mc)\langle\mathbf{v}'\times\mathbf{H}'\rangle$, $\langle N'\mathbf{v}'\rangle$, which are associated with the fluctuations and are smaller than the terms remaining in the right-hand sides of the set (3) by a factor of at least $\langle(\delta N/N_0)^2\rangle$.

We note that account of the small quantities $\langle\mathbf{v}'\delta N\rangle$ and $\langle N'\delta N\rangle$ in (3) is important, because, as will be seen from what follows, just such terms enter into the damping of the mean field. These terms are linear in the fluctuations. The equation for the fluctuating quantities,

[1)]We considered the case of high-frequency fields, when the motion of the ions can be neglected.

*$[\mathbf{vH}] \equiv \mathbf{v}\times\mathbf{H}$.

if we neglect the nonlinearity (this is valid with accuracy $\langle(\delta N/N_0)^2\rangle \ll 1$), takes the form

$$\frac{\partial \mathbf{v}'}{\partial t}+\frac{e}{m}\mathbf{E}'+\frac{v_T^2}{N_0}\operatorname{grad}N'=\frac{v_T^2}{N_0^2}\operatorname{grad}(\delta N\langle \tilde N\rangle),$$
$$\operatorname{rot}\mathbf{E}'+\frac{1}{c}\frac{\partial \mathbf{H}'}{\partial t}=0,$$
$$\operatorname{rot}\mathbf{H}'-\frac{1}{c}\frac{\partial \mathbf{E}'}{\partial t}-\frac{4\pi e}{c}N_0\mathbf{v}'=\frac{4\pi e}{c}\delta N\langle \mathbf{v}\rangle, \tag{4}$$
$$\frac{\partial N}{\partial t}+N_0\operatorname{div}\mathbf{v}'=-\operatorname{div}(\langle \mathbf{v}\rangle\delta N).$$

As is usually done in the corresponding linear problem,[8,9] small terms of the form $N'\delta N - \langle|N'\delta N|\rangle$ are also not taken into account. These would lead to the appearance of quantities of the order of $\sim\langle(\delta N/N_0)^3\rangle$, $\langle(\delta N/N_0)^4\rangle$, etc, on the left side of the equations for the mean field. Thus, the approximating system of equations (3) and (4), which allows us to obtain closed nonlinear equations for the averaged quantities, is valid under the conditions in which corrections $\sim\langle(\delta N/N_0)^2\rangle$ are taken into account in the linear terms and are discarded in the nonlinear ones.

In the following, we shall consider the interaction of plane plasma and transverse waves propagating along the x axis (with dependence $e^{i(\omega t - kx)}$). Therefore, for the problems of interest to us, we need the value only of the Fourier components of the terms in (3) associated with fluctuations of N:

$$\mathbf{v}_1(\omega,k)=\frac{v_T^2}{N_0^2}\int \operatorname{grad}\langle(\delta N(x)N')\rangle e^{-i(\omega t-kx)}\,dt\,dx,$$
$$\nu_2(\omega,k)=\int \operatorname{div}\langle\delta N(x)v'\rangle e^{-i(\omega t-kx)}\,dt\,dx, \tag{5}$$
$$\mathbf{v}_3(\omega,k)=\frac{4\pi e}{c}\int\langle\delta N(x)\mathbf{v}'\rangle e^{-i(\omega t-kx)}\,dt\,dx.$$

Substituting the solutions of the set of equations (4) and (5), and assuming that the correlation function of the fluctuations of N has the form[2)]

$$\langle\delta N(x)\delta N(x+\xi)\rangle=\langle(\delta N)^2\rangle\exp(-|\xi|/l_0),$$

we get

$$\nu_1(\omega,k_{\rm pl})=-i\omega\langle v_x(\omega,k_{\rm pl})\rangle\left\langle\left(\frac{\delta N}{N_0}\right)^2\right\rangle\frac{(k_{\rm pl}l_0)^2}{1+2ik_{\rm pl}l_0}$$
$$+ik_{\rm pl}v_T^2\left\langle\left(\frac{\delta N}{N_0}\right)^2\right\rangle\frac{\langle N(\omega,k_{\rm pl})\rangle}{N_0}\left[1+ik_{\rm pl}l_0\frac{1+ik_{\rm pl}l_0}{1+2ik_{\rm pl}l_0}\right],$$
$$\nu_2(\omega,k_{\rm pl})=\frac{1}{N_0}\left\langle\left(\frac{\delta N}{N_0}\right)^2\right\rangle\langle v_x(\omega,k_{\rm pl})\rangle\left[ik_{\rm pl}-\frac{\omega^2 l_0}{2v_T^2}\frac{1+ik_{\rm pl}l_0}{1+2ik_{\rm pl}l_0}\right]$$
$$+i\omega\left\langle\left(\frac{\delta N}{N_0}\right)^2\right\rangle\langle N(\omega,k_{\rm pl})\rangle\frac{(k_{\rm pl}l_0)^2}{1+2ik_{\rm pl}l_0},$$
$$\nu_3(\omega,k_{\rm em})=i\frac{2\pi e}{cN_0}\left\langle\left(\frac{\delta N}{N_0}\right)^2\right\rangle\frac{\omega_0^2}{\omega^2-\omega_0^2}\cdot$$
$$\times\langle v_y(\omega,k_{\rm em})\rangle\frac{k_{\rm em}l_0(1+ik_{\rm em}l_0)}{1+2ik_{\rm em}l_0}, \tag{6}$$

where

$$k_{\rm em}=\frac{\sqrt{\omega^2-\omega_0^2}}{c},\quad k_{\rm pl}=\frac{\sqrt{\omega^2-\omega_0^2}}{v_T},\quad \omega_0^2=\frac{4\pi e^2N_0}{m}.$$

The expressions (6) are valid for arbitrary relations among $k_{\rm pl}$, $k_{\rm em}$, and l_0 and can be materially simplified under conditions of small-scale ($k_{\rm pl}l_0$, $k_{\rm em}l_0 \ll 1$) and large-scale ($k_{\rm pl}l_0$, $k_{\rm em}l_0 \gg 1$) inhomogeneities.

[2)]The choice of such a correlation function is not a matter of principle, and is determined only by the simplicity of the mathematical calculations. For another form of $\langle\delta N(x)\delta N(x+\xi)\rangle$, the difference will be only in numerical factors of the order of unity.

2. CONTRACTED EQUATIONS FOR THE AMPLITUDES AND PHASES OF THE INTERACTING WAVES AND THEIR INVESTIGATION

As was pointed out above, the nonlinearities in (3) are assumed to be small. Then, for the description of the interaction of two transverse (of one polarization) and plasma waves, which satisfy the condition of synchronism[1]

$$k_{\rm pl}+k_2=k_1,\quad \omega_{\rm pl}+\omega_2=\omega_1 \tag{7}$$

(the indices 1 and 2 refer to the electromagnetic waves) we can use the asymptotic method.[4,10] However, the solution of the resultant set of equations for the slowly changing amplitudes A, B_1, B_2 and phases $\varphi_{\rm pe}$, φ_1, φ_2 of the plasma and two transverse waves, respectively, is difficult in the general case.

Therefore, we consider the following problem: a plane electromagnetic pump wave with frequency ω_1 and a signal wave with frequency ω_2, which can be found from the conditions of synchronism, are incident from the vacuum on a semi-infinite plasma.[3)] We shall investigate the distribution of the electromagnetic field and the field of the excited longitudinal wave in the plasma. In the stationary case ($\partial/\partial t = 0$), the set of equations takes the form

$$\frac{dA}{dx}=\frac{\sigma_{\rm pl}}{v_{\rm pl}}B_1B_2\cos\Phi-\frac{\nu_{\rm pl}}{v_{\rm pl}}A,$$
$$\frac{dB_1}{dx}=-\frac{\sigma_1}{v_1}AB_2\cos\Phi-\frac{\nu^{(1)}}{v_1}B_1,$$
$$\frac{dB_2}{dx}=\frac{\sigma_2}{v_2}AB_1\cos\Phi-\frac{\nu^{(2)}}{v_2}B_2,$$
$$\frac{d\Phi}{dx}=\sin\Phi\left(\frac{\sigma_2}{v_2}\frac{AB_1}{B_2}-\frac{\sigma_1}{v_1}\frac{AB_2}{B_1}+\frac{\sigma_{\rm pl}}{v_{\rm pl}}\frac{B_1B_2}{A}\right) \tag{8}$$
$$+\frac{\delta_{\rm em}^{(2)}}{v_2}-\frac{\delta_{\rm em}^{(1)}}{v_1}-\frac{\delta_{\rm pl}}{v_{\rm pl}}$$

Here

$$\sigma_{\rm pl}=\frac{e\omega_0^2k_{\rm pl}}{2m\omega_{\rm pl}\omega_1\omega_2},\quad \sigma_{1,2}=\frac{ek_{1,2}}{2m\omega_{1,2}},\quad \Phi=\varphi_2-\varphi_1-\varphi_{\rm pl},$$

$v_{\rm pl}$ and $v_{1,2}$ are the group velocities of the plasma and electromagnetic waves, $\nu_{\rm pl}$, $\nu_{\rm em}^{(1,2)}$, and $\delta_{\rm em}^2$ denote damping and phase shift, respectively, due to the presence of fluctuations of N in the medium. The specific form of these can be obtained from formulas for ν_1, ν_2, and ν_3; it will be given below in the various limiting cases.

For simplicity then, we shall assume that the inequality

$$\omega_{1,2}\gg\omega_0.$$

is satisfied. Then

$$\omega_{\rm pl}\approx\omega_0+v_T^2\omega_0/2c^2,\quad v_1\approx v_2=v\approx c,\quad \sigma_1\approx\sigma_2=\sigma,$$

$$\nu^{(1)}\approx\nu^{(2)}=\nu\approx\begin{cases}\frac{1}{4}\left\langle\left(\frac{\delta N}{N_0}\right)^2\right\rangle\frac{\omega_0^4}{\omega^3}k_{\rm em}l_0 \text{ for } k_{\rm em}l_0\ll 1\\ \frac{1}{8}\left\langle\left(\frac{\delta N}{N_0}\right)^2\right\rangle\frac{\omega_0^4}{\omega^3}k_{\rm em}l_0 \text{ for } k_{\rm em}l_0\gg 1.\end{cases} \tag{9}$$

[3)]The equations mentioned above can be analyzed also for stimulated Mandel'shtam-Brillouin scattering (SMBS), when the incoming signal wave is amplified. Of course, such a process of SMBS also takes place relative to thermal noise. However, we shall neglect it, inasmuch as conditions can always be chosen for which the level of thermal fluctuations in the boundary layer of the plasma is sufficiently low.

Taking it into account that[4)] $v_{pl} = k_{pl}v_T^2/\omega_0 \ll c$, we have

$$\frac{\sigma_{pl}}{v_{pl}} \Big/ \frac{\sigma}{v} \approx \frac{\omega_0}{\omega}\frac{c^2}{v_T^2}, \qquad (10)$$

i.e., for $c^2/v_T^2 \gg \omega/\omega_0$ this ratio is large. Consequently, the amplitude of the plasma wave goes over rapidly into the "equilibrium" state ($dA/dx = 0$), which is determined by the relation

$$A = \frac{B_1 B_2 \sigma_{pl}}{\nu_{pl}} \cos \Phi. \qquad (11)$$

Substituting (11) in (8) with account of (9), we get

$$\frac{dB_1}{dx} = -\frac{\sigma}{v} B_1 B_2^2 \frac{\sigma_{pl}}{v_{pl}} \cos^2 \Phi - \frac{\nu}{v} B_1,$$
$$\frac{dB_2}{dx} = \frac{\sigma\sigma_{pl}}{v v_{pl}} B_2 B_1^2 \cos^2 \Phi - \frac{\nu}{v} B_2, \qquad (12)$$
$$\frac{d\Phi}{dx} = -\frac{\sigma\sigma_{pl}}{2v\nu_{pl}} (B_1^2 - B_2^2) \sin 2\Phi - \frac{\nu_{pl}}{v_{pl}} \operatorname{tg} \Phi - \frac{\delta_{pl}}{v_{pl}}.$$

It is necessary to investigate this system in the phase space B_1, B_2, and Φ, which presents significant difficulties. Therefore, we limit ourselves to the approximate analysis of (12). In the equilibrium state, the phase Φ "tracks" the amplitudes of the interacting waves ($d\Phi/dx = 0$). Here Φ is found from the equation

$$\frac{\operatorname{tg} \Phi}{1 + \operatorname{tg}^2 \Phi} \left[\frac{\sigma\sigma_{pl}}{\nu_{pl}^2} \frac{v_{pl}}{v} (B_2^2 - B_1^2) \right] - \operatorname{tg} \Phi = \frac{\delta_{pl}}{\nu_{pl}}. \qquad (13)$$

As analysis shows, for $\delta_{pl}/\nu_{pl} = 0$ corresponds to stable equilibrium. The presence of fluctuations $\delta N(x)$ shifts this state by an amount $\sim \delta_{pl}/\nu_{pl} \ll 1$, which we shall neglect. We estimate the conditions for which this is possible.

In the case of small-scale inhomogeneities ($k_{pl}l_0 \ll 1$) the values of δ_{pl} and ν_{pl} are equal to[5)]

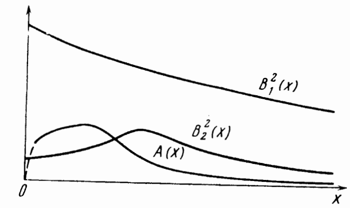
$$\delta_{pl} \approx \langle (\delta N / N_0)^2 \rangle \omega_{pl}, \qquad (14)$$

$$\nu_{pl} \approx {}^1\!/_4 \langle (\delta N / N_0)^2 \rangle (\omega_0 / k_{pl} v_T)^2 \omega_0 (k_{pl} l_0). \qquad (15)$$

Their ratio

$$\frac{\delta_{pl}}{\nu_{pl}} \approx \frac{4}{k_{pl} l_0} \left(\frac{k_{pl} v_T}{\omega_0} \right)^2,$$

as is easy to see, can be made rather small. We note that, in the opposite limiting case ($k_{pl}l_0 \gg 1$) the value of δ_{pl}/ν_{pl} is always small. Thus, the phase correction, associated with δ_{pl}/ν_{pl}, can be neglected with a great degree of accuracy. In Eqs. (12) for the amplitudes,



this can even more readily be neglected, since it enters into the right hand side of (12) in quadratic fashion.

Taking this into account, we can write the solutions of (12) in the form[12]

$$\frac{B_2^2(x)}{B_2^2(0)} = \frac{(1+\beta) e^{-\nu x/v}}{\beta + \exp[-\gamma(1+\beta)(1-e^{-\nu x/v})]}, \qquad (16)$$
$$B_1^2(x) + B_2^2(x) = [B_1^2(0) + B_2^2(0)] e^{-\nu x/v},$$

where $B_1(0)$ and $B_2(0)$ are the values of the amplitudes of the fields on the boundary of the plasma $x = 0$;

$$\beta = \frac{B_2^2(0)}{B_1^2(0)}, \quad \gamma = \frac{\sigma\sigma_{pl}}{\nu\nu_{pl}} B_1^2(0).$$

The qualitative form of the distribution of the field amplitudes is shown in the drawing, from which it is evident that for large x the amplitudes of all the fields are damped, in contrast with the case of a transparent one-dimensional medium, when reverse pumping of the interacting waves exists.[1]

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[4)]In the quasihydrodynamic consideration, Landau damping is not taken into account, which is valid for $k_{pl}v_T/\omega_0 \ll 1$ [11].

[5)]In the limiting case of large-scale inhomogeneities $k_{pl}l_0 \gg 1$ the factor 1/2 appears in Eq. (14) and the factor 1/4 is replaced by 3/8 in (15).