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CALCULATION OF THE PION-DEUTERON SCATTERING LENGTH

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A formula for the pion-deuteron scattering length is derived by summation of an infinite series of nonrelativistic Feynman diagrams. The formula is an extension of the Brueckner formula. It is shown that terms corresponding to scattering of nucleons in the intermediate state need not be taken into account. The πd elastic scattering length is -0.06 F.

1. INTRODUCTION

In calculating the pion-nucleus scattering length, allowance for multiple scattering of the pion serves as a correction to the impulse approximation, since the ratio of the πN-scattering amplitude to the average distance between nucleons ($r_0 = 1.2$ F) is a small quantity:

$$a_{\pi N} = a_0 + a_1 \mathbf{I}\boldsymbol{\tau}_N, \qquad (1)$$

where $a_0 = -0.013 \pm 0.005$ F, $a_1 = -0.154 \pm 0.05$ F, I is the pion isospin operator, and τ_N is the usual (Pauli) isospin operator for the nucleon. In the case of pion-deuteron scattering, it is necessary to take multiple scattering into account exactly; the impulse approximation is a small quantity by virtue of the smallness of a_0 ($a_{\pi d}^{\text{imp}} = 2a_0$) and is comparable with the effect of multiple scattering (the contribution of the multiple scattering is of the order a_1^2/R_d, where R_d is the deuteron radius).

There are two different approaches to this problem. One was proposed by Brueckner[1]. Brueckner assumed that owing to the smallness of the pion mass compared with the nucleon mass ($\mu/m \approx 1/7$), the nucleons will experience a small recoil when scattering the pion. Therefore the amplitude of the πd scattering can be regarded as the amplitude for scattering by a system of two immobile centers[1,2] averaged over the deuteron wave function. If the momentum of the incoming pion is k, the zero-angle πd-scattering amplitude is given, according to Brueckner, by

$$f_{\pi d} = \int [\tilde{\Psi}_d(r)]^2 \left\{ \frac{f_1 + f_2 + 2f_1 f_2 r^{-1} e^{i\mathbf{kr}+ikr}}{1 - f_1 f_2 r^{-2} e^{2ikr}} \right\} d\mathbf{r}. \qquad (2)$$

Here $\tilde{\Psi}_d(r)$ is the normalized wave function of the deuteron, f_1 and f_2 are the amplitudes for the scattering by a neutron and by a proton (Brueckner disregarded charge exchange).

An advantage of formula (2) is that it takes into account all the rescatterings of the pion. Brueckner, however, did not give any criteria for the applicability of this formula.

Another approach to multiple pion scattering was developed by a number of workers[3-7] and reduces essentially to allowance for double scattering, which corresponds in diagram language to allowance for the diagrams of Fig. 1b and Fig. 4 (see below). However, the expansion of the amplitude in terms of the scattering multiplicity (the series of diagrams of Fig. 1) is in no ways the expansion of the total amplitude in powers of the parameters $a_{\pi N}/R_d$; this will be demonstrated in Sec. 2. For example, in the diagram of Fig. 1a, the amplitude $a_{\pi N}$ becomes dimensionless because of the factor $\langle 1/r \rangle_d$. This quantity is not determined by the dimension of the deuteron and depends significantly on the behavior of the deuteron wave function at short distances.

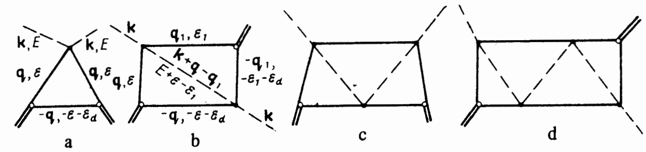

FIG. 1

In the present article, to take multiple scattering of the pion into account, we use the method of nonrelativistic diagram technique[1]. In Sec. 2 we sum the series of diagrams which gives the main contribution to the πd-scattering length. The result is a certain modification of Brueckner's formula (2). The discarded terms are estimated in Sec. 3. In Sec. 4 we average the obtained amplitude over the isotopic variables and obtain the final result.

In the present paper we do not touch upon the calculation of the contribution made by the absorption to the πd-scattering length. This contribution is present because of the $\pi d \to NN$ reaction and was estimated in[7,10]. Different authors obtained different values of Re $f_{\pi d}^{\text{abs}}$, which range from +0.085 F[10] to −0.007 F[7].

2. πd SCATTERING AMPLITUDE

In this section we sum the series of diagrams which gives the main contribution to the πd-scattering amplitude. This series of diagrams is shown in Fig. 1[2]. During the calculation, the πN-scattering amplitudes are assumed to be constant. In this section we also disregard the fact that the πN-scattering amplitude (1) is a matrix in the isotopic indices. The diagrams of Fig. 1 will be calculated with accuracy μ/m, while the ratio of the πN-scattering amplitude to the dimension of the deuteron can, in principle, be arbitrary. We

[1] See the reviews [8,9] concerning the nonrelativistic diagram technique.

[2] In all figures, the solid lines correspond to nucleon propagators and, the dashed lines to pion propagators, and a double line to the deuteron wave function.

shall henceforth calculate the zero-angle scattering amplitude[3].

We now proceed to calculate each of the diagrams of Fig. 1. Figure 1a shows a triangular diagram. This diagram corresponds to the following invariant amplitude $M^{(1)}_{\pi d}$:

$$M^{(1)}_{\pi d} = \int \frac{i^2(-2im)^3 M_1^2 F(q) F(q) A_1 \, d\mathbf{q} \, d\varepsilon}{(2\pi)^4 (\mathbf{q}^2 - 2m\varepsilon - i0)(\mathbf{q}^2 + 2m\varepsilon + 2m\varepsilon_d - i0)(\mathbf{q}^2 - 2m\varepsilon - i0)} \quad (3)$$

Here $M_1^2 = 8\pi\alpha/m^2$ ($\alpha^2 = m\epsilon_d$, ϵ_d is the deuteron binding energy); $A_1(A_2)$ is the (invariant) amplitude for scattering of the pion by the first (second) nucleon, F(q) is the deuteron form factor, and

$$\frac{F(q)}{q^2+\alpha^2} = \frac{1}{4\pi}\int e^{i\mathbf{q}\mathbf{r}} \Psi_d(r) \, d\mathbf{r}, \quad (4)$$

where the deuteron wave function is so normalized that

$$\int [\Psi_d(r)]^2 \, d\mathbf{r} = \frac{2\pi}{\alpha}.$$

Taking the integral with respect to energy in (3), we obtain

$$M^{(1)}_{\pi d} = \frac{\alpha A_1}{\pi^2} \int \frac{F^2(q)}{(q^2+\alpha^2)^2} \, d\mathbf{q}.$$

Using (4), we finally arrive at the expression

$$M^{(1)}_{\pi d} = A_1 \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \, d\mathbf{r}.$$

Let us proceed now to examination of double scattering (Fig. 1b). Taking the integral with respect to $d\epsilon$ and $d\epsilon_1$, we obtain

$$M^{(2)}_{\pi d} = \frac{\mu\alpha A_1 A_2}{2^2\pi^5} \int d\mathbf{q}\, d\mathbf{q}_1 F(q) F(q_1) \Big\{ (q^2+\alpha^2) \Big[(\mathbf{k}+\mathbf{q}-\mathbf{q}_1)^2 + 2\mu(\varepsilon_d - E) + \frac{\mu}{m} q^2 + \frac{\mu}{m} q_1^2 \Big] (q_1^2+\alpha^2) \Big\}^{-1}. \quad (5)$$

Let us consider, for concreteness, the case $E < \epsilon_d$. Since we consider diagrams with accuracy μ/m, we can omit terms of the type $\mu q^2/m$ and $\mu q_1^2/m$ in the pion propagator. These terms correspond to allowance for the kinetic energy of the nucleons in the intermediate state. It appears at first glance that it is necessary simultaneously to omit also the term $2\mu\epsilon_d$ in the same propagator, since it has the same order of magnitude ($m\epsilon_d \sim \overline{q}^2 \sim \overline{q}_1^2$). However, by discarding the terms $\mu q^2/m$ and $\mu q_1^2/m$, we increase the value of the integral (5). The expression $2\mu\epsilon_d$ is also positive, and therefore, by omitting it, we also increase the value of this integral, i.e., we incur a large error. The indicated procedure has therefore primarily a quantitative meaning: neglect of the terms $2\mu\epsilon_d$ and $\mu(q^2+q_1^2)/m$ causes to exaggerate the true value of (5) by 70% at $E = 0$. By omitting $\mu(q^2+q_1^2)/m$ but retaining $2\mu\epsilon_d$, we make the error in (5) only 35%.

Using the transformation (4) and also the fact that

$$\frac{1}{\mathbf{s}^2+\varkappa^2} = \frac{1}{4\pi}\int \frac{e^{i\mathbf{s}\mathbf{r}-\varkappa r}}{r} \, d\mathbf{r}, \quad (6)$$

we arrive at a final expression for $M^{(2)}_{\pi d}$:

$$M^{(2)}_{\pi d} = \frac{\alpha A_1}{2\pi} \frac{\mu A_2}{2\pi} \int [\Psi_d(r)]^2 \frac{e^{i\mathbf{k}\mathbf{r}-\varkappa r}}{r} \, d\mathbf{r},$$

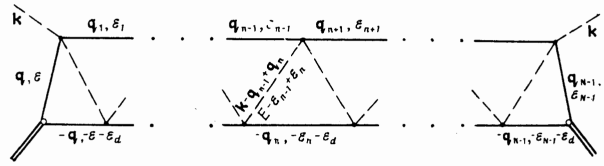

FIG. 2

where $\kappa^2 = 2\mu(\epsilon_d - E)$. Proceeding analogously, we can show that the amplitudes $M^{(3)}_{\pi d}$ and $M^{(4)}_{\pi d}$, shown in Figs. 1c and 1d, are expressed as follows:

$$M^{(3)}_{\pi d} = \frac{\alpha A_1}{2\pi} \frac{\mu A_2}{2\pi} \frac{\mu A_1}{2\pi} \int [\Psi_d(r)]^2 \frac{e^{-2\varkappa r}}{r^2} \, d\mathbf{r},$$

$$M^{(4)}_{\pi d} = \frac{\alpha A_1}{2\pi} \left(\frac{\mu A_2}{2\pi}\right)^2 \frac{\mu A_1}{2\pi} \int [\Psi_d(r)]^2 \frac{e^{i\mathbf{k}\mathbf{r}-3\varkappa r}}{r^3} \, d\mathbf{r}.$$

We now consider an arbitrary term of the series. It corresponds to the diagram of Fig. 2 (we choose odd N). Taking the integral with respect to the energy, we obtain

$$M^{(N)}_{\pi d} = \frac{\mu^{N-1} \alpha A_1^{(N+1)/2} A_2^{(N-1)/2}}{2^{2N-2}\pi^{3N-1}} \int d\mathbf{q}\, d\mathbf{q}_1 \ldots d\mathbf{q}_{N-1} \cdot \frac{F(q) F(q_{N-1})}{q^2+\alpha^2} \times \frac{1}{[(\mathbf{k}+\mathbf{q}-\mathbf{q}_1)^2+\varkappa^2] \ldots [(\mathbf{k}+\mathbf{q}_{N-1}-\mathbf{q}_{N-2})^2+\varkappa^2](q_{N-1}^2+\varkappa^2)}$$

Using the transformations (4) and (6), we have the final result

$$M^{(N)}_{\pi d} = \frac{A_1\alpha}{2\pi} \left(\frac{\mu A_1}{2\pi}\right)^{(N-1)/2} \left(\frac{\mu A_2}{2\pi}\right)^{(N-1)/2} \int [\Psi_d(r)]^2 \frac{e^{-(N-1)\varkappa r}}{r^{N-1}} \, d\mathbf{r}.$$

Analogously for even N

$$M^{(N)}_{\pi d} = \frac{A_1\alpha}{2\pi} \left(\frac{\mu A_1}{2\pi}\right)^{N/2-1} \left(\frac{\mu A_2}{2\pi}\right)^{N/2} \int [\Psi_d(r)]^2 \frac{e^{i\mathbf{k}\mathbf{r}-(N-1)\varkappa r}}{r^{N-1}} \, d\mathbf{r}.$$

We shall find it convenient to change from the invariant amplitudes $M^{(N)}_{\pi d}$ and A_i ($i = 1, 2$) to the amplitudes $f^{(N)}_{\pi d}$ and f_i ($i = 1, 2$):

$$f^{(N)}_{\pi d} = \frac{\mu}{2\pi} M^{(N)}_{\pi d}, \quad f_i = \frac{\mu}{2\pi} A_i.$$

In terms of this notation, the amplitude

$$f_{\pi d} = \sum_n f^{(n)}_{\pi d}$$

is determined by the following expression:

$$f_{\pi d} = \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \left\{ \Phi + \frac{f_1 f_2 e^{-2\varkappa r}}{r^2} \Phi + \left(\frac{f_1 f_2 e^{-2\varkappa r}}{r^2}\right)^2 \Phi + \ldots \right\} d\mathbf{r}, \quad (7)$$

where

$$\Phi = f_1 + f_2 + 2\frac{f_1 f_2}{r} e^{i\mathbf{k}\mathbf{r}-\varkappa r}.$$

Transforming the sum in the curly brackets, we finally obtain

$$f_{\pi d} = \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \left\{ \Phi \left(1 - \frac{f_1 f_2}{r^2} e^{-2\varkappa r}\right)^{-1} \right\} d\mathbf{r}. \quad (8)$$

In the case $E > \epsilon_d$, we can perform similar calculations and the amplitude $f_{\pi d}$ takes the form

$$f_{\pi d} = \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \left(f_1 + f_2 + 2\frac{f_1 f_2}{r} e^{i\mathbf{k}\mathbf{r}+ipr} \right) \left(1 - \frac{f_1 f_2}{r^2} e^{2ipr} \right)^{-1} d\mathbf{r}, \quad (9)$$

where $p^2 = 2\mu(E - \epsilon_d)$. Formulas (8) and (9) take into account three-particle unitarity: the amplitude $f_{\pi d}$ is real below the deuteron disintegration threshold and

[3]The final formula for the amplitude of scattering through a nonzero angle will differ from the expression (8) obtained by us in the presence of a factor $\exp(i\Delta r)$ in the integrand (Δ is the momentum transfer).

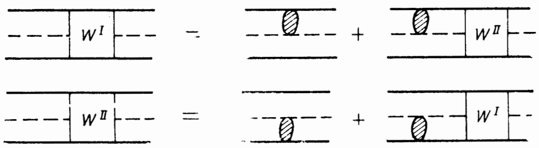

FIG. 3

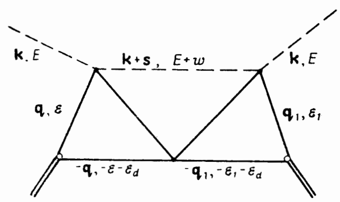

FIG. 4

complex above. These formulas are a natural generalization of the Brueckner formula (2).

As will be shown below, the contributions of all the remaining diagrams have a literal smallness and can be neglected.

We turn now to formulas (8) and (9). It follows from them that the integral that determines the total amplitude converges even in the case when the deuteron wave function is chosen in the approximation of the zero effective radius of the forces. On the other hand, the diagrams of Fig. 1 when taken individually (b, c, etc.) diverge in this approximation. The reason for these divergences can be readily seen. The amplitude (8) is the amplitude for the scattering by a system of two immobile centers averaged over the wave function of the deuteron—the expression in the curly brackets of formula (8). The latter expression is meaningful at any ratio of the scattering amplitude f_1 or f_2 to the distance r. On the other hand, when the diagrams of Fig. 1 are taken individually, they represent averaging, over the deuteron wave function, of the terms of the expansion of the amplitude in the curly brackets of formula (8) in powers of the ratio f_i/r—an expansion valid only when $r \gg f_i$ ($i = 1, 2$). But since the wave function of the deuteron is not small when $r \gg f_i$, the calculation of each of the diagrams of the series of Fig. 1 separately is an erroneous operation: we are averaging over the region of all r a formula that is meaningful only in the region of large r.

The need for interchanging the order of summation and integration in (7) can likewise be understood by starting from the Faddeev equations. The summed diagrams correspond to a system of two integral equations for the amplitudes, which are shown graphically in Fig. 3 (there is no need to include the third amplitude W^{III} in this system, as will be shown below). After solving this system (solution of this system leads us to Brueckner's formula) it is necessary to integrate the obtained amplitudes over the external nucleon lines. We average in this case, over the deuteron wave function, the total amplitudes W^I and W^{II}, and not their perturbation-theory iterations (this corresponds to calculating each of the diagrams of the series of Fig. 1 separately).

3. ESTIMATE OF THE DISCARDED TERMS

In this section we estimate the contribution of all the other diagrams to the πd-scattering length. We prove first that any diagram containing nucleon rescattering tends to zero as $\mu/m \to 0$. The simplest of these diagrams is shown in Fig. 4. This diagram corresponds to the amplitude M^I. Integrating with respect to the energy in the amplitude M^I, we obtain

$$M^I = \frac{a_{NN} f_1 f_2}{\pi^5 m}\int d\mathbf{q}\, d\mathbf{q}_1\, d\mathbf{s}\, F(q) F(q_1)\cdot$$
$$\times (q^2+\alpha^2)^{-1}\left[(\mathbf{k}+\mathbf{s})^2+\frac{\mu}{m}(\mathbf{q}-\mathbf{s})^2+\frac{\mu}{m}q^2+\varkappa^2\right]^{-1}$$
$$\times\left[(\mathbf{k}+\mathbf{s})^2+\frac{\mu}{m}(\mathbf{q}_1-\mathbf{s})^2+\frac{\mu}{m}q_1^2+\varkappa^2\right]^{-1}(q_1^2+\varkappa^2)^{-1}$$

where a_{NN} is the triplet NN-scattering length. There is no need to take charge exchange into account in the diagram of Fig. 4, since the amplitude for s-scattering of nucleons with total isotopic spin 1 and total spin 1 is equal to zero by virtue of the Pauli principle. The integral with respect to ds can be readily evaluated in this expression by using the fact that the equality

$$(\mathbf{k}+\mathbf{s})^2+\frac{\mu}{m}(\mathbf{q}-\mathbf{s})^2+\frac{\mu}{m}q^2=\left(\mathbf{k}+\mathbf{s}-\frac{\mu}{m}\mathbf{q}\right)^2+2\frac{\mu}{m}\left(\mathbf{q}+\frac{\mathbf{k}}{2}\right)^2$$

holds true accurate to μ/m. Indeed, using the transformation (6), we obtain

$$J=\int\frac{d\mathbf{s}}{[(\mathbf{k}+\mathbf{s}-\mu\mathbf{q}/m)^2+\gamma^2(\mathbf{q})][(\mathbf{k}+\mathbf{s}-\mu\mathbf{q}_1/m)^2+\gamma^2(\mathbf{q}_1)]}$$
$$=\frac{(2\pi)^3}{(4\pi)^2}\int\exp\left\{-i\frac{\mu}{m}\mathbf{r}(\mathbf{q}-\mathbf{q}_1)-[\gamma(\mathbf{q})+\gamma(\mathbf{q}_1)]r\right\}\frac{d\mathbf{r}}{r^2}$$
$$=2\pi^2\frac{m}{\mu|\mathbf{q}-\mathbf{q}_1|}\operatorname{arctg}\frac{\mu|\mathbf{q}-\mathbf{q}_1|}{m[\gamma(\mathbf{q})+\gamma(\mathbf{q}_1)]}, \quad (10)$$

where

$$\gamma^2(\mathbf{q})=\left[2\frac{\mu}{m}\left|\mathbf{q}+\frac{\mathbf{k}}{2}\right|^2+\varkappa^2\right].$$

It should be noted furthermore that the following inequality holds true:

$$\operatorname{arctg}\frac{\mu|\mathbf{q}-\mathbf{q}_1|}{m[\gamma(\mathbf{q})+\gamma(\mathbf{q}_1)]}<\operatorname{arctg}\sqrt{\frac{\mu}{2m}}\frac{|\mathbf{q}-\mathbf{q}_1|}{|\mathbf{q}+\mathbf{k}/2|+|\mathbf{q}_1+\mathbf{k}/2|}\leqslant\sqrt{\frac{\mu}{2m}}. \quad (11)$$

Using (10) and (11), we obtain the following estimate for the amplitude $f^I = \mu M^I/2\pi$:

$$f^I < f_1 C\frac{\alpha}{2\pi}\int[\Psi_d(r)]^2\frac{f_1 a_{NN}}{r^2}d\mathbf{r};\quad C=\frac{4}{\pi}\sqrt{\frac{\mu}{2m}}$$

As will be shown in the Appendix (formula 23), the sum of all the diagrams in which there is only one rescattering of the nucleons, f^I, has the following upper bound:

$$f^I<\frac{\alpha}{2\pi}C\int[\Psi_d(r)]^2\{\mathbf{r};\mathbf{k}\}^2\frac{a_{NN}}{r^2}d\mathbf{r}, \quad (12)$$

where

$$\{\mathbf{r};\mathbf{k}\}=\left(f_1+f_2+2\frac{f_1f_2}{r}e^{i\mathbf{k}\mathbf{r}-\varkappa r}\right)\left(1-\frac{f_1f_2}{r^2}e^{-2\varkappa r}\right)^{-1}. \quad (13)$$

In exactly the same way, we can obtain an estimate of the sum of all the diagrams with two rescattered nucleons (see the Appendix, formula (24)), etc. Summing the amplitudes f_N and the amplitude $f_{\pi d}$ (8), we obtain

$$f_{\pi d}=\frac{\alpha}{2\pi}\int[\Psi_d(r)]^2\{\mathbf{r};\mathbf{k}\}\left[1+\frac{C\{\mathbf{r};\mathbf{k}\}a_{NN}}{r^2}+\left(\frac{C\{\mathbf{r};\mathbf{k}\}a_{NN}}{r^2}\right)^2+\ldots\right]d\mathbf{r}$$

$$= \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \{r; k\} \left[\frac{r^2}{r^2 - C\{r; k\} a_{NN}}\right] dr. \qquad (14)$$

To estimate the integral (14), we take $\{\mathbf{r}; \mathbf{k}\}$ in the impulse approximation, i.e., we put $\{\mathbf{r}; \mathbf{k}\} = (f_1 + f_2)$. Substituting in (14) the real values of the πN amplitudes (1), we obtain

$$f_{\pi d} = \frac{\alpha}{2\pi}(f_1 + f_2) \int [\Psi_d(r)]^2 \left(\frac{r^2}{r^2 - C(f_1 + f_2)a_{NN}}\right) dr$$
$$= (f_1 + f_2)\left[1 + 2{,}15\sqrt{\frac{\mu}{2m}}\,\alpha(f_1 + f_2)\right]. \qquad (15)$$

Since $2.15\sqrt{\mu/2m\alpha}(f_1 + f_2) = -0.0035$, we can neglect this quantity completely, taking into account the fact that the accuracy with which the entire amplitude is calculated is μ/m.

Thus, allowance for multiple scattering of the pion gives for the πd-scattering length a correction of the order of $\alpha f_1 f_2$, as calculated in the impulse approximation; allowance for all the diagrams with rescattering of the nucleons gives for this quantity a correction of the order of $\sqrt{\mu/m\alpha}(f_1 + f_2)^2$. Since the $\sqrt{\mu/m}$ is not a sufficiently small parameter for pion scattering, this correction can be neglected in comparison with $\alpha f_1 f_2$ only because of the smallness of $(f_1 + f_2)$.[4)]

4. CONCLUDING REMARKS

It is first necessary to take into account the effect of charge exchange in the πN amplitudes

$$f_{\pi N} = f_0 + f_1 \mathbf{I}\boldsymbol{\tau}_N. \qquad (16)$$

Taking this into account, the amplitudes $f_{\pi d}^{(N)}$ take the form

$$\hat{f}_{\pi d}^{(1)} = \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \hat{P}_1\, dr,$$

$$\hat{f}_{\pi d}^{(2)} = \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \frac{e^{ikr - \varkappa r}}{r} \hat{P}_2\, dr$$

etc. The symbols $\hat{P}_N$ stand for the following operators:

$$\hat{P}_1 = (f_0 + f_1 \mathbf{I}\boldsymbol{\tau}_{N1}) + (f_0 + f_1 \mathbf{I}\boldsymbol{\tau}_{N2}),$$

$$\hat{P}_2 = (f_0 + f_1 \mathbf{I}\boldsymbol{\tau}_{N1})(f_0 + f_1 \mathbf{I}\boldsymbol{\tau}_{N2}) + (f_0 + f_1 \mathbf{I}\boldsymbol{\tau}_{N2})(f_0 + f_1 \mathbf{I}\boldsymbol{\tau}_{N1})$$

etc. Taking (16) into account, the amplitude $f_{\pi d}$ (7) is given by

$$f_{\pi d} = \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \left\{P_1 + \frac{e^{ikr - \varkappa r}}{r} P_2 + \frac{e^{-2\varkappa r}}{r^2} P_3 + \ldots\right\} dr,$$

where P_n ($n = 1, 2, 3$, etc.) are the mean values of the operators $\hat{P}_n$. The values of P_n were obtained in[6] [5)]. It was proved there that at zero isotopic spin of the nucleus, the quantities P_n are connected by the recurrence relation

$$P_{n+2} = P_n(f_0^2 - 2f_1^2). \qquad (17)$$

Using (17), and also the fact that $P_1 = 2f_0$ and $P_2 = 2(f_0^2 - 2f_1^2)$, we obtain for the πd-scattering amplitude

$$f_{\pi d} = \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \left\{\left(2f_0 + 2[f_0^2 - 2f_1^2]\frac{e^{ikr - \varkappa r}}{r}\right)\right.$$
$$\left. \times \left(1 - \frac{[f_0^2 - 2f_1^2]e^{-2\varkappa r}}{r^2}\right)^{-1}\right\} dr \qquad (18)$$

[4)]The other terms in the integral (15) are of the order of $\sqrt{\mu/M}$ $(f_1 + f_2)\alpha^2 f_1 f_2$.

[5)]In [6] they considered the general case of pion scattering by a nucleus with isotopic spin I.

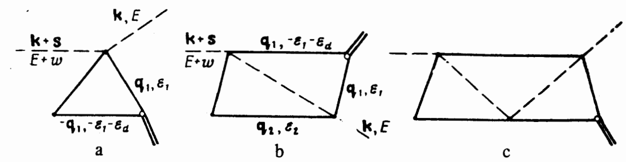

FIG. 5

Formula (18) is the final expression for the πd-scattering amplitude (in the case when $E > \epsilon_d$ it is necessary to replace κ by $-ip$). It should be noted that allowance for the isotopic spin in estimating the contributions of the remaining diagrams leads us to a formula similar to (14) for $f_{\pi d}$, with replacement of $\{\mathbf{r}, \mathbf{k}\}$ by the expression in the curly brackets of (18). It should also be noted that formula (18) is valid only when the p-wave in the πN scattering can be neglected ($E \lesssim$ (5–6) MeV). But at such low energies, the s-wave amplitudes of πN scattering (16) can be successfully replaced by the πN-scattering lengths (1).

Taking the last remark into account, we obtain the following expression for the πd-scattering length ($k = 0$):

$$a_{\pi d} = \int [\tilde{\Psi}_d(r)]^2 \left\{\left(2a_0 + 2[a_0^2 - 2a_1^2]\frac{e^{-\varkappa_0 r}}{r}\right)\left(1 - [a_0^2 - 2a_1^2]\frac{e^{-2\varkappa_0 r}}{r^2}\right)^{-1}\right\} dr, \qquad (19)$$

where $\kappa_0^2 = 2\mu\epsilon_d$. For the calculation, $\tilde{\Psi}_d(r)$ was chosen in the form proposed by Hulthen[11]

$$\tilde{\Psi}_d(r) = \sqrt{\frac{\alpha\beta(\alpha + \beta)}{2\pi(\beta - \alpha)^2}}\,\frac{e^{-\alpha r} - e^{-\beta r}}{r}.$$

The results of the calculations yield the following πd-scattering length:

$$a_{\pi d} = -0{,}06 \text{ F}. \qquad (20)$$

It should be noted that (19) does not include the terms corresponding to the kinetic energy of the nucleons in the intermediate state (the discarded terms of (8)). Owing to the motion of the nucleons in the deuteron, a contribution of the p-wave πN scattering should be present in the πd-scattering amplitude even at $k = 0$. Allowance for these factors should lead to a certain decrease of the πd-scattering length (20)[6)]. Further work in this direction is now under way.

In conclusion, I wish to thank I. S. Shapiro for suggesting the topic and useful discussions, L. A. Kondrayuk and Yu. A. Simonov for a useful discussion, and V. M. Kolybasov who attentively read the manuscript and made a number of valuable remarks.

APPENDIX

We calculate first the sum of all diagrams that contain one rescattering each. We proceed in the following manner: we first sum the parts of all the diagrams to the right of the nucleon rescattering amplitude (Fig. 5), combine into a block all the parts of the diagrams to the left of nucleon rescattering, and then write down the summary expression.

Figure 5a shows the part of the simplest diagram with rescattering of the nucleons. It corresponds to the expression $L_{(1)}^{\text{right}}$. Integrating with respect to the energy and using the transformation (4), we obtain

[6)]The need for taking these factors into account for the calculation of the πd-scattering length was noted by V. M. Kolybasov.

$$L_1^{\text{right}} = \frac{m^2 A_1 M_1}{(2\pi)^4} \int \frac{e^{i\mathbf{q}_1\mathbf{r}_1} \Psi_d(r_1)\, d\mathbf{r}_1\, d\mathbf{q}_1}{[\mathbf{q}_1; \mathbf{s}]}$$

where

$$[\mathbf{q}_1; \mathbf{s}] = [(\mathbf{q}_1 - \mathbf{s})^2 + q_1^2 + 2m(w + \varepsilon_d)].$$

We now consider the part of the diagram shown in Fig. 5b. Taking the integrals with respect to $d\epsilon_1$ and $d\epsilon_2$ and using the transformation (4), we obtain the following expression:

$$L_2^{\text{right}} = \frac{m^2 A_1 M_1}{(2\pi)^4} \int \frac{e^{i\mathbf{q}_1\mathbf{r}_1} f_2 e^{i\mathbf{k}\mathbf{r}_1 - \varkappa r_1} \Psi_d(r_1)}{[\mathbf{q}_1; \mathbf{s}]\, r}\, d\mathbf{r}_1\, d\mathbf{q}_1.$$

Now we sum all L_n^{right} (Fig. 5):

$$L^{\text{right}} = \sum_n L_n^{\text{right}} = \frac{m^2 M_1}{(2\pi)^3 \mu} \int \frac{e^{i\mathbf{q}_1\mathbf{r}} \{\mathbf{r}_1; \mathbf{k}\} \Psi_d(r_1)}{[\mathbf{q}_1; \mathbf{s}]}\, d\mathbf{r}_1\, d\mathbf{q}_1, \tag{21}$$

with $\{\mathbf{r}; \mathbf{k}\}$ determined by expression (13). It is possible to sum in the same manner all the L_n^{left}:

$$L^{\text{left}} = \sum_{n=1}^{\infty} L_n^{\text{left}} = \frac{m^2 M_1}{(2\pi)^3 \mu} \int \Psi_d(r) \{\mathbf{r}; \mathbf{k}\} \frac{e^{-i\mathbf{q}\mathbf{r}}\, d\mathbf{q} d\mathbf{r}}{[\mathbf{q}; \mathbf{s}]}. \tag{22}$$

Using (21) and (22), we readily write the sought sum of all the diagrams $\tilde{M}^{\text{I}}$:

$$\tilde{M}^{\text{I}} = M_{NN} \frac{m^4 M_1^2 (-2i\mu)}{(2\pi)^6 \mu^2 (2\pi)^4} \int \Psi_d(r) \frac{\{\mathbf{r}; \mathbf{k}\} e^{-i\mathbf{q}\mathbf{r}}}{[\mathbf{q}; \mathbf{s}]} \cdot \times \frac{e^{i\mathbf{q}_1\mathbf{r}_1} \{\mathbf{r}_1; \mathbf{k}\} \Psi_d(r_1)\, dw\, ds\, d\mathbf{q}\, d\mathbf{q}_1}{(\mathbf{k}; \mathbf{s})\, [\mathbf{q}_1; \mathbf{s}]}\, d\mathbf{r}\, d\mathbf{r}_1.$$

Here $(\mathbf{k}; \mathbf{s}) = [(\mathbf{k} + \mathbf{s})^2 - 2\mu(E + w)]$.

The integration with respect to dw and ds is carried out in exactly the same manner as in the derivation of (12). We therefore obtain finally for $\tilde{f}^{\text{I}} = \mu \tilde{M}^{\text{I}} / 2\pi$

$$f^{\text{I}} < \frac{\alpha}{2\pi} C \int [\Psi_d(r)]^2 \{\mathbf{r}; \mathbf{k}\}^2 \frac{a_{NN}}{r^2}\, d\mathbf{r}. \tag{23}$$

We calculate in a similar manner the sum with two rescatterings of the nucleons. In addition to the blocks L^{right} and L^{left} summed above, this sum will contain also a certain block L^{mid}, located between the two amplitudes with nucleon rescattering. This block is calculated in the same manner as L^{right}. Taking L^{mid} into account, the amplitude M^{II} is given by

$$\tilde{M}^{\text{II}} = M_{NN}^2 \frac{m^4 M_1^2}{(2\pi)^6 \mu^2} \frac{m^2}{2^3 \pi^5 \mu} \frac{(-2i\mu)^2}{(2\pi)^8} \int \Psi_d(r) \frac{\{\mathbf{r}; \mathbf{k}\} e^{-i\mathbf{q}\mathbf{r}}}{[\mathbf{q}; \mathbf{s}]\, (\mathbf{k}; \mathbf{s})} \frac{e^{i\mathbf{q}_1\mathbf{r}_1} \{\mathbf{r}_1; \mathbf{k}\}}{[\mathbf{q}_1; \mathbf{s}]} \times \frac{e^{-i\mathbf{q}_2\mathbf{r}_1}}{[\mathbf{q}_2; \mathbf{s}_1]' (\mathbf{k}; \mathbf{s}_1)'} \frac{e^{i\mathbf{q}_3\mathbf{r}_2} \{\mathbf{r}_2; \mathbf{k}\}}{[\mathbf{q}_3; \mathbf{s}_1]} \Psi_d(r_2)\, dw\, dw'\, ds\, ds_1\, d\mathbf{q}\, d\mathbf{q}_1\, d\mathbf{q}_2\, d\mathbf{q}_3\, d\mathbf{r}\, d\mathbf{r}_1\, d\mathbf{r}_2,$$

where the quantities $[\mathbf{q}; \mathbf{s}]'$ and $(\mathbf{k}; \mathbf{s})'$ are obtained from the corresponding unprimed quantities by making the substitution $w \to w'$.

The final expression for the estimate of $\tilde{f}^{\text{II}} = \mu \tilde{M}^{\text{II}} / 2\pi$ is

$$f^{\text{II}} < \frac{\alpha}{2\pi} C^2 \int [\Psi_d(r)]^2 \frac{a_{NN}^2 \{\mathbf{r}; \mathbf{k}\}^3}{r^4}\, d\mathbf{r}. \tag{24}$$

We can estimate analogously an arbitrary term of the series $\tilde{f}^N$. The total amplitude

$$\left(\tilde{f} = f_{nd} + \sum f^N\right)$$

will then be determined by the inequality

$$\tilde{f} < \frac{\alpha}{2\pi} \int [\Psi_d(r)]^2 \{\mathbf{r}; \mathbf{k}\} \left[1 + \frac{C a_{NN} \{\mathbf{r}; \mathbf{k}\}}{r^2} + \left(\frac{C a_{NN} \{\mathbf{r}; \mathbf{k}\}}{r^2}\right)^2 + \ldots\right] d\mathbf{r},$$

where $C = 4\pi^{-1} \sqrt{\mu / 2m}$.

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