SOME GENERAL RELATIONS FOR WAVES AT THE MOVING BOUNDARY BETWEEN TWO MEDIA

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Boundary conditions in Lagrangian form are derived for an arbitrary field at the boundary between two dispersive media. It is shown that in the general case the form of the boundary conditions depends on the internal parameters of the boundary region. An invariant relation is obtained for normal actions (number of quanta) of the field in quasimonochromatic wave packets interacting at the boundary. The amplification properties of such packets are discussed for "superluminal" motion of the boundary.

INTRODUCTION

PROCESSES of wave transformation on a moving boundary between media having different reactive parameters have been investigated many times in connection with problems of general character, and also with the possibility of conversion of the frequency and duration of the signals. The moving boundary can be produced by displacement of a medium (mirror) in a vacuum or in some other medium[1-4] as well as by the sharp front of the field of a specified wave ("pump") which changes the parameters of a nonlinear medium relative to the signal in question[5-8]. Many problems of this type were solved recently, especially for electromagnetic waves, but these solutions are connected as a rule with some concrete model of the medium and do not make it possible to establish some general laws suitable for other systems.

In the present paper we derive some general relations characterizing the transformation of waves by a moving discontinuity of reactive parameters of a dispersive medium, using the Lagrangian form of the field equations. Just as in many other cases, such an approach permits a unified description of a rather wide class of systems, regardless of their concrete realization. The boundary conditions themselves turn out to depend not only on medium parameters that are external with respect to the boundary, but also on the internal parameters such as the dimension and possibly also the structure of the real transition region approximated by the discontinuity.

By using boundary conditions in Lagrangian form, it is possible to obtain a universal relation for the energy quantities (normal actions proportional to the number of field quanta) in bounded wave packets interacting on the moving boundary. This relation is independent, in a rather general case, of the structure of the boundary layer and plays approximately the same role in the processes under consideration as the well known Manley-Rowe formulas for monochromatic oscillations. In particular, it makes it possible to assess the conditions under which the field energy becomes amplified; an increase in the number of field quanta becomes possible if the velocity of the boundary exceeds the phase velocity of at least one of the waves[1)].

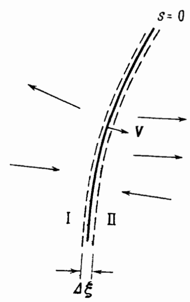

FIG. 1

1. BOUNDARY CONDITIONS

Let us consider an arbitrary field described by the Lagrangian[2)]

$$\mathscr{L} = \mathscr{L}(q, \dot{q}, q_{x_s}, p), \tag{1}$$

where $q = q_1 \ldots q_N$ is the set of generalized coordinates, $\dot{q} = \partial q/\partial t$; $q_{x_s} = \partial q/\partial x_s$; $x_s = x, y, z$ are the spatial coordinates, and $p(x_i, t)$ is the set of parameters of the medium. The corresponding field equations form a system of order 2N:

$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} + \frac{\partial}{\partial x_s}\frac{\partial \mathscr{L}}{\partial q_{ix_s}} - \frac{\partial \mathscr{L}}{\partial q_i} = 0. \tag{2}$$

Let the parameters p vary quite sharply in the vicinity $\Delta\xi$ of a certain moving surface $s(x_s, t) = 0$ (Fig. 1). We integrate (2) in the space (r, t) with respect to the normal to this surface. We choose the coordinate system such that the local velocity of the boundary V is directed along $x_1 = x$. It is then clear that in the $\Delta\xi$ scale we need take into consideration only the dependence of all the quantities on the variable

[1)]Such a motion will be called for brevity superluminal, although the field under consideration need not necessarily be electromagnetic.

[2)]This notation is quite general (see [9]); even if $\mathscr{L}$ contains derivatives of order higher than the first of certain q, it is always possible to arrive at an equivalent variational problem with a Lagrangian of the type (1) by making the change of variables customarily employed for variational methods.

$\xi = x - Vt$, since the variation as a function of the remaining variables is much slower; consequently, in the vicinity of the boundary we have $\partial/\partial t \approx -V\partial/\partial\xi$, $\partial/\partial x \approx \partial/\partial\xi$. Integrating (2) with respect to ξ in the small vicinity $\Delta\xi \to 0$, we obtain

$$\frac{\partial\mathscr{L}}{\partial q_{ix}} - V\frac{\partial\mathscr{L}}{\partial\dot{q}_i} = \text{const.} \tag{3}$$

We recognize further that in the vicinity of the boundary, the left-hand side of (3) depends on q and on $\partial q/\partial\xi$. Solving (3) with respect to $\partial q/\partial\xi$, we obtain

$$\partial q_i/\partial\xi = f(q, r, t), \tag{4}$$

where f stands for bounded functions. Integration of (4) with respect to ξ yields $q_i \equiv$ const over a small integration region. From this and from (3) there follow 2N boundary conditions:

$$\left\{\frac{\partial\mathscr{L}}{\partial q_{ix}} - V\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right\} = 0, \tag{5}$$

$$\{q_i\} = 0, \tag{6}$$

where the curly brackets denote the difference between the corresponding values on the two sides of the surface s = 0 and in its immediate vicinity. Sometimes it is more convenient to use in place of (6) the equivalent condition obtained by differentiating (6) with respect to t along the trajectory of the boundary:

$$\left\{\frac{\partial q_i}{\partial t} + V\frac{\partial q_i}{\partial x}\right\} = 0. \tag{6'}$$

These boundary conditions are also valid for nonlinear media, provided only Eqs. (3) are satisfied over the entire transition region replaced by the "discontinuity."

Let us also estimate the work performed by the boundary on the field. The expressions for the energy density W and for the energy flux **S** of the field, as is well known, are given by[9]

$$W = \dot{q}_i\frac{\partial\mathscr{L}}{\partial\dot{q}_i} - \mathscr{L}, \quad S_\alpha = \dot{q}_i\frac{\partial\mathscr{L}}{\partial q_{ix_\alpha}}. \tag{7}$$

From (1), (2), and (7) we obtain the energy-transport equation in the usual manner; integration of the latter in the vicinity of the boundary $\Delta t = V\Delta\xi$ yields

$$V\{W\} - \{S_x\} = -V\int_{\Delta t}\frac{\partial\mathscr{L}}{\partial p}\frac{\partial p}{\partial t}dt. \tag{8}$$

Consequently, the work performed by the boundary on the field is connected, as it should be, only with the time variation of the parameters, i.e., in this case with the motion of the boundary. Positive work corresponds to changes of p such that $\mathscr{L}$ decreases with time (for fixed q, $\dot{q}$, and q_x).

2. ROLE OF INTERNAL PARAMETERS OF THE BOUNDARY

The foregoing elementary derivation of the boundary conditions admits of important exceptions, which make it necessary to examine more thoroughly the physical properties of the transition layer approximated by the discontinuity.

Let one or several of the generalized coordinates ($q = q_m$) be contained in the Lagrangian (1) without their derivatives. Then the corresponding equation of motion (2) takes the form

$$\partial\mathscr{L}/\partial q_m = 0, \tag{9}$$

i.e., the total order of the system is reduced. It is clear that the boundary conditions (5) and (6) are not satisfied with respect to q_m, and instead the values of the coordinate q_m on each side of the boundary are determined from relation (9), which is algebraic with respect to q_m, i.e., it has a local character. The coordinate q_m then experiences in the general case a discontinuity on the boundary. The situation here is, in a certain sense, the same as in classical mechanics, namely, the variable on which the kinetic energy of the system does not depend can be replaced by a discontinuity.

It is easily seen now that the solution of the problem of the variation of the field on the boundary actually depends on the real spatial (Δx) and temporal ($\Delta t = \Delta x/V$) dimensions of the boundary region. Indeed, the "discontinuity hypothesis" itself requires only that Δx and Δt be small compared with the characteristic length (λ) and with the period (τ) of the waves outside the discontinuity (external condition). However, if $\partial\mathscr{L}/\partial q \neq 0$, the field equations (2) contain also proper scales with dimensions of the coordinate (x_0) and of the time (t_0), determined by the relations between the different terms of these equations; these scales can themselves be small compared with λ and τ. Then if Δx and Δt are nevertheless smaller than x_0 and t_0 (abrupt boundary), then the first two terms in (2) are large compared with the third, and formulas (3)—(6) are valid. If, however, an inequality of the type $x_0, t_0 \ll \Delta x, \Delta t \ll \lambda, \tau$, is satisfied for some x_0, and t_0, then this means that the field on the boundary changes smoothly enough to permit neglect of all the differential operators compared with $\partial\mathscr{L}/\partial q$ in the corresponding equation of the system (2). In this case, some of the boundary conditions are replaced by Eq. (9). Thus, an important role is played not only by the external conditions but also by the internal conditions that are connected with the variation of the field inside the boundary region.

Let us illustrate the foregoing using as an example an electromagnetic field. The usual conditions for the intensities (**E**, **H**) and the inductions (**D**, **B**) of the electric and magnetic fields on the boundary reduce (in the absence of free surface currents and charges) to continuity of the normal components of **D** and **B** and of the tangential components of the vectors $\mathbf{E} + \mathbf{V}\times\mathbf{B}/c$ and $\mathbf{H} - \mathbf{V}\times\mathbf{D}/c$, where c is the velocity of light in vacuum[10]. For them to be valid it suffices that the thickness of the boundary layer be small compared with the length and with the period of the wave. If the dimensions of the boundary region are small also with respect to all the vibrational and relaxational processes in the medium, then a still stronger statement can be made: regardless of the concrete form of the material equations, all four of the vectors **E**, **H**, **B**, and **D** are continuous on such a boundary. This could be demonstrated by using a Lagrangian that contains variable fields (potentials) and also generalized coordinates characterizing the polarization of the molecules of the medium. Here, however, simpler considerations will suffice. Indeed, as is well known[10], a loss of continuity of the field in any real medium propagates with a velocity c, since the polarization does not have time to change (this circumstance is connected with the hyperbolic character of the equations of electrodynamics). Therefore the field should remain strictly continuous on any boundary that moves with a velocity different from c. This almost obvious statement has apparently never been taken into account directly, although its validity can be verified for all the problems considered in the literature in which concrete models of dispersive media are considered (see, for example,[2,3]. We note that the use of the field continuity conditions makes it possible in many cases to simplify the solution of the problem considerably.

In addition, in the literature there are frequently considered problems involving the motion of the boundary in a nondispersive medium, when the material equations relating the vectors **E**, **D**, **H**, and **B** have a local character corresponding to (9). It then follows from the general boundary conditions that the field, generally speaking, experiences a discontinuity in the vicinity of the boundary[5,7]. It is clear from the statements made above that such problems are meaningful if the real thickness of the boundary is sufficiently large to permit neglect of dispersion over the entire transition region, i.e., so that the polarization can follow the variation of the field.

These singularities in the variation of the electromagnetic field are obviously in full agreement with the general considerations given above.

Another case in which the term with $\partial\mathscr{L}/\partial q$ enters in the boundary conditions pertains to a δ-type layer, the "density" of which increases with decreasing thickness. Integration of (3) with respect to ξ yields in place of (5) the equation

$$\left\{\frac{\partial\mathscr{L}}{\partial q_{ix}} - V\frac{\partial\mathscr{L}}{\partial \dot{q}_i}\right\} = \lim_{\Delta\xi\to 0}\int\frac{\partial\mathscr{L}}{\partial q_i}d\xi = p_{0i}. \quad (10)$$

Here P_{0i} is assumed to be finite; then the conditions (6) remain in force. For a linear medium, when the Lagrangian has the form (11) (we put for simplicity $b = c = 0$), we obviously have

$$p_{0i} = q_k \lim_{\Delta\xi\to 0}\int d_{ik}(\xi)\,d\xi$$

and $p_{0i} \neq 0$ if $d_{ik} \sim 1/\Delta\xi$; then the "thickness" of the layer can be characterized by finite parameters $p_{ik} = \lim_{\Delta\xi\to 0}\int d_{ik}d\xi$, which do not depend on q.

Notice should also be taken of the following possible inapplicability of conditions (6). In the derivation of (6) it was assumed that all f in (4) are finite or are at least definite-integrable as $\Delta\xi \to 0$. This assumption, however, may also not be satisfied. Thus, substituting in (3) the Lagrangian (11) of a field in a linear medium and retaining only the derivatives $\partial q/\partial\xi$, we find that all the f are proportional to D^{-1}, where

$$D = \mathrm{Det}\,\|a_{ik}^{11} + V^2 a_{ik}^{44} + V(a_{ik}^{14} + a_{ik}^{41})\|.$$

Consequently, Eq. (6) may not hold if $D = 0$ at some value $p = p_1$ from the region $\Delta\xi$ (with (5) remaining in force). The result now, generally speaking, depends on the value of p_1, i.e., again on the internal parameters of the "discontinuity." It is easily seen that at $D = 0$ the boundary is a characteristic surface of the system (2) at $p = p_1$, and V is the corresponding characteristic velocity in the x direction (to this end the system (2) should be of the hyperbolic or mixed type). This is precisely the cause of the mathematical singularity of such problems; from the physical point of view this singularity is obviously connected with the synchronism between the motion of the boundary and the perturbations produced by it. Such cases have already been considered for electromagnetic waves in a nondispersive dielectric[7]. These include all the problems involving the interaction of small perturbations with shock waves in the "discontinuity" approximation[11,5]; as is well known, to solve such problems it is necessary to take into account the perturbations of the shock-front velocity. They will be excluded from the analysis that follows.

3. ENERGY RELATIONS FOR WAVE PACKETS

We proceed to consider harmonic and quasiharmonic waves in linear media without dissipation. In the general case the field Lagrangian of such a medium is written in quadratic form:

$$\mathscr{L} = a_{ik}^{sl}\frac{\partial q_i}{\partial x_s}\frac{\partial q_k}{\partial x_l} + b_{ik}^{s}q_i\frac{\partial q_k}{\partial x_s} + c_{ik}^{s}\frac{\partial q_i}{\partial x_s}q_k + d_{ik}q_iq_k, \quad i, k = 1, \ldots, N; \quad s, l = 1, 2, 3, 4, \quad (11)$$

where, to simplify the notation, we put $x_{1,2,3} = x, y, z$ and $x_4 = t$[3)]; summation over repeated indices is implied. The parameters a and c which change jumpwise on the moving boundary, satisfy the symmetry conditions

$$a_{ik}^{sl} = a_{ki}^{ls}, \quad b_{ik}^{s} = c_{ki}^{s}, \quad d_{ik} = d_{ki}. \quad (12)$$

In each of the regions on either side of the boundary, the field can be represented by a superposition of plane normal waves:

$$q_i = \sum_\nu A^\nu e^{j(\omega_\nu t - \mathbf{k}_\nu \mathbf{r})} + \text{c.c.} = \sum_\nu A_i^\nu e^{j\varkappa^\nu \mathbf{R}} + \text{c.c.}, \quad (13)$$

where, obviously, $\boldsymbol{\varkappa} = (\mathbf{k}, -\omega)$ and $\mathbf{R} = (\mathbf{r}, t)$; all the $\boldsymbol{\varkappa}^\nu$ are assumed to be real. Substituting (13) in the equation of motion (2), we obtain

$$[a_{ik}^{sl}\varkappa_s^\nu\varkappa_l^\nu + j\varkappa_s^\nu(c_{ik}^s - b_{ik}^s) + d_{ik}]A_k^\nu = 0. \quad (14)$$

From this, in particular, follows the dispersion equation

$$\mathrm{Det}\,\|\varkappa_s^\nu\varkappa_l^\nu a_{ik}^{sl} + j\varkappa_s^\nu(c_{ik}^s - b_{ik}^s) + d_{ik}\| = 0 \quad (15)$$

(Relations (12) ensure hermiticity of the matrix (15)).

We assume for concreteness that one of the monochromatic waves on one side of the boundary (region I) is given and is incident on the boundary (which we denote by the index 0); the remaining waves are secondary and move away from the boundary in regions I and II. It then follows from the boundary conditions (5) and (6) for the instantaneous values of q, in particular, that the phases of all the waves on the boundary are equal (apart from arbitrary additive constants), leading to certin frequency relations that are valid for any pair of waves

$$\frac{\omega_\nu}{\omega_0} = \frac{(1 - \mathbf{V}\mathbf{v}_{\text{ph}}/v_{\text{ph}}^2)_0}{(1 - \mathbf{V}\mathbf{v}_{\text{ph}}/v_{\text{ph}}^2)_\nu}, \quad [(\mathbf{k}_\nu - \mathbf{k}_0)\mathbf{V}] = 0, \quad (16)$$

3) There is no need to change over here to the relativistic variable jct, since we do not use the relativistic invariance of the Lagrangian anywhere.

where $v_{ph} = \omega k/k^2$ are the phase velocities of the waves. These relations, together with (15), define implicitly (in the presence of dispersion) the frequencies and the wave numbers of all the waves.

Now let the interacting waves have the form of quasimonochromatic homogeneous wave packets with narrow frequency $(\Delta\omega)$ and angle $(\Delta\overline{k})$ spectra. We neglect the dispersion distortions of the packets during the interaction time, and then each of the packets propagates as a unit with group velocity $u(\omega, \overline{k}) = \partial\omega/\partial\mathbf{k}$ and duration $T \sim (\Delta\omega)^{-1}$ (T is defined in the direction of the group velocity). By varying (16) with respect to ω, we obtain

$$\frac{T_\nu}{T_0} = \frac{\Delta\omega_0}{\Delta\omega_\nu} = \frac{(1 - \mathbf{V}u/u^2)_\nu}{(1 - \mathbf{V}u/u^2)_0}. \tag{17}$$

It is clear that in a dispersive medium the durations of the pulses are transformed in a manner different from the periods of the carrier frequency, in view of the difference between the group and phase velocities. The volumes Γ of the packets are transformed in proportion to their dimensions l_x in a direction normal to the boundary, namely

$$\frac{\Gamma_\nu}{\Gamma_0} = \left|\frac{u_x^\nu - V}{u_x^0 - V}\right|. \tag{18}$$

We shall furthermore assume that the medium is transparent with respect to all the waves under consideration, i.e., all the ω_ν and $\mathbf{k}_\nu$ are real. For the energy densities and for the energy flux of each of the monochromatic waves entering in the sum (13) we have from (8)

$$\begin{aligned} \overline{W}^\nu &\sim a_{ik}{}^{4l}\varkappa_4{}^\nu\varkappa_l{}^\nu \operatorname{Re}(A_i{}^{\nu*}A_k{}^\nu) - c_{ik}{}^4\varkappa_4 \operatorname{Im}(A_i{}^{\nu*}A_k{}^\nu), \\ \overline{S}_s{}^\nu &\sim a_{ik}{}^{sl}\varkappa_4{}^\nu\varkappa_l{}^\nu \operatorname{Re}(A_i{}^{\nu*}A_k{}^\nu) - c_{ik}{}^s\varkappa_4 \operatorname{Im}(A_i{}^{\nu*}A_k{}^\nu) = u_s\overline{W}^\nu, \end{aligned} \tag{19}$$

where the bar denotes averaging over t, and u, as above, is the group-velocity vector. The expression for $\overline{W}$ takes into account the fact that in a monochromatic traveling wave we have $\overline{\mathscr{L}} = 0$ (this can be readily shown here by multiplying (14) by $A_l^{\nu*}$ and averaging).

To obtain the connection between the energy quantities on the boundary, let us multiply pairwise the equations in (6) and (6′) corresponding to the same q_i after first substituting (13) in them, average, and sum over i:

$$\Big\{\sum_{\nu,\nu'} (\varkappa_4 + V\varkappa_1)^{\nu'}[\varkappa_l{}^\nu(a_{ik}{}^{1l} - Va_{ik}{}^{4l})\operatorname{Re}(A_i{}^{\nu'*}A_k{}^\nu) - (c_{ik}{}^1 - Vc_{ik}{}^4)\operatorname{Im}(A_i{}^{\nu'*}A_k{}^\nu)]\Big\} = 0. \tag{20}$$

We assume the boundary to be plane (at least when measured in the scale of a wave packet), and the boundary velocity to be directed along $x_1 = x$; by virtue of (16), all the factors $(\kappa_4 + V\kappa_1)^{\nu'}$ are the same here.

Using (14), we can show (see the Appendix) that all the terms with $\nu \neq \nu'$ in the sum (20) vanish. Then, multiplying and dividing each term of (20) by κ_4^ν and using (19), we obtain

$$\sum_\nu \left\{\frac{\overline{S}_x{}^\nu - V\overline{W}^\nu}{\omega^\nu}\right\} = 0. \tag{21}$$

To determine the meaning of this equation, let us consider a homogeneous wave packet with finite volume Γ_0 incident on the boundary[4]. Since $S_x^\nu = u_x^\nu W^\nu$, we get from (21) and (18)

$$\left\{\sum_{\nu=0}^{n+1} \pm \frac{E_\nu}{\omega^\nu}\right\} = 0. \tag{22}$$

Here $E_\nu = \int_{\Gamma_\nu} \overline{W}_\nu d\Gamma_\nu$ is the total energy of the wave and n is the number of secondary waves.

The sign in (22) coincides with the sign of the difference $u_x^\nu - V$. We designate the half-spaces separated by the boundary as regions I and II, with the x axis directed towards region II. It is clear that if the incident wave packet approaches the boundary from region I, then we have for it $u_x^0 - V > 0$, whereas in region II we have $u_x^0 - V < 0$. To the contrary, we should have $u_x^\nu - V < 0$ for all the secondary waves in region I (reflected), and $u_x^\nu - V > 0$ for the waves in region II (transmitted); only then can these waves move away from the discontinuity (the radiation condition). Consequently, (22) can be rewritten in the form

$$\frac{E_0}{\omega_0} = \sum_{\nu\neq 0}^{n} \frac{E_\nu}{\omega_\nu}, \tag{22′}$$

where the index 0 pertains to the incident wave and the remaining $\nu \neq 0$ to the secondary waves.

It must be borne in mind, however, that the sign of ω_ν can be different for different waves. Indeed, according to (16), the sign of the terms in (22) coincides with the sign of the quantity $\gamma = (|v_{ph}| - |V| \cos\theta)$, where θ is the angle between v_{ph} and the normal to the boundary. We then obtain ultimately

$$(\operatorname{sign}\gamma_0) I_0 = \sum_{\nu\neq 0} (\operatorname{sign}\gamma_\nu) I_\nu, \tag{23}$$

where $I_\nu = |E_\nu/\omega_\nu|$ is the normal action for the ν-th wave, and is proportional to the number of field quanta (quasiparticles) in it[5].

We note that if some of the conditions (5) and (6) are replaced by (9), i.e., the coordinate q_m experiences a discontinuity, then the applicability of (23) is not affected, since the corresponding amplitudes A_m simply do not enter in (18)—(23) (all $a_{mk} = 0$). The latter are true also for the δ-layer described by formula (10), since the function p_0 oscillates in quadrature with $\partial\mathscr{L}/\partial x$, $\partial\mathscr{L}/\partial t$ and makes no contribution to (20) when averaged.

It is clear from (23) that if $\gamma > 0$ for all the interacting waves, then the transformation of the wave packet takes place with conservation of the total number of quanta. This is valid also in the case when all γ (including γ_0) are negative. On the other hand, if at $\gamma_0 > 0$ there exist secondary waves with $\gamma < 0$, then the total number of quanta is larger in the secondary waves than in the incident wave. The condition $\gamma < 0$ denotes that the given wave lies within the limits of the Cerenkov cone relative to the velocity of the boundary. Consequently, the production of quanta on the boundary occurs in those cases when some of the interacting waves lie in the region of the normal Doppler effect, and others in the region of the anomalous effect. This conclusion does not depend on whether the medium

[4]We can also consider plane waves of infinite dimension, and then E will henceforth denote the energy of the section of the wave with fixed transverse dimensions (relative to the x axis).

[5]Relations of the type (23) were already cited for the particular case of electromagnetic waves in an immobile medium without dispersion, where the moving discontinuity of the parameters was produced by the front of an external "pump" field [8].

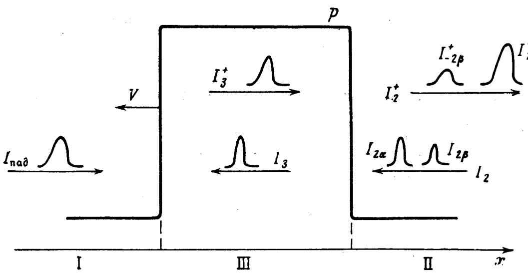

FIG. 2

moves or a parameter wave propagates in an immobile medium[6)].

4. AMPLIFICATION OF WAVES UPON INTERACTION WITH A MOVING LAYER

In conclusion, let us briefly discuss the following example, which illustrates the features of the amplification of continuous signals and short wave packets. We consider a rectangular layer made up of two discontinuities and moving with velocity V in the direction opposite to the incident wave $I_1^+ = I_{inc}$ (Fig. 2). Applying (23) in succession to the two boundaries, we can easily verify that this relation connects the waves on the two sides of the layer. Assume that in each of the regions outside and inside the layer there exist two normal waves (+ and −) propagating in opposite directions. We note that the frequency and duration of the wave I_2^+ transmitted through the layer are always the same as for the incident wave. In the case of "subluminal" motion of the boundary (relative to the group and phase velocities) there are produced one reflected wave (I_1^-) and one transmitted one (I_2^+), with $I_{inc} = I_1^- + I_2^+$ (several problems of this type were considered for electromagnetic waves[13,14]). On the other hand, if the motion of the layer is "superluminal," then there is no reflected wave, but instead there is excited a second wave I_2^- behind the layer (Fig. 2), and for this wave $\gamma < 0$. Consequently, we now have $I_{inc} = I_2^+ - I_2^-$, i.e., $I_2^+ > I_{inc}$, and since their frequencies are equal, there is also amplification of the power of the wave I_2^+ compared with the incident wave. An exception is a "resonant" layer, the length of which is such that $I_2^- = 0$ and $I_2^+ = I_{inc}$.

We note that amplification is also possible for a thin (compared with the wavelength) moving layer corresponding to (10). Such a problem can be regarded as a "planar analog" of the problem of induced scattering of a wave by a moving particle; in the case of subluminal motion, the scattering occurs with conservation of the number of quanta[12], but the emission of the anomalous Doppler waves is connected with two-quantum processes and the total number of field quanta increases.

If we change over from one layer to a periodic sequence of layers, then, subject to satisfaction of certain resonance conditions that ensure in-phase interference of the re-reflected waves, cumulative parametric amplification of the traveling wave takes place (see[15]).

[6)] It is assumed, however, that in the moving medium there is no distributed amplification or absorption of the waves, which in principle can occur in the presence of the anomalous Doppler effect [12].

Then formula (22) or (23), valid for each "elementary" layer, can be regarded as a generalization of the known Manley-Rowe relations[16], which are valid only for a steady-state process in an immobile region of parameter variation[15,17].

Worthy of special discussion is the case of passage through a layer of length L short in comparison with the L of the wave packet; then the successive interactions on the boundaries of the layer occur independently. In the "subluminal" case the pulses are rereflected many times and a series of pulses emerges from the layer. In the "superluminal" case, as can be readily verified, only two pulses (α and β), corresponding to the waves I_2^+ and I_2^- (Fig. 2), are produced in each case. Relation (23) is again satisfied for the total number of quanta in the + and − waves, i.e.,

$$I_{inc} = (I_{2\alpha}^+ + I_{2\beta}^+) - (I_{2\alpha}^- + I_{2\beta}^-).$$

However, even in one pulse $I_{2\alpha}^+$ the number of quanta is always larger than in the incident pulse; indeed, from (23) we obtain $I_{inc} = I_{2\alpha}^+ - I_{2\alpha}^- - I_3^-$ (I_3^- corresponds to a pulse passing through the first boundary, see Fig. 2). If again we change over from a single layer to a sequence of layers, then, in view of the independence of the re-reflection processes, the pulse $I_{2\alpha}^+$ becomes exponentially amplified without imposition of any resonance conditions (the phase relations are immaterial), provided the process is repeated many times. The carrier frequency and the pulse duration remain unchanged (so long as one can neglect the dispersion spreading). From the spectral point of view, such a nonresonant effect is connected with the resonant amplification of the individual components in the spectrum of the pulse and with simultaneous redistribution of the energy among the different spectral components.

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APPENDIX

We shall show that all the terms with $\nu \neq \nu'$ in (20) vanish. To this end we separate in (20) the sum of terms that differ in the permutations $\nu \rightleftarrows \nu'$ and $i \rightleftarrows k$:

$$[\varkappa_l^\nu(a_{ik}^{1l} - Va_{ik}^{4l}) + \varkappa_l^{\nu'}(a_{ki}^{1l} - Va_{ki}^{4l})]\operatorname{Re}(A_i^{\nu'*}A_k^\nu) - (c_{ik}^1 - Vc_{ik}^4 - c_{ki}^1 + Vc_{ki}^4)\operatorname{Im}(A_i^{\nu'*}A_k^\nu). \tag{A.1}$$

(We have left out here the factors $\kappa_4 + V\kappa_1$, which are the same for all waves.)

Taking the symmetry relations (12) into account, we rewrite (A.1) in the form

$$[a_{ik}^{11}(\varkappa_1^\nu + \varkappa_1^{\nu'}) - Va_{ik}^{44}(\varkappa_4^\nu + \varkappa_4^{\nu'}) + \varkappa_m^\nu(a_{ik}^{1m} + a_{ik}^{m1} + Va_{ik}^{4m} - Va_{ik}^{m4}) + a_{ik}^{14}(\varkappa_4^\nu - V\varkappa_1^{\nu'}) + a_{ik}^{41}(\varkappa_4^{\nu'} - V\varkappa_1^\nu)]\operatorname{Re}(A_i^{\nu'*}A_k^\nu) + (b_{ik}^1 - Vb_{ik}^4 - c_{ik}^1 + Vc_{ik}^4)\operatorname{Im}(A_i^{*\nu'}A_k^\nu), \tag{A.2}$$

where m = 2 and 3. From (16) it follows that the sum of the terms with a^{14} and a^{41} in (A.2) is given by

$$(a_{ik}^{14} + a_{ik}^{41})(\varkappa_4^\nu - V\varkappa_1^{\nu'})\operatorname{Re}(A_i^{\nu'*}A_k^\nu). \tag{A.2a}$$

We consider further Eqs. (14), which are valid for plane waves on each side of the boundary. We multiply

(14) by $A_i^{\nu'*}$ and sum over i, and then subtract the analogous equality for the wave ν' with the replacement $i \rightleftarrows k$. As a result we obtain

$$(a_{ik}^{sl}+a_{ik}^{ls})(\varkappa_s^{\nu}\varkappa_l^{\nu}-\varkappa_s^{\nu'}\varkappa_l^{\nu'})\,\mathrm{Re}(A_{ik}^{\nu'*}A_k^{\nu}) + (b_{ik}^{s}-c_{ik}^{s})(\varkappa_s^{\nu}-\varkappa_s^{\nu'})\,\mathrm{Im}(A_i^{\nu'*}A_k^{\nu})=0. \quad (A.3)$$

We also use the identity

$$\varkappa_s^{\nu}\varkappa_l^{\nu}-\varkappa_s^{\nu'}\varkappa_l^{\nu'}=\varkappa_s^{\nu}(\varkappa_l^{\nu}-\varkappa_l^{\nu'})+\varkappa_l^{\nu'}(\varkappa_s^{\nu}-\varkappa_s^{\nu'}).$$

Since, according to (16), the differences in the last expression are different from zero only at $l = 1$ and 4 or $s = 1$ and 4, and $\kappa_4^{\nu}-\kappa_4^{\nu'}=V(\kappa_1^{\nu'}-\kappa_1^{\nu})$, we can readily rewrite (A.3) in the form

$$(\varkappa_1^{\nu}-\varkappa_1^{\nu'})[(a_{ik}^{1s}+a_{ik}^{s1}-Va_{ik}^{s4}-Va_{ik}^{4s})\varkappa_s^{\nu} + (a_{ik}^{1l}+a_{ik}^{l1}-Va_{ik}^{4l}-Va_{ik}^{l4})\varkappa_l^{\nu'}]\,\mathrm{Re}(A_i^{\nu'*}A_k^{\nu}) + (\varkappa_1^{\nu}-\varkappa_1^{\nu'})(b_{ik}^{1}-Vb_{ik}^{4}-c_{ik}^{1}+Vc_{ik}^{4})\,\mathrm{Im}(A_i^{\nu'*}A_k^{\nu})=0. \quad (A.4)$$

From this we have at $\kappa_1^{\nu}\neq\kappa_1^{\nu'}$

$$[2a_{ik}^{11}(\varkappa_1^{\nu}+\varkappa_1^{\nu'})-2Va_{ik}^{44}(\varkappa_4^{\nu}+\varkappa_4^{\nu'})+2\varkappa_m^{\nu}(a_{ik}^{m1} + a_{ik}^{1m}-Va_{ik}^{m4}-Va_{ik}^{4m})+(a_{ik}^{41}+a_{ik}^{14})(\varkappa_4^{\nu}-V\varkappa_1^{\nu}+\varkappa_4^{\nu'}-V\varkappa_1^{\nu'})] \times \mathrm{Re}(A_i^{\nu'*}A_k^{\nu})+(b_{ik}^{1}-Vb_{ik}^{4}-c_{ik}^{1}+Vc_{ik}^{4})\,\mathrm{Im}(A_i^{\nu'*}A_k^{\nu})=0. \quad (A.5)$$

Here again m = 2 and 3. It is clear from (16) that the sum of the terms with a^{14} and a^{41} in (A.5) is equal to $2(a_{ik}^{14}+a_{ik}^{41})(\kappa_4^{\nu}-V\kappa_1^{\nu'})$ (see (A.2a)). Thus, the left-hand side of (A.5) coincides with (A.2), thus proving that all the terms of (A.1) vanish at $\nu\neq\nu'$.

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