INTERACTION BETWEEN A CAVITY AND A VORTEX IN A SUPERCONDUCTOR OF THE SECOND KIND

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The free energy of a superconducting vortex in a superconductor of the second kind interacting with a hollow cylindrical channel of radius $r \ll \delta_0$ parallel to it is calculated (δ_0 is the penetration depth). It is shown that on this assumption, capture of only a single vortex by the channel is energetically favorable. The pinning force is calculated to be $f_p = H_{cm}^2 \xi(T)/2$, where H_{cm} is the critical thermodynamic field strength.

1. INTRODUCTION AND FORMULATION OF PROBLEM

THE large values of the critical currents in rigid superconductors are usually attributed to the sticking of superconducting vortices to different inhomogeneities in a superconductor of the second kind, i.e., to pinning of the vortices. Pinning of the vortices on the external surface of the superconductor, which should also be treated as an inhomogeneity, is considered in[1,2]. In this paper we make the next step in the construction of the theory of vortex pinning in superconductors of the second kind, namely, we investigate the interaction between a superconducting vortex and a cavity inside the superconductor, i.e., we investigate now the role of an internal surface in a superconductor of the second kind.

The simplest case of such a system (vortex and surface) is an infinite-length, round, and empty channel in a superconductor, interacting with a superconducting vortex parallel to it. In this paper we obtain the energy of such a system, calculate the pinning force, and estimate the critical current determined by this force.

Thus, let us consider an infinite cylindrical cavity of radius r in an infinite ideally homogeneous superconductor of the second kind, and a superconducting vortex parallel to this cavity and located at a distance ρ_0 from its center. Let the magnetic field inside the cavity be H_0 and directed parallel to the axis of the cavity. It is assumed that the constant of the Ginzburg-Landau (GL) theory is $\kappa \gg 1$, and that the radius of the cavity satisfies the inequality $\kappa^{-1} \ll r \ll 1$. Here and throughout we use the relative units of the GL theory: the unit of length is the depth of penetration of the weak magnetic field $\delta_0(T)$, and the unit of magnetic field intensity is $\sqrt{2}H_{cm}$, where H_{cm} is the critical thermodynamic magnetic field. Assuming for the field in the cavity $H_0 \ll H_{c2}$, we can write the GL equation for the field in the superconductor in the form

$$\mathbf{H}(\rho) + \operatorname{rot}\operatorname{rot}\mathbf{H}(\rho) = \frac{2\pi}{\varkappa}\delta(\rho-\rho_0)\mathbf{e}, \quad \mathbf{H}|_s = H_0. \tag{1}$$

Here S is the surface of the cavity of radius r, **e** is a unit vector directed along the vortex, and ρ_0 is the coordinate of the vortex.

It is required to find the dependence of the free energy of this system (vortex and cavity) on the distance ρ_0. From this we obtain directly an expression for the pinning force.

2. SOLUTION OF EQUATION

We change over to cylindrical coordinates (ρ, φ) with center on the cavity axis. If the vortex filament is located at a point with coordinates $(\rho_0, 0)$, then Eq. (1) takes the form

$$\frac{\partial^2 H}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial H}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 H}{\partial\varphi^2} - H = -\frac{2\pi}{\varkappa\rho}\delta(\varphi)\delta(\rho-\rho_0), \tag{2}$$
$$H(r,\varphi) = H_0, \quad H(\infty,\varphi) = 0.$$

We seek the solution of this equation in the form

$$H = H_1(\rho) + H_2(\rho, \varphi),$$

where H_1 and H_2 satisfy respectively the following equations with the following boundary conditions:

$$\frac{\partial^2 H_1}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial H_1}{\partial\rho} - H_1 = 0, \tag{3}$$
$$H_1(r) = H_0, \quad H_1(\infty) = 0;$$

$$\frac{\partial^2 H_2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial H_2}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 H_2}{\partial\varphi^2} - H_2 = -\frac{2\pi}{\varkappa\rho}\delta(\varphi)\delta(\rho-\rho_0), \tag{4}$$
$$H_2(r,\varphi) = 0, \quad H_2(\infty,\varphi) = 0.$$

The solution of (3) is obtained immediately:

$$H_1(\rho) = H_0 K_0(\rho) / K_0(r),$$

where K_0 is a Hankel function of zero order and imaginary argument. Expanding $H_2(\rho, \varphi)$ in a Fourier series with respect to the variable φ:

$$H_2(\rho,\varphi) = \sum_{k=-\infty}^{\infty} H_k(\rho)e^{ik\varphi}. \tag{5}$$

and substituting this expansion in (5), we obtain the following equation for $H_k(\rho)$:

$$\frac{\partial^2 H_k}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial H_k}{\partial\rho} - \left(1+\frac{k^2}{\rho^2}\right)H_k = -\frac{1}{\varkappa\rho}\delta(\rho-\rho_0),$$
$$H_k(r) = 0, \quad H_k(\infty) = 0.$$

The solution of this equation is

$$H_k(\rho) = \frac{1}{\varkappa}\frac{K_k(\rho_0)}{K_k(r)}[I_k(\rho)K_k(r) - I_k(r)K_k(\rho)],$$
$$r \leqslant \rho \leqslant \rho_0,$$

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$$H_k(\rho)=\frac{1}{\varkappa}\frac{K_k(\rho)}{K_k(r)}\left[I_k(\rho_0)K_k(r)-I_k(r)K_k(\rho_0)\right], \qquad (6)$$
$$\rho_0\leqslant\rho\leqslant\infty.$$

Thus, the total magnetic field inside the superconductor takes the final form

$$H(\rho,\varphi)=H_0\frac{K_0(\rho)}{K_0(r)}+\sum_{k=-\infty}^{\infty}H_k(\rho)e^{ik\varphi} \qquad (7)$$

where $H_k(\rho)$ is given by (6).

To solve the problem completely, it remains to determine the field H_0 in the cavity. By virtue of the condition of quantization of the magnetic flux in superconductors, the field H_0 cannot be arbitrary, and must be determined. Under our initial assumptions ($\kappa \gg 1$, $H_0 \ll H_{c2}$), the second GL equation can be written in the form

$$\operatorname{rot}\mathbf{H}=\varkappa^{-1}\nabla\theta-\mathbf{A},$$

where θ is the phase of the wave function of the GL theory, and $\mathbf{A}$ is the vector potential. Integrating this equation along the circular contour of our cavity, i.e., along a circle of radius r, we have

$$r\int_0^{2\pi}\operatorname{rot}_\varphi\mathbf{H}d\varphi=\frac{2\pi}{\varkappa}n-\pi r^2H_0, \quad n=0,1,2,\ldots$$

Substituting here Eqs. (6) and (7), we obtain after simple transformations

$$H_0=\frac{1}{\varkappa r}\frac{K_0(\rho_0)+nK_0(r)}{K_1(r)+{}^1/_2rK_0(r)}.$$

Since, by definition, $r \ll 1$, we get $K_1(r) \sim r^{-1}$, $rK_0(r) \sim -r\ln r$, and therefore the second term in the denominator can be neglected and we obtain ultimately

$$H_0=\frac{K_0(\rho_0)}{\varkappa}+n\frac{K_0(r)}{\varkappa}. \qquad (8)$$

This shows clearly the makeup of the field in the cavity. The first term is the field produced in the cavity by the nearby vortex filament. This quantity, naturally, is not quantized. The second term is the field determined by the number n of magnetic-flux quanta captured by the cavity. This quantity is quantized and equals the field remaining in the cavity if the vortex filament is removed to infinity. Expression (8) also shows clearly how the field H_0 increases with increasing n. Such an increase does not occur jumpwise—the field H_0 increases continuously as the filament comes closer to the cavity. At the instant when ρ_0 becomes equal to r, the filament vanishes, but H_0 becomes equal to $(n + 1)K_0(\rho_0)/\kappa$. This means that the cavity has captured one more quantum of the magnetic flux.

3. CALCULATION OF THE FREE ENERGY

The free energy of the system in question is determined by the formula

$$\mathcal{F}=\int(H^2+(\operatorname{rot}\mathbf{H})^2)dV. \qquad (9)$$

Let us find the free energy of a superconductor layer of unit thickness, contained between the planes z = 0 and z = 1. Substituting Eqs. (6), (7), and (8) in (9), using the theory proved in the appendix of[2], and performing simple calculations, we obtain

$$\mathcal{F}=2\pi\frac{H_0^2}{K_0(r)}+\frac{2\pi}{\varkappa}H_2(\rho_0-\varkappa^{-1},0), \qquad (10)$$

if the vortex filament is at the point $(\rho_0, 0)$.

Let us find $H_2(\rho_0 - \kappa^{-1}, 0)$. To this end we substitute (6) in (5):

$$H_2(\rho_0-\varkappa^{-1},0)=\frac{1}{\varkappa}\sum_{k=-\infty}^{\infty}\Big[K_k(\rho_0)I_k(\rho_0-\varkappa^{-1})-\frac{I_k(r)}{K_k(r)}K_k^2(\rho_0)\Big]. \qquad (11)$$

In the last term, the sum converges if $\rho = \rho_0$, and we therefore put immediately $\rho = \rho_0$ in place of $\rho = \rho_0 - \kappa^{-1}$.

We consider first the case $\rho_0 \ll 1$. Then the sum in (11) can be easily calculated and

$$H_2(\rho_0-\varkappa^{-1},0)=\frac{K_0(\varkappa^{-1})}{\varkappa}-\frac{K_0^2(\rho_0)}{\varkappa K_0(r)}+\frac{1}{\varkappa}\ln\left(1-\frac{r^2}{\rho_0^2}\right). \qquad (12)$$

Substituting now in formula (10) the expression (8) for H_0 and (12) for $H_2(\rho_0 - \kappa^{-1}, 0)$, we obtain the following expression for the free energy of the system:

$$\mathcal{F}=\mathcal{F}_0+\frac{4\pi}{\varkappa^2}\left[\frac{1}{2}\ln\left(1-\frac{r^2}{\rho_0^2}\right)+nK_0(\rho_0)\right], \quad \rho_0\ll1, \qquad (13)$$

$$\mathcal{F}_0=\frac{4\pi}{\varkappa}\Big[H_{c1}+\frac{n^2}{2\varkappa}K_0(r)\Big]. \qquad (14)$$

Here $\mathcal{F}_0$ denotes that part of the system free energy which is the sum of the self-energies of the vortex filament and of the cavity with n quanta of the magnetic flux. The second term of (13) gives the energy of interaction between the filament and the cavity. In formula (14) we use the well-known expression for the first critical field H_{c1} at $\kappa \gg 1$: $H_{c1} = (2\kappa)^{-1}\ln\kappa$.

In the second limiting case $\rho_0 \gg 1$, it is easy to show that

$$\mathcal{F}=\mathcal{F}_0+\frac{4\pi}{\varkappa^2}nK_0(\rho_0), \quad \rho_0\gg1, \quad n\neq0. \qquad (15)$$

Let us turn to formula (13). If $r \ll \rho_0 \ll 1$, then it is easy to see that the logarithmic term will be much smaller than the quantity $nK_0(\rho_0)$ if $n \neq 0$. Therefore, comparing formulas (13) and (15), we can state that at any ρ_0 (if $\rho_0 \gg r$) the expression for the free energy will be determined by formula (15). On the other hand, if $\rho_0 \sim r$, it is necessary to use (13).

4. DISCUSSION OF RESULTS

The plots of $\mathcal{F}(\rho_0)$ are shown schematically in the figure. As seen from the figure, the $\mathcal{F}(\rho_0)$ curve is monotonic only if n = 0, i.e., a cavity in which not a single flux quantum is captured always attracts the superconducting vortex to itself. If $n \geq 1$, then a qualitative change takes place in $\mathcal{F}(\rho_0)$. There is now a potential barrier between the vortex and the cavity. Its position ρ_{0n} can be readily determined from the equation $\partial\mathcal{F}/\partial\rho_0 = 0$ by using (13):

$$\rho_{0n}=r[(1+n)/n]^{1/2}. \qquad (16)$$

From (16) it follows that with increasing n the position of the maximum of $\mathcal{F}(\rho_0)$ approaches the surface of the cavity. The barrier then disappears when the maximum reaches the surface of the cavity. The equations used in this paper are valid when the core of the vortex is far (compared with κ^{-1}) from the surface, i.e., at $\rho_0 - r \gg \kappa^{-1}$. However, the correct order of magnitude can be obtained also for a vortex on the surface of the cavity, by putting for this case $\rho_0 = r + \kappa^{-1}$. Therefore the condition for the disappearance of the barrier is written in the form $\rho_{on} = r + \kappa^{-1}$. The number of quanta captured by the cavity then reaches saturation and becomes equal to n_s. Putting $n_s \gg 1$, we get from (16)

$$n_s \sim \varkappa r / 2.$$

The cavity cannot retain a larger number of flux quanta.

Let us find now the energy of the vortex located on the cavity surface itself, i.e., $\mathscr{F}(r)$. Using (13), we get

$$\mathscr{F}(r) = \mathscr{F}_0 + \frac{2\pi}{\varkappa^2}\ln\frac{2}{\varkappa r} + \frac{4\pi n}{\varkappa^2}\ln\frac{2}{\gamma r}, \tag{17}$$

where $\gamma = e^C \approx 1.78$. We have used here the condition $r \ll 1$. From (17) it follows that $\mathscr{F}(r)$ can be either larger or smaller than $\mathscr{F}_0$. At $n = 0$ we have $\mathscr{F}(r) < \mathscr{F}_0$ (see the figure). Therefore, indeed, the capture of one quantum is always favored. But after this single quantum is captured by the cavity, the capture of the next quantum will be energywise favored or not, depending on whether the point $\mathscr{F}(r)$ lies below or above the level $\mathscr{F}_0$.

Let us estimate the number of quanta n_0 starting with which the capture of the $(n_0 + 1)$-st quantum comes to be not favored energywise. We put to this end in (17) $\mathscr{F}(r) = \mathscr{F}_0$ and solve the resultant equation with respect to n_0:

$$n_0 = \frac{1}{2}\ln\frac{\varkappa r}{2} \Big/ \ln\frac{2}{\gamma r}.$$

We see therefore that n_0 does not exceed unity greatly. Indeed, if $n_0 = 1$, then $r \approx 0.3$ at $\kappa \approx 100$. This means that for a cavity whose radius is one-third of the depth of penetration or less, capture of the second quantum is no longer favored. And for a cavity with $r \approx 0.5$, capture of the third quantum is not favored.

Since in our calculations we assumed $r \ll 1$, we shall henceforth assume the capture of only one quantum to be favored energywise.

We now find the mechanical force of interaction between the vortex filament and the cavity, i.e., the pinning force. The force per unit filament length is

$$f_p = -\frac{\partial \mathscr{F}}{\partial \rho_0}\bigg|_{\rho_0 = r + \varkappa^{-1}}$$

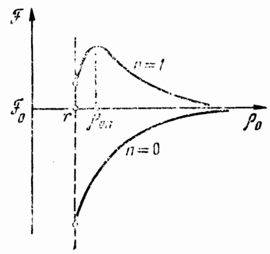
On the basis of (13) we have

$$|f_p| = \frac{2\pi}{\varkappa}\left(1 - \frac{2n}{\varkappa r}\right). \tag{18}$$

We see therefore that the maximum pinning is possessed by a cavity free of captured flux. The pinning force becomes equal to zero when the number of captured quanta reaches saturation (n_s). The maximum pinning force at $n = 0$ is $f_{pmax} = 2\pi/\kappa$ and thus is independent of r (when $r \ll 1$).

Changing over to absolute Gaussian units, we obtain

$$f_{p\,max} = {}^1\!/_2 H_{cm}{}^2 \xi(T). \tag{19}$$



Typical quantities for superconductors of the second kind are $H_{cm} \sim 10^3$ Oe and $\xi(T) \sim 10^{-6}$ cm; then $f_{pmax} \sim 0.5$ dyne/cm. Let us use now formula (19) to estimate the force of pinning of a vortex by a spherical cavity of radius r. Since f_{pmax} (19) is the pinning force per unit length of the vortex, it follows that, for all the lack of rigor in our approach, we can still hope to obtain an estimate of the required force of pinning of the vortex by a spherical cavity by using the expression

$$F_p \sim 2rf_{p\,max} = H_{cm}{}^2 \xi r. \tag{20}$$

The force of pinning of a vortex by a spherical cavity of radius r is estimated in[3,4]. Formula (20) coincides with the results of these papers with logarithmic accuracy, if we assume in the latter results $r \sim \xi(T)$. Thus, from our point of view, the results of[3,4] underestimate the pinning forces.

Let us estimate in conclusion the transport superconducting current necessary to overcome the pinning force (19). Using the well-known expression for the Lorentz force acting on the vortices during the flow of transport current

$$f_L = \frac{1}{c} j_{tr} \Phi_0,$$

where Φ_0 is the quantum of the magnetic flux, and equating this force to the pinning force (19), we obtain

$$j_c \simeq \frac{cH_{cm}{}^2\xi(T)}{2\Phi_0} \simeq \frac{cH_{cm}}{2\sqrt{2}\delta_0(T)}.$$

Using the previous estimates for H_{cm} and ξ, we obtain $j_c \sim 2 \times 10^7$ A/cm^2.

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