FLUCTUATIONS OF PHOTON FLUX IN A MEDIUM WITH RANDOM INHOMOGENEITIES OF THE DIELECTRIC CONSTANT

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Propagation of light in a medium with a fluctuating dielectric constant is considered from the standpoint of quantum optics. An expression is derived for the rms fluctuations of the number of photons passing through a surface Σ during a time T. For a coherent monochromatic source, this quantity consists of two terms, of which one is identical with the classical expression and the other is due to quantum fluctuations, but also depends on the fluctuations of the dielectric constant. It is shown that for small values of Σ and T the probability distribution for the number of coherent radiation photons passing through the medium without fluctuation is not Poissonian.

1. FORMULATION OF PROBLEM

FLUCTUATIONS of the intensity and of other parameters of light propagating in a medium with random inhomogeneities (for example, in a turbulent atmosphere) have recently attracted much attention in connection with the use of lasers. In many problems of this group, the classical analysis is perfectly satisfactory (see, for example, the review of the theoretical papers in this field in [1]). In some cases, however, quantum effects may turn out to be significant. Their role increases in the case of low radiation intensities, when the receivers resolve individual photons. The probability distribution for the number of photoelectrons was considered by Glauber[2] and by Klauder and Sudarshan[3] for a homogeneous medium. An attempt was made in [4] to find the analogous distribution for a medium with random inhomogeneities, by starting from semiclassical representations. The result obtained there, which will subsequently be shown to be not completely accurate, will be discussed in the concluding part of this paper.

A consistent study of photon statistics in a medium with random inhomogeneities should be based on the equations of quantum electrodynamics with allowance for the interaction between the field and the medium. In such a formulation, the problem entails very great difficulties. We therefore consider the much simpler problem, in which the medium can be described phenomenologically with the aid of a dielectric constant. In a medium with random inhomogeneities, the function $\epsilon(\mathbf{r}, t)$ is a random function of the coordinates and of the time.

The second simplification which we shall use is connected with the assumption that the wavelength is short compared with the inhomogeneity scales. In this case there is hardly any depolarization of the propagating light, and we can consider only a single nonvanishing component of the electric field.

Finally, the third essential limitation is the assumption that the temporal variations of $\epsilon(\mathbf{r}, t)$ are slow compared with the period of the optical oscillations. In this case one can use a quasistationary approximation in which $\epsilon = \epsilon(\mathbf{r})$ and the dependence of ϵ on t is taken into account parametrically.

Under these assumptions, the propagation of the light is described by the equation

$$\frac{\varepsilon(\mathbf{r})}{c^2}\frac{\partial^2 E(\mathbf{r},t)}{\partial t^2} - \Delta E(\mathbf{r},t) = 0, \tag{1}$$

which follows from Maxwell's equations; E is the only nonvanishing component of the electric field.

In the succeeding sections of the paper, Eq. (1) will be quantized and the statistical characteristics of the photon-number flux $\mathbf{J}(\mathbf{r}, t)$ will be determined. We do not consider here the interaction between the radiation and the photodetector, assuming the latter to be ideal, i.e., responding to each incident photon. Allowance for nonideality of the photodetector can be carried out in part by introducing its quantum efficiency directly in the final results.

2. CLASSICAL SOLUTION AND RESOLUTION OF FIELD INTO MODES

Equation (1) can be obtained from the action

$$S = \frac{1}{2}\int dt \int d^3r \left\{\varepsilon(\mathbf{r}) \left(\frac{\partial E}{\partial t}\right)^2 - c^2(\nabla E)^2\right\} \tag{2}$$

by variation with respect to $E(\mathbf{r}, t)$. We take $E(\mathbf{r}, t) = q(\mathbf{r}, t)$ to be the generalized coordinate. Then the momentum $p(\mathbf{r}, t)$ canonically conjugate to q is

$$p(\mathbf{r},t) = \frac{\delta S}{\delta q(\mathbf{r},t)} = \varepsilon(\mathbf{r})\dot{E}(\mathbf{r},t) \tag{3}$$

and the Lagrange-function density is

$$\mathscr{L} = {}^1/_2\{\varepsilon\dot{E}^2 - c^2(\nabla E)^2\}$$

We then obtain for the Hamiltonian density $\mathscr{H}(\mathbf{r}, t)$

$$\mathscr{H}(\mathbf{r},t) = p\dot{q} - \mathscr{L} = {}^1/_2[\varepsilon(\mathbf{r})\dot{E}^2(\mathbf{r},t) + c^2(\nabla E)^2], \tag{4}$$

$$H = \int \mathscr{H}(\mathbf{r},t)\,d^3r. \tag{4a}$$

We seek a real solution of Eq. (1) in the form

$$E(\mathbf{r},t) = \int_0^\infty [E(\mathbf{r},\omega)\exp(-i\omega t) + E^*(\mathbf{r},\omega)\exp(i\omega t)]\,d\omega. \tag{5}$$

We then obtain for $E(\mathbf{r}, \omega)$ the equation

$$\Delta E(\mathbf{r},\omega) + \frac{\omega^2\varepsilon(\mathbf{r})}{c^2}E(\mathbf{r},\omega) = 0. \tag{6}$$

Equation (6) has solutions that depend on the vector parameter **k**:

$$\Delta u(\mathbf{k},\mathbf{r})+\frac{\omega^2\varepsilon(\mathbf{r})}{c^2}u(\mathbf{k},\mathbf{r})=0. \qquad (7)$$

In the case ϵ = const = $\bar{\epsilon}$, the solutions (7) can be chosen in the form of plane waves

$$u_0(\mathbf{k},\mathbf{r})=(8\pi^3\bar{\varepsilon})^{-1/2}\exp(i\mathbf{kr}) \qquad (8)$$

and in order for (8) to be a solution it is necessary to satisfy the relation $\omega=\omega(\mathbf{k})$, where

$$\omega(\mathbf{k})=c(\bar{\varepsilon})^{-1/2}|\mathbf{k}|. \qquad (9)$$

In the case of arbitrary $\epsilon(\mathbf{r})$ we can construct solutions that generalize (8). We put $\epsilon(\mathbf{r})=\bar{\epsilon}(1+\tilde{\epsilon}(\mathbf{r}))$, where $\bar{\epsilon}$ is the mean value of the random function $\epsilon(\mathbf{r})$($\bar{\epsilon}$ = const) and $\tilde{\epsilon}=(\epsilon-\bar{\epsilon})/\bar{\epsilon}$ are the relative fluctuations. Then Eq. (7) takes the form

$$\Delta u(\mathbf{k},\mathbf{r})+\frac{\omega^2\bar{\varepsilon}}{c^2}u(\mathbf{k},\mathbf{r})=-\frac{\omega^2\bar{\varepsilon}}{c^2}\tilde{\varepsilon}(\mathbf{r})u(\mathbf{k},\mathbf{r}).$$

Putting $\omega^2\bar{\epsilon}/c^2=k^2$ in accordance with (9) i.e., leaving the connection between ω and **k** the same as for a homogeneous medium, and assuming

$$G(k,\rho)=-\exp(ik\rho)/4\pi\rho$$

we can transform the last equation into the integral equation

$$u(\mathbf{k},\mathbf{r})=u_0(\mathbf{k},\mathbf{r})-k^2\int G(k,\mathbf{r}-\mathbf{r}')\tilde{\varepsilon}(\mathbf{r}')u(\mathbf{k},\mathbf{r}')d^3r' \qquad (10)$$

where u_0 is given by (8). Equation (10) defines uniquely the function $u(\mathbf{k},\mathbf{r})$, with the parameter **k** introduced via the function $u_0(\mathbf{k},\mathbf{r})$ and $\omega=\omega(\mathbf{k})$ determined by relation (9), where $\bar{\epsilon}$ is the mean value of $\epsilon(\mathbf{r})$. Multiplying (7) by $u^*(\mathbf{k}',\mathbf{r})$, subtracting from the obtained equation its complex conjugate with **k** and **k**′ interchanged, and integrating, we obtain an orthogonality condition with weight $\epsilon(\mathbf{r})$,

$$\int\varepsilon(\mathbf{r})u(\mathbf{k},\mathbf{r})u^*(\mathbf{k}',\mathbf{r})d^3r=0, \text{ if } \omega^2\neq\omega'^2$$

i.e., with allowance for (9), if $|\mathbf{k}|\neq|\mathbf{k}'|$. If $|\mathbf{k}|=|\mathbf{k}'|$, there exist solutions that differ in two parameters; they can also be orthogonalized, and by suitable normalization it is possible to satisfy the condition

$$\int\varepsilon(\mathbf{r})u(\mathbf{k},\mathbf{r})u^*(\mathbf{k}',\mathbf{r})d^3r=\delta(\mathbf{k}-\mathbf{k}'). \qquad (11)$$

The functions $u(\mathbf{k},\mathbf{r})$, defined by Eq. (10), satisfy the condition (11). This can be directly verified accurate to $\tilde{\epsilon}^2$ by substituting in (11) the iteration series for Eq. (10). The first term of the series u_0, when multiplied by $\bar{\epsilon}$, then yields $\delta(\mathbf{k}-\mathbf{k}')$, and all the terms linear in $\tilde{\epsilon}$ (which result from either $\epsilon(\mathbf{r})$ or $u(\mathbf{k},\mathbf{r})u^*(\mathbf{k}',\mathbf{r})$) add up to zero.

The subsequent calculations give rise also to integrals of $u(\mathbf{k},\mathbf{r})u(\mathbf{k}',\mathbf{r})$. On the basis of (7) it is easy to show that, just as for uu^* at $\omega\neq\omega'$, i.e., at $|\mathbf{k}|\neq|\mathbf{k}'|$, we have orthogonality:

$$\int\varepsilon(\mathbf{r})u(\mathbf{k},\mathbf{r})u(\mathbf{k}',\mathbf{r})d^3r=0 \text{ for } |\mathbf{k}'|\neq|\mathbf{k}|. \qquad (11a)$$

In the case ϵ = const, the functions $u_0(\mathbf{k},\mathbf{r})$ with different **k** form a complete system of functions in the space of the solutions of (6). We shall assume the same also with respect to the functions $u(\mathbf{k},\mathbf{r})$. In this case

$$f(\mathbf{r})=\int u(\mathbf{k},\mathbf{r})\check{f}(\mathbf{k})d^3k. \qquad (12)$$

Multiplying by $\epsilon(\mathbf{r})u^*(\mathbf{k}',\mathbf{r})$ and integrating with respect to **r**, we obtain, with allowance for (11)

$$\check{f}(\mathbf{k})=\int f(\mathbf{r}')u^*(\mathbf{k},\mathbf{r}')\varepsilon(\mathbf{r}')d^3\mathbf{r}'.$$

Substituting this expression for $\check{\mathbf{f}}(\mathbf{k})$ in (12), we get

$$f(\mathbf{r})=\int d^3r'\,\varepsilon(\mathbf{r}')f(\mathbf{r}')\int u(\mathbf{k},\mathbf{r})u^*(\mathbf{k},\mathbf{r}')d^3k,$$

whence

$$\varepsilon(\mathbf{r}')\int u(\mathbf{k},\mathbf{r})u^*(\mathbf{k},\mathbf{r}')d^3k=\delta(\mathbf{r}-\mathbf{r}'). \qquad (13)$$

It can be directly verified that condition (13) is actually satisfied accurate to terms of order $\tilde{\epsilon}^2$.

The solution of Eq. (6) can be expanded in functions of the modes $u(\mathbf{k},\mathbf{r})$ in an integral of the type (12). Since (6) pertains to a fixed frequency ω, the expansion of $E(\mathbf{r},\omega)$ in terms of $u(\mathbf{k},\mathbf{r})$ should contain only those **k** for which $\omega(\mathbf{k})=\omega$:

$$E(\mathbf{r},\omega)=\int A(\mathbf{k})\delta(\omega(\mathbf{k})-\omega)u(\mathbf{k},\mathbf{r})d^3k. \qquad (14)$$

Substituting this expression in (5) and integrating with respect to ω, we obtain

$$E(\mathbf{r},t)=\int d^3k\{A(\mathbf{k})u(\mathbf{k},\mathbf{r})\exp[-i\omega(\mathbf{k})t]+A^*(\mathbf{k})u^*(\mathbf{k},\mathbf{r})\exp[i\omega(\mathbf{k})t]\}. \qquad (15)$$

The integration with respect to **k** extends here already over all of space. The expansion (15) replaces the usual plane-wave expansion that appears in the case ϵ = const.

3. QUANTIZATION

The canonically-conjugate variables $E(\mathbf{r},t)$ and $\epsilon(\mathbf{r})\dot{E}(\mathbf{r},t)$ are replaced in the Heisenberg representation by Hermitian operators with commutation relations

$$[E(\mathbf{r},t),\varepsilon(\mathbf{r}')\dot{E}(\mathbf{r}',t)]=i\hbar\delta(\mathbf{r}-\mathbf{r}'). \qquad (16)$$

The equations of motion are

$$\frac{\partial E(\mathbf{r},t)}{\partial t}=\frac{1}{i\hbar}[E(\mathbf{r},t),H],\quad \frac{\partial\varepsilon(\mathbf{r})\dot{E}(\mathbf{r},t)}{\partial t}=\frac{1}{i\hbar}[\varepsilon(\mathbf{r})\dot{E}(\mathbf{r},t),H].$$

Using (4), (4a), and (16) we obtain, after calculating the commutators

$$\frac{\partial E(\mathbf{r},t)}{\partial t}=\dot{E}(\mathbf{r},t),\quad \varepsilon(\mathbf{r})\frac{\partial\dot{E}(\mathbf{r},t)}{\partial t}=c^2\Delta E(\mathbf{r},t). \qquad (17)$$

Both the commutation relations (16) and the equations of motion (17) can be satisfied by seeking $E(\mathbf{r},t)$ in the form of an expansion analogous to (15):

$$E(\mathbf{r},t)=\int d^3k\sqrt{\frac{\hbar}{2\omega(\mathbf{k})}}[u(\mathbf{k},\mathbf{r})a(\mathbf{k},t)+u^*(\mathbf{k},\mathbf{r})a^+(\mathbf{k},t)]. \qquad (18)$$

The annihilation operators for the photons of the **k**-th mode $a(\mathbf{k},t)$, and the corresponding Hermitian-adjoint creation operators $a^+(\mathbf{k},t)$, should satisfy the conditions

$$\dot{a}(\mathbf{k},t)=\frac{1}{i\hbar}[a,H]=-i\omega(\mathbf{k})a(\mathbf{k},t),\quad \dot{a}^+(\mathbf{k},t)=i\omega(\mathbf{k})a^+(\mathbf{k},t) \qquad (19)$$

and

$$[a(\mathbf{k},t),a^+(\mathbf{k}',t)]=\delta(\mathbf{k}-\mathbf{k}'),$$
$$[a(\mathbf{k},t),a(\mathbf{k}',t)]=[a^+(\mathbf{k},t),a^+(\mathbf{k}',t)]=0. \qquad (20)$$

It follows from (19) that

$$\dot{E}(\mathbf{r},t) = -i\int d^3k\sqrt{\frac{\hbar\omega(\mathbf{k})}{2}}[u(\mathbf{k},\mathbf{r})a(\mathbf{k},t) - u^*(\mathbf{k},\mathbf{r})a^+(\mathbf{k},t)]. \quad (21)$$

Substituting (18) and (21) in (16), we verify that this equation is satisfied by virtue of the commutation relations (20) and relation (13).

Substituting (18) and (21) in (4) and (4a) and integrating with respect to **r** (it is necessary here to use Eqs. (11), (11a), and (7)), we can obtain the usual expression

$$H = \frac{1}{2}\int d^3k\hbar\omega(\mathbf{k})\,[a^+(\mathbf{k},t)a(\mathbf{k},t) + a(\mathbf{k},t)a^+(\mathbf{k},t)]. \quad (22)$$

Using this representation, we easily verify that relation (19) is indeed satisfied.

Calculating the quantity $\dot{\mathscr{H}}(\mathbf{r}, t) = (i\hbar)^{-1}[\mathscr{H}, H]$, we can obtain the relation

$$\dot{\mathscr{H}}(\mathbf{r}, t) + \operatorname{div}\Pi(\mathbf{r}, t) = 0$$

where the energy flux density operator Π is given by

$$\Pi(\mathbf{r}, t) = -{}^1/_2c^2\{\dot{E}(\mathbf{r}, t)\nabla E(\mathbf{r}, t) + \nabla E(\mathbf{r}, t)\dot{E}(\mathbf{r}, t)\}. \quad (23)$$

The representation of Π in terms of the operators a and a^+ is quite cumbersome and will not be given here.

We consider the operators

$$a(\mathbf{r},t) = \sqrt{\varepsilon(\mathbf{r})}\int d^3k u(\mathbf{k},\mathbf{r})a(\mathbf{k},t),$$
$$a^+(\mathbf{r},t) = \sqrt{\varepsilon(\mathbf{r})}\int d^3k u^*(\mathbf{k},\mathbf{r})a^+(\mathbf{k},t). \quad (24)$$

By virtue of (20) and (13), they satisfy the commutation relations

$$[a(\mathbf{r}, t), a^+(\mathbf{r}', t)] = \delta(\mathbf{r} - \mathbf{r}'),$$
$$[a(\mathbf{r}, t), a(\mathbf{r}', t)] = [a^+(\mathbf{r}, t), a^+(\mathbf{r}', t)] = 0, \quad (25)$$

making it possible to interpret $a(\mathbf{r}, t)$ and $a^+(\mathbf{r}, t)$ as the operators for the annihilation and creation of a photon at the point (r, t). We introduce the photon-number density operator

$$n(\mathbf{r},t) = a^+(\mathbf{r},t)a(\mathbf{r},t) = \varepsilon(\mathbf{r})\iint d^3k\,d^3k'u(\mathbf{k},\mathbf{r})u^*(\mathbf{k}',\mathbf{r})a^+(\mathbf{k}'\ t)a(\mathbf{k},t). \quad (26)$$

Integrating (26) with respect to **r** and using (11), we obtain the photon-number operator

$$N = \int d^3r n(\mathbf{r},t) = \int d^3k a^+(\mathbf{k},t)a(\mathbf{k},t). \quad (27)$$

We now construct the photon-flux density operator. To this end we determine $\dot{n}(\mathbf{r}, t)$. Using the readily proved relation

$$\frac{1}{i\hbar}[a^+(\mathbf{k}_1,t)a(\mathbf{k}_2,t), H] = i[\omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)]a^+(\mathbf{k}_1,t)a(\mathbf{k}_2,t)$$

we obtain on the basis of (26)

$$\dot{n}(\mathbf{r},t) = i\varepsilon(\mathbf{r})\int d^3k\int d^3k'u(\mathbf{k},\mathbf{r})u^*(\mathbf{k}',\mathbf{r})\,[\omega(\mathbf{k}') - \omega(\mathbf{k})]a^+(\mathbf{k}',t)a(\mathbf{k},t) \quad (28)$$

Using (7), we can readily prove the relation

$$\varepsilon(r)\,[\omega(\mathbf{k}) - \omega(\mathbf{k}')]u(\mathbf{k}, \mathbf{r})u^*(\mathbf{k}', \mathbf{r})$$
$$= \operatorname{div}\{c^2[u(\mathbf{k}, \mathbf{r})\nabla u^*(\mathbf{k}', \mathbf{r}) - u^*(\mathbf{k}', \mathbf{r})\nabla u(\mathbf{k}, \mathbf{r})]/(\omega(\mathbf{k}) + \omega(\mathbf{k}'))\}$$

with the aid of which (28) can be rewritten in the form of the photon-number conservation law:

$$\dot{n}(\mathbf{r}, t) + \operatorname{div}\mathbf{J}(\mathbf{r}, t) = 0. \quad (29)$$

Here $\mathbf{J}(\mathbf{r}, t)$ is the photon flux density operator:

$$\mathbf{J}(\mathbf{r}, t) = 2c^2\int d^3k_1\int d^3k_2\frac{\mathbf{m}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{r})}{\omega(k_1) + \omega(\mathbf{k}_2)}a^+(\mathbf{k}_2, t)a(\mathbf{k}_1, t), \quad (30)$$

$$\mathbf{m}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{r}) = \frac{1}{2i}[u^*(\mathbf{k}_2, \mathbf{r})\nabla u(\mathbf{k}_1, \mathbf{r}) - u(\mathbf{k}_1, \mathbf{r})\nabla u^*(\mathbf{k}_2, \mathbf{r})]. \quad (31)$$

We note that the operator $\mathbf{J}(\mathbf{r}, t)$ is more convenient in some respects than the energy flux density Π, since the latter has a much more complicated appearance than **J**, and, unlike **J**, its application to vacuum states gives rise to divergent expressions. At the same time, the operator **J** determines the number of photons registered by an ideal (with unity quantum efficiency) detector, i.e., it has a direct physical meaning.

4. COHERENT-RADIATION PHOTON FLUX DENSITY AND ITS FLUCTUATIONS

As is well known (see, for example, [2,3]) coherent states are eigenfunctions of the operator $a(\mathbf{k}, t)$:

$$a(\mathbf{k}, t)|z\rangle = z(\mathbf{k})\exp[-i\omega(\mathbf{k})t]|z\rangle,$$
$$\langle z|a^+(\mathbf{k}, t) = z^*(\mathbf{k})\exp[i\omega(\mathbf{k})t]\langle z|. \quad (32)$$

If we represent the operator $E(\mathbf{r}, t)$ in the form of a sum of operators containing only $a(\mathbf{k}, t)$, or $a^+(\mathbf{k}, t)$, i.e., $E(\mathbf{r}, t) = E_-(\mathbf{r}, t) + E_+(\mathbf{r}, t)$,

$$E_-(\mathbf{r},t) = \int d^3k\sqrt{\frac{\hbar}{2\omega(\mathbf{k})}}u(\mathbf{k},\mathbf{r})a(\mathbf{k},t)$$
$$E_+(\mathbf{r},t) = \int d^3k\sqrt{\frac{\hbar}{2\omega(\mathbf{k})}}u^*(\mathbf{k},\mathbf{r})a^+(\mathbf{k},t) \quad (33)$$

with $(E_-)^+ = E_+$, then, using (32), we obtain

$$E_-(\mathbf{r},t)|z\rangle = \int d^3k\sqrt{\frac{\hbar}{2\omega(\mathbf{k})}}u(\mathbf{k},\mathbf{r})z(\mathbf{k})\exp[-i\omega(\mathbf{k})t]|z\rangle.$$

Putting

$$A(\mathbf{k}) = \sqrt{\frac{\hbar}{2\omega(\mathbf{k})}}z(\mathbf{k}), \quad V(\mathbf{r},t) = \int d^3kA(\mathbf{k})u(\mathbf{k},\mathbf{r})\exp[-i\omega(\mathbf{k})t] \quad (34)$$

we get

$$E_-(\mathbf{r}, t)|z\rangle = V(\mathbf{r}, t)|z\rangle. \quad (35)$$

Comparing the expression for V with (15), we get $E(\mathbf{r}, t) = 2\operatorname{Re} V(\mathbf{r}, t)$.

As is well known (see [2,3]), the number of photons in one mode has a Poisson distribution in the coherent state. This is connected with the fact that, on the one hand, the probability of an n-particle state of the **k**-th mode, as follows from the definition of the coherent state, is

$$|\langle n|z\rangle|^2 = \frac{|z(\mathbf{k})|^{2n}}{n!}e^{-|z(\mathbf{k})|^2}$$

and, on the other hand, the n-particle state is an eigenvector of the photon-number operator N. It is difficult, however, to measure the number of photons in a given mode, since the photon coordinates at a specified **k** are indeterminate and the measuring instrument must monitor a considerable region of space. What can be measured is the number of photons passing within a given time T through a given area Σ. This quantity is determined by the integral of $\mathbf{J}(\mathbf{r}, t)$ with respect to t and $\mathbf{r}_\perp$. It is easy to see, however, that the n-photon states of the **k**-th mode

$$|n, \mathbf{k}\rangle = \frac{1}{\sqrt{n!}}[a^+(\mathbf{k}, 0)]^n|0\rangle \tag{36}$$

are not eigenvectors of the operator **J** defined by formula (30), since in (30) the operators a^+ and a have different arguments $\mathbf{k}_2$ and $\mathbf{k}_1$. Therefore the number of photons passing through a given area Σ in a time T does not have a Poisson distribution. It will be shown below that when T and Σ are increased this distribution tends to become a Poisson distribution nevertheless.

Using (32), we obtain the mean value of **J** in the coherent state $|z\rangle$:

$$\langle z|\mathbf{J}(r, t)|z\rangle = 2c^2 \iint d^3k_1\, d^3k_2 \frac{\mathbf{m}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{r})}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)} z^*(\mathbf{k}_2) z(\mathbf{k}_1) \exp[i(\omega(\mathbf{k}_2)-\omega(\mathbf{k}_1))t]. \tag{37}$$

If the coherent state is monochromatic, i.e., if the amplitude defined by (34) is $A(\mathbf{k}) = A_0\delta(\mathbf{k}-\mathbf{k}_0)$, then

$$z(\mathbf{k}) = (2\omega(\mathbf{k}_0)/\hbar)^{1/2} A_0 \delta(\mathbf{k}-\mathbf{k}_0)$$

and we obtain from (37)

$$\langle z(\mathbf{k}_0)|\mathbf{J}(\mathbf{r}, t)|z(\mathbf{k}_0)\rangle = \frac{2c^2|A_0|^2}{\hbar}\mathbf{m}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{r}). \tag{37a}$$

Formula (37a) can be obtained from a semiclassical analysis of the problem. Indeed, in a coherent monochromatic field

$$E(\mathbf{r}, t) = A_0 u(\mathbf{k}_0, \mathbf{r}) \exp[-i\omega(\mathbf{k}_0)t] + A_0^* u^*(\mathbf{k}_0, \mathbf{r}) \exp[i\omega(\mathbf{k}_0)t].$$

We then have for the energy flux density $\Pi = -c^2\dot{E}\nabla E$

$$\Pi = 2\omega(\mathbf{k}_0)c^2|A_0|^2\mathbf{m}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{r}) + \mathbf{A}_1\exp(2i\omega t) + \mathbf{A}_1^*\exp(-2i\omega t).$$

The value of Π averaged over the period contains only the first term. Then the photon flux density can be obtained from Π by dividing by the energy of one photon $\hbar\omega(\mathbf{k}_0)$, which leads to formula (37a).

We now find the quantity $\mathbf{J}(\mathbf{r}, t) \otimes \mathbf{J}(\mathbf{r}', t')$, where ⊗ denotes the tensor product of the vectors. Substituting (30), we get

$$\mathbf{J}(\mathbf{r}, t)\otimes\mathbf{J}(\mathbf{r}', t') = 4c^4 \iiiint d^3k_1\, d^3k_2\, d^3k_1'\, d^3k_2' \cdot \frac{\mathbf{m}(\mathbf{k}_1, k_2, \mathbf{r})\otimes\mathbf{m}(\mathbf{k}_1', \mathbf{k}_2', \mathbf{r}')}{[\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)][\omega(\mathbf{k}_1')+\omega(\mathbf{k}_2')]} a^+(\mathbf{k}_2, t)a(\mathbf{k}_1, t)a^+(\mathbf{k}_2', t')a(\mathbf{k}_1', t').$$

After applying the commutation rules (20) we get

$$a^+(\mathbf{k}_2, t)a(\mathbf{k}_1, t)a^+(\mathbf{k}_2', t')a(\mathbf{k}_1', t') = a^+(\mathbf{k}_2, t)a^+(\mathbf{k}_2', t')a(\mathbf{k}_1, t)a(\mathbf{k}_1', t') + \delta(\mathbf{k}_1 - k_2')\exp[i\omega(\mathbf{k}_1)(t'-t)]a^+(\mathbf{k}_2, t)a(\mathbf{k}_1', t').$$

Calculating now the mean value over the coherent state $|z\rangle$, we readily find that

$$\langle z|\mathbf{J}(r, t)\otimes\mathbf{J}(\mathbf{r}', t')|z\rangle = \langle z|\mathbf{J}(\mathbf{r}, t)|z\rangle\otimes\langle z|\mathbf{J}(\mathbf{r}', t')|z\rangle + 4c^4\iiint d^3k_1\, d^3k_2\, d^3k_1' \frac{\mathbf{m}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{r})\otimes\mathbf{m}(\mathbf{k}_1', \mathbf{k}_1, \mathbf{r}')}{[\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)][\omega(\mathbf{k}_1)+\omega(\mathbf{k}_1')]} \times z^*(\mathbf{k}_2)z(\mathbf{k}_1')\exp\{i[\omega(\mathbf{k}_2)-\omega(\mathbf{k}_1)]t + i[\omega(\mathbf{k}_1)-\omega(\mathbf{k}_1')]t'\}.$$

In the particular case of a monochromatic signal $|z\rangle = |z(\mathbf{k}_0)\rangle$ we have

$$\langle z(\mathbf{k}_0)|\mathbf{J}(\mathbf{r}, t)\otimes\mathbf{J}(\mathbf{r}', t')|z(\mathbf{k}_0)\rangle = \frac{4c^4|A_0|^4}{\hbar^2}\mathbf{m}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{r})\otimes\mathbf{m}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{r}') + \frac{8c^4\omega(\mathbf{k}_0)|A_0|^2}{\hbar}\int d^3\varkappa \frac{\mathbf{m}(\varkappa, \mathbf{k}_0, \mathbf{r})\otimes\mathbf{m}(\mathbf{k}_0, \varkappa, \mathbf{r}')}{[\omega(\mathbf{k}_0)+\omega(\varkappa)]^2}\exp\{i[\omega(\varkappa)-\omega(\mathbf{k}_0)](t'-t)\}. \tag{38a}$$

The second term in (38) or (38a), due to the fact that the operators a^+ and a do not commute, gives the quantum fluctuations of the photon flux.

Let us consider the expression

$$D(\mathbf{r}, t; \mathbf{r}', t') = \overline{\langle z|\mathbf{J}(\mathbf{r}, t)\otimes\mathbf{J}(\mathbf{r}', t')|z\rangle} - \overline{\langle z|\mathbf{J}(\mathbf{r}, t)|z\rangle}\otimes\overline{\langle z|\mathbf{J}(\mathbf{r}', t')|z\rangle},$$

where the superior bar denotes averaging over the fluctuations of ε. The tensor D is the space-time correlation function of the photon flux density. Substituting (37a) and (38a), we obtain for monochromatic coherent fields

$$D = \frac{4c^4|A_0|^4}{\hbar^2}\{\overline{\mathbf{m}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{r})\otimes\mathbf{m}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{r}')} - \overline{\mathbf{m}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{r})}\otimes\overline{\mathbf{m}(\mathbf{k}_0, \mathbf{k}_0, \mathbf{r}')}\} + \frac{8c^4|A_0|^2\omega(\mathbf{k}_0)}{\hbar}\int d^3\varkappa \frac{\overline{\mathbf{m}(\varkappa, \mathbf{k}_0, \mathbf{r})\otimes\mathbf{m}(\mathbf{k}_0, \varkappa, \mathbf{r}')}}{[\omega(\mathbf{k}_0)+\omega(\varkappa)]^2}\exp\{i[\omega(\varkappa)-\omega(\mathbf{k}_0)] \times (t'-t)\} = D^{\mathrm{cl}} + D^{\mathrm{qu}} \tag{39}$$

The first term in (39) is of purely classical origin and can be obtained by semiclassical reasoning analogous to that used above to explain formula (37a). The second term is of purely quantum origin, but it also depends on the fluctuations of ε.

We note that instead of $\mathbf{m}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{r})$ and $\mathbf{m}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{r}')$ we should write $\mathbf{m}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{r}, t)$ and $\mathbf{m}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{r}', t')$, noting explicitly the quasistationary dependence of ε on t. We shall henceforth bear in mind that **m** depends on t.

It should also be noted that even in the case when a single mode $\mathbf{k}_0$ is excited, the fluctuations depend on fields of other frequencies, since the contribution to D^{qu} is determined by the integral over all $\varkappa$.

5. FLUCTUATIONS OF THE NUMBER OF PHOTONS CROSSING AN AREA Σ IN A TIME T

If we put in (39) $\mathbf{r}' = \mathbf{r}$ and $t' = t$, then the second term of D^{qu} turns out to be infinite, i.e., the quantum fluctuations of the photon flux at the point (**r**, t) are infinite. Let us examine the number of photons crossing an area Σ perpendicular to the vector $\mathbf{k}_0$ in a time T. Designating this quantity ν(Σ, T), we obtain the corresponding operator

$$\nu(\Sigma, T; \mathbf{r}, t) = \iint_\Sigma d^2\rho \int_{-T/2}^{T/2} \frac{\mathbf{k}_0}{k_0}\mathbf{J}(\mathbf{r}+\boldsymbol{\rho}, t+\tau)\,d\tau, \tag{40}$$

where the vector $\boldsymbol{\rho}$ is normal to $\mathbf{k}_0$: $\mathbf{k}_0\boldsymbol{\rho} = 0$. If we integrate expressions (37a) and (39) so as to obtain ν(Σ, T), then we obtain the mean values $\overline{\nu(\Sigma, T)}$ and $\overline{(\Delta\nu(\Sigma, T))^2}$. To integrate with respect to $\boldsymbol{\rho}$, however, it is necessary to know the dependence of **m** on **r**.

We represent the field u(**k**, **r**) in the form

$$u(\mathbf{k}, \mathbf{r}) = u_0(\mathbf{k}, \mathbf{r})\exp\{\Phi(\mathbf{k}, \mathbf{r})\} \tag{41}$$

where Re Φ = χ is the logarithm of the ratio of the amplitude u to the amplitude of the incident wave, and Im Φ = φ − **k**·**r** represents the fluctuations of the phase of the wave. Substituting (41) in (31), we obtain

$$\mathbf{m}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{r})=\left[\frac{\mathbf{k}_1+\mathbf{k}_2}{2}+\frac{\nabla\Phi(\mathbf{k}_1,\mathbf{r})-\nabla\Phi^*(\mathbf{k}_2,\mathbf{r})}{2i}\right]u(\mathbf{k}_1,\mathbf{r})u^*(\mathbf{k}_2,\mathbf{r}). \tag{42}$$

Since we are considering large-scale inhomogeneities (compared with λ), the direction of light propagation fluctuates in this case very weakly, i.e., $|\mathbf{k}_1+\mathbf{k}_2| \gg |\nabla\Phi_1-\nabla\Phi_2^*|$, so that the second term in the bracket can be neglected. At the same time, the quantity $u(\mathbf{k}_1,\mathbf{r})u^*(\mathbf{k}_2,\mathbf{r})$ can differ greatly from $u_0(\mathbf{k}_1,\mathbf{r})u_0^*(\mathbf{k}_2,\mathbf{r})$, i.e., the factor $\exp[\Phi(\mathbf{k}_1,\mathbf{r})+\Phi^*(\mathbf{k}_2,\mathbf{r})]$ can differ greatly from unity:

$$\mathbf{m}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{r})\approx {}^1/_2(\mathbf{k}_1+\mathbf{k}_2)u(\mathbf{k}_1,\mathbf{r})u^*(\mathbf{k}_2,\mathbf{r})$$
$$=\frac{\mathbf{k}_1+\mathbf{k}_2}{16\pi^3\bar{\varepsilon}}\exp\{i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}+\Phi(\mathbf{k}_1,\mathbf{r})+\Phi^*(\mathbf{k}_2,\mathbf{r})\}. \tag{42a}$$

In the integration of $\mathbf{m}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{r}+\rho)$ with respect to ρ, we take into account the fact that the most rapidly varying factor in (42a) is $\exp\{i(\mathbf{k}_1-\mathbf{k}_2)\mathbf{r}\}$. If the dimension Σ of the integration region is small compared with the correlation radius L_Φ of the quantity Φ (this correlation radius coincides either with the dimension of the inhomogeneities $l \gg \lambda$, or with the radius of the first Fresnel zone (see [5])) and is large compared with the wavelength, i.e.,

$$\lambda^2 \ll \Sigma \ll L_\Phi^2,$$

then the factor $\exp[\Phi(\mathbf{k}_1,\mathbf{r}+\rho)+\Phi^*(\mathbf{k}_2,\mathbf{r}+\rho)]$ can be regarded as constant when integrating with respect to ρ. Similarly, if τ_Φ is the correlation time of Φ and the conditions

$$[\omega(\mathbf{k}_0)]^{-1}\ll T\ll\tau_\Phi$$

are satisfied, then the quantity $\mathbf{m}$ can be regarded as constant when integrating with respect to the time.

Substituting (42a) in (37a) and integrating with respect to ρ and t, we obtain

$$\iint_\Sigma d^2\rho\int_{-T/2}^{T/2}d\tau\left\langle z(\mathbf{k}_0)\,|\mathbf{J}(\mathbf{r}+\rho,t+\tau)|\,z(\mathbf{k}_0)\right\rangle=\frac{2c^2|A_0|^2}{\hbar}\mathbf{m}(\mathbf{k}_0,\mathbf{k}_0,\mathbf{r})T\Sigma$$
$$=\frac{\mathbf{k}_0c^2|A_0|^2T\Sigma}{4\pi^3\hbar\bar{\varepsilon}}\exp\{2\chi(\mathbf{k}_0,\mathbf{r})\}.$$

Multiplying this equation by $\mathbf{k}_0/k_0$ and averaging over the fluctuations of ϵ, we obtain the average number of photons crossing the area Σ in a time T

$$\overline{\nu(\Sigma,T)}=\frac{k_0c^2|A_0|^2T\Sigma}{4\pi^3\hbar\bar{\varepsilon}}. \tag{43}$$

We have left out from (43) the factor $\overline{\exp\{2\chi\}}$, which in the case of a plane incident wave and statistically homogeneous fluctuations of ϵ is identically equal to unity by virtue of the energy conservation law (see [5]).

We now find the quantity $\overline{(\Delta\nu(\Sigma,T))^2}$, which is expressed in terms of D with the aid of the relation

$$\overline{(\Delta\nu(\Sigma,T))^2}$$
$$=\frac{k_i^0k_j^0}{k_0^2}\iint_\Sigma d^2\rho_1\iint_\Sigma d^2\rho_2\int_{-T/2}^{T/2}d\tau_1\int_{-T/2}^{T/2}d\tau_2 D_{ij}(\mathbf{r}+\rho_1,t+\tau_1;\ \mathbf{r}+\rho_2,t+\tau_2). \tag{44}$$

We find first the contribution from the first (classical) term in (39). Substituting $\mathbf{m}(\mathbf{k}_0,\mathbf{k}_0,\mathbf{r})=(\mathbf{k}_0/8\pi^3\bar{\epsilon})\times\exp\{2\chi(\mathbf{k}_0,\mathbf{r})\}$, we obtain after integration, which in this case ($\Sigma \ll L_\Phi^2$, $T\ll\tau_\Phi$) reduces to multiplication by T and Σ,

$$\overline{(\Delta\nu(\Sigma,T))_{\text{кл}}^2}=\left[\frac{k_0c^2|A_0|^2T\Sigma}{4\pi^3\hbar\bar{\varepsilon}}\right]^2\overline{\{\exp[4\chi(\mathbf{k}_0,\mathbf{r})]-1\}} \tag{45}$$

where again we took into account the equality $\overline{\exp 2\chi}=1$. We now find the contribution from the integral term. Integration with respect to time reduces to calculation of the integral

$$A(\Omega)=\int_{-T/2}^{T/2}d\tau_1\int_{-T/2}^{T/2}d\tau_2\exp[i\Omega(\tau_1-\tau_2)],\qquad \Omega=\omega(\varkappa)-\omega(\mathbf{k}_0)$$

since the time dependence of the factors $\mathbf{m}$ can be neglected. We have

$$A(0)=T^2,\qquad \int_{-\infty}^{\infty}A(\Omega)\,d\Omega=2\pi T.$$

Consequently, the function $A(\Omega)/2\pi T=\delta_T(\Omega)$ has at $T\gg\omega^{-1}$ the properties of a δ function, i.e., we can put

$$A(\Omega)=2\pi T\delta_T(\Omega). \tag{46}$$

We now consider the integral of D_{ij} with respect to ρ_1 and ρ_2. The dependence on the coordinates enters only via the factors $\mathbf{m}$; substituting (42a), we obtain

$$\frac{k_i^0k_j^0}{k_0^2}m_i(\varkappa,\mathbf{k}_0,\mathbf{r}+\rho_1)m_j(\mathbf{k}_0,\varkappa,\mathbf{r}+\rho_2)$$
$$=\frac{(k_0^2+\varkappa\mathbf{k}_0)^2}{(16\pi^3\bar{\varepsilon}k_0)^2}\exp\{i(\varkappa-\mathbf{k}_0)(\rho_1-\rho_2)+2[\chi(\varkappa,\mathbf{r})+\chi(\mathbf{k}_0,\mathbf{r})]\}.$$

In the right-hand side of the last expression we neglected the changes of the functions $\chi(\mathbf{k},\mathbf{r})$ when $\mathbf{r}$ changes by an amount ρ that is small compared with the characteristic scale of the function χ. Integration of the last expression with respect to ρ_1 and ρ_2 gives rise to the function

$$A(\varkappa_\perp)=\iint_\Sigma d^2\rho_1\iint_\Sigma d^2\rho_2\exp\{i\varkappa_\perp(\rho_1-\rho_2)\}$$

where we took into account the fact that $\mathbf{k}_0(\rho_1-\rho_2)=0$ and $\kappa_\perp$ denotes the projection of κ on a plane normal to $\mathbf{k}_0$. The function $A(\kappa_\perp)$ has the following properties:

$$A(0)=\Sigma^2,\qquad \iint_{-\infty}^{\infty}A(\varkappa_\perp)\,d^2\varkappa_\perp=4\pi^2\Sigma.$$

Consequently, if $\Sigma\gg\lambda^2$, it can be assumed that

$$A(\varkappa_\perp)=4\pi^2\Sigma\delta_\Sigma(\varkappa_\perp). \tag{47}$$

Substituting (46) and (47) in the second term of (44), we obtain

$$\overline{(\Delta\nu(\Sigma,T))_{\text{qu}}^2}=(c^4\omega(\mathbf{k}_0)|A_0|^2T\Sigma/4\pi^3\hbar\bar{\varepsilon}^2)$$
$$\times\int d^3\varkappa\frac{(k_0+\varkappa_\parallel)^2}{[\omega(\mathbf{k}_0)+\omega(\varkappa)]^2}\overline{\exp\{2[\chi(\varkappa,\mathbf{r})+\chi(\mathbf{k}_0,\mathbf{r})]\}}\,\delta_T(\omega(\mathbf{k}_0)-\omega(\varkappa))\,\delta_\Sigma(\varkappa_\perp)$$

which leads after integration to the final formula

$$\overline{(\Delta\nu(\Sigma,T))_{\text{qu}}^2}=\frac{k_0c^2|A_0|^2T\Sigma}{4\pi^3\hbar\bar{\varepsilon}}\overline{\exp[4\chi(\mathbf{k}_0,\mathbf{r})]}. \tag{48}$$

Taking (43) into account, the final expressions can be written in the form

$$\overline{(\Delta\nu(\Sigma,T))^2} = \overline{(\Delta\nu(\Sigma,T))_{cl}^2} + \overline{(\Delta\nu(\Sigma,T))_{qu}^2},$$
$$\overline{(\Delta\nu(\Sigma,T))_{cl}^2} = \overline{(\nu(\Sigma,T))^2}\{\overline{\exp[4\chi(\mathbf{k}_0,\mathbf{r})]}-1\}, \quad (49)$$
$$\overline{(\Delta\nu(\Sigma,T))_{qu}^2} = \overline{\nu(\Sigma,T)}\,\overline{\exp[4\chi(\mathbf{k}_0,\mathbf{r})]}.$$

We introduce the relative fluctuations of the number of photons.

$$\beta^2 = \frac{\overline{(\Delta\nu(\Sigma,T))^2}}{(\overline{\nu(\Sigma,T)})^2}, \quad \beta_{cl}^2 = \frac{\overline{(\Delta\nu(\Sigma,T))_{cl}^2}}{(\overline{\nu(\Sigma,T)})^2}, \quad \beta_{qu}^2 = \frac{\overline{(\Delta\nu(\Sigma,T))_{qu}^2}}{(\overline{\nu(\Sigma,T)})^2}.$$

We then obtain from (49)

$$\beta^2 = \beta_{cl}^2 + \beta_{qu}^2, \quad \beta_{cl}^2 = \overline{\exp[4\chi]} - 1, \quad \beta_{qu}^2 = \frac{\overline{\exp[4\chi]}}{\overline{\nu(\Sigma,T)}}.$$

If we eliminate from these relations the quantity $\overline{\exp[4\chi]}$, which we express in terms of β_{cl}^2, then we get

$$\beta_{qu}^2 = \frac{1+\beta_{cl}^2}{\overline{\nu(\Sigma,T)}}, \quad \beta^2 = \beta_{cl}^2 + \frac{1+\beta_{cl}^2}{\overline{\nu(\Sigma,T)}}. \quad (50)$$

We note that in the derivation of (50) we made use of no concrete approximate solution for Φ or χ, using only the representation of the solution in the form (41) and the smallness of the fluctuations of the direction of propagation of the light. Consequently, relations (50) are valid also in the so-called region of strong intensity fluctuations, where $\beta_{cl}^2 \sim 1$ and where it is not convenient to calculate this quantity by the perturbation method.[6]

Notice should be taken of one more important circumstance. If we consider a medium with constant ϵ, then $\beta_{cl}^2 = 0$ and formula (50) becomes

$$\beta^2 = 1/\overline{\nu(\Sigma,T)}.$$

This relation corresponds to a Poisson distribution that results from averaging the density of the photon flux through an area Σ in a time T. If one of the relations $\Sigma \gg \lambda^2$ or $\omega T \gg 1$ is not satisfied, then $\nu(\Sigma, T)$ does not have a Poisson probability distribution.

6. SEMICLASSICAL ANALYSIS

In the already cited paper,[4] the probability distribution of the number of registered photons in a randomly inhomogeneous medium was calculated by using the following reasoning. A coherent light source is characterized by a Poisson distribution

$$p(n) = \nu^n \exp(-\nu)/n! \quad (51)$$

where ν is the average number of photons reaching a photodetector having an area Σ in a time T. In a randomly inhomogeneous medium, the quantity ν fluctuates, and therefore the mean values calculated with the aid of (51) must be additionally averaged over the probability distribution of ν. At fixed ν, the conditional averages over the Poisson distribution are

$$\bar{n}|_\nu = \nu, \quad \overline{n^2}|_\nu = \nu + \nu^2.$$

The unconditional averages are obtained by averaging over ν:

$$\bar{n} = \bar{\nu}, \quad \overline{n^2} = \bar{\nu} + \overline{\nu^2}.$$

Hence

$$\overline{(\Delta n)^2} = \overline{n^2} - (\bar{n})^2 = \overline{\nu^2} - (\bar{\nu})^2 + \bar{\nu}$$

and

$$\beta^2 = \frac{\overline{(\Delta n^2)}}{(\bar{n})^2} = \frac{\overline{\nu^2} - (\bar{\nu})^2}{(\bar{\nu})^2} + \frac{1}{\bar{\nu}} = \beta_{cl}^2 + \frac{1}{\bar{\nu}}.$$

This expression differs from formula (50) by an amount β_{cl}^2/ν.

We can propose another reasoning which, generally speaking, is founded to just as small a degree as the preceding one, but which gives a correct value of β^2. We describe the source, as before, by the distribution (51), in which ν is replaced by $\overline{\nu(\Sigma, T)}$, but the inhomogeneous medium is described by introducing the transition probability $p(m|n)$. This quantity is the probability of registration of m photons under the condition that n photons were emitted in the receiver direction in the time T. Then

$$\bar{m} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} mp(m|n)p(n), \quad \overline{m^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m^2 p(m|n)p(n). \quad (52)$$

With respect to $p(m|n)$ we make the following assumptions that are natural for a linear medium:

$$\bar{m}|_n = \sum_{m=0}^{\infty} mp(m|n) = \alpha_1 n, \quad \overline{m^2}|_n = \sum_{m=0}^{\infty} m^2 p(m|n) = \alpha_2 n^2. \quad (53)$$

These relations denote that the average number of photons $\bar{m}|_n$ at a fixed number of emitted photons n is proportional to n; the same pertains also to $\overline{m^2}|_n$. Substituting (53) in (52), we obtain

$$\bar{m} = \alpha_1 \sum_{n=0}^{\infty} np(n) = \alpha_1 \overline{\nu(\Sigma,T)},$$
$$\overline{m^2} = \alpha_2(\overline{\nu(\Sigma,T)} + \overline{(\nu(\Sigma,T))^2}). \quad (54)$$

The quantities α_1 and α_2 are expressed in terms of β_{cl}^2. Indeed, at fixed n (i.e., in the absence of quantum fluctuations) we have

$$\beta^2|_n = \beta_{cl}^2 = \frac{\overline{m^2}|_n - (\bar{m}|_n)^2}{(\bar{m}|_n)^2} = \frac{\alpha_2 n^2 + \alpha_1^2 n^2}{\alpha_1^2 n^2} = \frac{\alpha_2}{\alpha_1^2} - 1$$

whence $\alpha_2 = \alpha_1^2(1 + \beta_{cl}^2)$. Substituting these expressions in (54), we obtain

$$\beta^2 = \frac{\overline{m^2} - (\bar{m})^2}{(\bar{m})^2} = \beta_{cl}^2 + \frac{1+\beta_{cl}^2}{\overline{\nu(\Sigma,T)}}.$$

This formula coincides with (50). Thus, in semiclassical calculations it is preferable to use the second reasoning scheme. It is possible here, in particular, to take into account the quantum efficiency of the photodetector, since it is known that convolution of the Poisson and binomial distributions leads again to a Poisson distribution in which $\overline{\nu(\Sigma, T)}$ is replaced by $\overline{\gamma\nu(\Sigma, T)}$, where γ is the quantum efficiency of the photodetector.

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