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INJECTION OF A RELATIVISTIC ELECTRON BEAM INTO A PLASMA

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The response of a plasma to injection of a low density relativistic electron beam is investigated. It is shown that if $r_0 > c/\omega_p$ where r_0 is the beam radius and ω_p the plasma frequency, all plasma perturbations are localized in the region of the electron beam itself; all currents induced in the plasma tend to cancel out the magnetic field of the beam, facilitating in this way its injection into the plasma. If the beam injection time is $\tau > \tau_0 = \nu^{-1}(r_0\omega_p/c)^2$ where ν is the plasma electron collision frequency, then magnetic field cancellation occurs at distances $z < z_0 = u\tau_0$ from the front of the beam, u being the directed electron velocity. Hence in the case of a high-current electron beam, when the magnetic energy of the beam current exceeds the electron kinetic energy, easy injection is possible only in a dense plasma providing $\omega_p > c/r_0$ and $\tau < \tau_0$.

1. INTRODUCTION

THE theory of the investigation of an electron beam with a plasma under stationary conditions has been the subject of a tremendous number of papers. There have been numerous investigations of the critical beam and plasma parameters at which the system becomes unstable, and studies of the spectra of the oscillations excited thereby and of the character of the relaxation of the two-stream instability (see the reviews[1-3] and the literature cited there). The establishment of the stationary state of an electron beam injected into a plasma, on the other hand, has been investigated relatively little. This question was dealt with in essence only in the paper of Hammer and Rostoker[4], in which the first attempt was made to solve the problem of beam injection into a plasma. Namely, they investigated the induced charges, currents, and electromagnetic waves produced when an electron beam is injected into a space occupied by a plasma. They noted a number of interesting effects connected with the charge and current compensation (“magnetic neutralization”) of the beam in definite regions of the plasma (see also[5]).

It should be noted that the asymptotic method of solving the plasma, suitable at long times exceeding the characteristic times of the problem, has prevented Hammer and Rostoker[4] from investigating the role of radiation when the beam is injected into a plasma. In addition, an unfortunate choice of the coordinate system has so complicated the calculations, that to simplify the problem they had to neglect the diffusion of the magnetic field into the plasma, allowance for which, as will be shown below, leads to quantitative changes in their results, and even to qualitative ones at large distances. We use a simple and natural method of solving the problem of perturbation of a plasma by a given electron beam, and the derived general formulas are suitable for a description of the system at any instant of time upon injection of an electron beam of finite length into a plasma. For a comparison with the results of[4], however, we investigated most thoroughly the asymptotic limit of long times.

2. FORMULATION OF PROBLEM

Assume that starting with the instant t = 0 and during a time τ an electron beam of radius r_0 is injected into an unbounded plasma at a point z = 0, with a velocity u parallel to the z axis. The beam charge density under conditions when its spreading can still be neglected, is written in the form

$$\rho^{(0)} = en\eta(r_0 - r)\begin{cases} \eta[z - u(t-\tau)] - \eta(z - ut), & t > \tau, \\ \eta(z) - \eta(z - ut), & t < \tau, \end{cases} \tag{2.1}$$

where $\eta(x) = 1$ at $x > 0$ and $\eta(x) = 0$ when $x < 0$. The electron beam perturbs the surrounding plasma and induces in it charges, currents, and electromagnetic fields described by the field equations

$$\begin{gathered} \operatorname{div}\mathbf{E} = 4\pi(\rho + \rho^{(0)}), \quad \operatorname{div}\mathbf{B} = 0, \\ \operatorname{rot}\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}(\mathbf{j} + \mathbf{j}^{(0)}), \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \\ \frac{\partial}{\partial t}(\rho + \rho^{(0)}) + \operatorname{div}(\mathbf{j} + \mathbf{j}^{(0)}) = 0. \end{gathered} \tag{2.2}$$

Here ρ and j are the densities of the charge and current induced in the plasma, and $\rho^{(0)}$ and $\mathbf{j}^{(0)}$ are the charge and current of the beam electrons.

Two formulation of the problem are now possible:

a) If we assume that $\mathbf{j}^{(0)} = \rho^{(0)}\mathbf{u}$, then we have the obvious relations

$$\frac{\partial\rho^{(0)}}{\partial t} + \operatorname{div}\rho^{(0)}\mathbf{u} = -\frac{\partial\rho}{\partial t} - \operatorname{div}\mathbf{j} = en\eta(r_0 - r)\delta\left(\frac{z}{u}\right)[\eta(t) - \eta(t-\tau)]. \tag{2.3}$$

In such a formulation of the problem, production of beam electrons and annihilation of plasma electrons take place at the point z = 0, and the total charge in the system is conserved.

b) On the other hand if we assume that

$$\frac{\partial\rho^{(0)}}{\partial t} + \operatorname{div}\rho^{(0)}\mathbf{u} = \frac{\partial\rho}{\partial t} + \operatorname{div}\mathbf{j} = 0, \tag{2.4}$$

then this means that production and annihilation of the electrons occurs neither in the volume of the plasma nor on the surface of the injector; these processes are possible only on an infinitely remote surface.

Finally, let us discuss briefly the employed plasma model. We assume that external fields exist in the plasma and, being interested only in rapid processes, we neglect the thermal motion of the particles. To describe the linear electromagnetic properties of such a plasma we use the following expression for the dielectric tensor:

$$\varepsilon_{ij}(\omega) = \delta_{ij}\varepsilon(\omega) = \delta_{ij}\left[1 - \frac{\omega_p^2}{\omega(\omega + i\nu)}\right], \quad (2.5)$$

where $\omega_p = [4\pi e^2 N/m]^{1/2}$ is the plasma frequency and ν is the plasma-electron collision frequency. We neglect the beam-electron collisions, assuming the ratio v_{Te}/u to be a small parameter (v_{Te} is the thermal velocity of the plasma electrons)[1].

Thus, the problem of injection of an electron beam into a plasma reduces to a solution of the field equations (2.2) with specified sources $\rho^{(0)}$ and $\mathbf{j}^{(0)}$ and with known plasma properties described by the dielectric constant (2.5). We note that such an approximation, which corresponds to neglecting the contribution of the perturbations to the dielectric constant of the system, is valid of the beam density is low, when $n \ll N$. This condition will be henceforth assumed satisfied.

3. FIELDS AND CURRENTS INDUCED IN THE PLASMA

To solve the field equations, we shall use the Fourier-Laplace transform method, assuming that at $t = 0$ there are no perturbations in the plasma:

$$A(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{-\infty+i\delta}^{+\infty+i\delta} d\omega e^{-i\omega t} \int d\mathbf{k} e^{i\mathbf{kr}} A(\omega, \mathbf{k}),$$

$$A(\omega, \mathbf{k}) = \int_0^\infty dt e^{i\omega t} \int d\mathbf{r} e^{-i\mathbf{kr}} A(\mathbf{r}, t). \quad (3.1)$$

The field-equation system (2.2) then reduces to

$$\left\{k^2\delta_{ij} - k_i k_j - \frac{\omega^2}{c^2}\varepsilon_{ij}(\omega, \mathbf{k})\right\} E_j(\omega, \mathbf{k}) = \frac{4\pi i\omega}{c^2} j_i^{(0)}(\omega, \mathbf{k}), \quad (3.2)$$

where $\mathbf{j}^{(0)}(\omega, \mathbf{k})$ is the Fourier transform of the beam current density[2]

$$\mathbf{j}^{(0)}(\omega, \mathbf{k}) = \frac{en(\omega, \mathbf{k})\mathbf{u}\omega}{(\omega; \mathbf{ku})} = \frac{2\pi enr_0 u\mathbf{u} J_1(k_\perp r_0)}{k_\perp(\omega - \mathbf{ku})(\omega; \mathbf{ku})}(e^{i\omega\tau} - 1). \quad (3.3)$$

Here $J_n(x)$ is a Bessel function, and the two denominators correspond to the two aforementioned formulations of the problem.

From (3.2), using (2.5), we readily obtain the Fourier transform of the electric field $\mathbf{E}$ induced in the plasma by the electron beam:

$$\mathbf{E}(\omega, \mathbf{k}) = \frac{4\pi i}{\omega} \frac{\omega^2 c^{-2}\varepsilon\mathbf{j}^{(0)} - \mathbf{k}(\mathbf{kj}^{(0)})}{\varepsilon(k^2 - \omega^2 c^{-2}\varepsilon)}. \quad (3.4)$$

Using the field equations (2.2), we can easily determine the Fourier transforms and the other perturbed quantities.

Substitution of (3.4) into the inverse-transform of (3.1) solves our problem of determining the fields and currents induced in the plasma at any instant of time and at any point of space.

We confine ourselves henceforth to an analysis of the asymptotic expressions, which are valid in the limit of long times, for the perturbed quantities. If we are interested here in the region near the electron beam, then it is necessary to take the limit as $t \to \infty$ in such a way as to leave invariant the time $t' = t - z/u$. According to the Laplace theorem we have for the asymptotic expressions in this case

$$\lim_{t\to\infty} A(\mathbf{r}, t) = -\frac{i}{(2\pi)^3}\int d\mathbf{k} \lim_{\omega\to\mathbf{ku}} (\omega - \mathbf{ku}) A(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{kr}}. \quad (3.5)$$

It is easily seen that in this limit the solutions of the field equations coincide identically for the two problems formulated above.

Substituting (3.4) in (3.5) we obtain for the asymptotic values of the induced fields and currents in the plasma the following expressions:

$$\mathbf{E}(r, t) = \frac{E_0}{2\pi}\int_0 dk_\perp \int_{-\infty}^{\infty} dk_z J_1(k_\perp r_0)\frac{\mathbf{g}}{\varepsilon}\chi,$$

$$\mathbf{B}(r, t) = \frac{iB_0}{2\pi}\int_0^\infty k_\perp dk_\perp \int_{-\infty}^{\infty} dk_z J_1(k_\perp r_0) J_1(k_\perp r)\frac{\chi}{k_z}\mathbf{e}_\varphi,$$

$$\mathbf{j}(r, t) = \frac{i\omega_p^2 r_0 j_0}{2\pi u}\int_0^\infty dk_\perp \int_{-\infty}^{\infty} dk_z J_1(k_\perp r_0)\frac{\mathbf{g}\chi}{\varepsilon(k_z u + i\nu)}, \quad (3.6)$$

$$\rho(r, t) + \rho^{(0)}(r, t) = \frac{i\rho_0 r_0}{2\pi}\int_0^\infty dk_\perp \int_{-\infty}^{\infty} dk_z J_1(k_\perp r_0) J_0(k_\perp r)\left(k^2 - \frac{\omega^2}{c^2}\varepsilon\right)\frac{\chi}{k_z\varepsilon};$$

$$\mathbf{g} = \left[\frac{ik_\perp}{k_z}J_1(k_\perp r)\mathbf{e}_r + \left(1 - \frac{u^2}{c^2}\varepsilon\right)J_0(k_\perp r)\mathbf{e}_z\right],$$

$$\chi = \left(k^2 - \frac{\omega^2}{c^2}\varepsilon\right)^{-1}(e^{ik_z z'} - e^{ik_z z_1}).$$

Here $\rho_0 = en$, $j_0 = \rho_0 u$, $E_0 = 4\pi\rho_0 r_0$, $B_0 = E_0 u/c$, e_i are unit vectors in a cylindrical coordinate system, and the quantities $z' = z - ut$ and $z_1 = z' + u\tau$ are the distances from the leading and trailing edges of the electron beam, respectively.

The integrals (3.6) are determined by the poles of the integrands, which coincide with the roots of the dispersion relations of the longitudinal ($\epsilon = 0$) and transverse ($k^2c^2 = \omega^2\epsilon$) waves in the plasma. Under the condition $u\omega_p/c \gg \nu$ these poles are located at the points

$$(k_z)_{1,2} = \frac{1}{u}\left(\pm\omega_p - i\frac{\nu}{2}\right), \quad (k_z)_{3,4} = \pm i\gamma\sqrt{k_\perp^2 + \frac{\omega_p^2}{c^2}}, \quad (3.7)$$

$$(k_z)_5 = -i\frac{\nu}{u}\frac{k_\perp^2}{k_\perp^2 + \omega_p^2/c^2} \qquad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

The first two roots describe the longitudinal plasma oscillations and the next two the transverse waves (which in this case experience polarization screening), while the last root corresponds to diffuse penetration of the transverse field into the plasma. This last pole is the one disregarded in[4].

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[1]We note that the entire analysis is valid during a time shorter than the free-path time of the beam electrons. Otherwise, owing to the beam scattering, the expression (2.1) and the entire model assumed above become meaningless.

[2]In writing down (4.4) we have assumed that the electron density of the beam is homogeneous in the region $r \leqslant r_0$. Generalization of the theory developed below to the case of an inhomogeneous beam n(r) entails no difficulty, but it leads to complications and to such cumbersome formulas, that the results are no longer clear. We note in addition that in real experiments, owing to the focusing action of the external and self-magnetic fields, the boundary of the beam is always sufficiently sharp.

As seen from (3.7), four poles lie in the lower half-plane of the complex variable k_z, and one in the upper half-plane. When integrating with respect to k_z in (3.6), the contour of integration at $z' > 0$ (or $z_1 > 0$) must be closed with an upper semicircle, and at $z' < 0$ (or $z_1 < 0$) with a lower semicircle. As a result of the integration we obtain

$$\rho = \rho_0\eta(r_0 - r)\{a_2[F(z') - 1] + a_3[F(z') - F(z_1)]\},$$

$$j_z = \frac{j_0\omega_p^2 r_0}{u^2}\Big[\frac{a_1}{2\gamma^2}[\Gamma_{02}(z') - \Gamma_{02}(z_1)] - a_2\Big[\frac{1}{2\gamma^2}(\Gamma_{02}(z') + \Gamma_{02}(z_1))$$
$$- \Psi_{00}(a)F(z') + \frac{u^2}{c^2}T_{00}(z')\Big] - a_3\Big[\frac{1}{2\gamma^2}(\Gamma_{02}(z') - \Gamma_{02}(z_1))$$
$$- \Psi_{00}(u)(F(z') - F(z_1)) + \frac{u^2}{c^2}(T_{00}(z') - T_{00}(z_1))\Big]\Big\},$$

$$j_r = \frac{j_0\omega_p^2 r_0}{u^2}\Big\{-\frac{a_1}{2\gamma^4}[\Gamma_{11}'(z') - \Gamma_{11}'(z_1)] + a_2\Big[\frac{1}{2\gamma^2}(\Gamma_{11}'(z') + \Gamma_{11}'(z_1))$$
$$+ \frac{u^2}{\omega_p^4}\Psi_{11}(u)F'(z') + \frac{u^2}{c^2}T_{1,-1}(z')\Big] + a_3\Big[\frac{1}{2\gamma^2}(\Gamma_{11}'(z') - \Gamma_{11}'(z_1))$$
$$+ \frac{u^2}{\omega_p^4}\Psi_{11}(u)(F'(z') - F'(z_1)) + \frac{u^2}{c^4}(T_{1,-1}'(z') - T_{1,-1}'(z_1))\Big]\Big\},$$

$$B_\varphi = B_0\Big\{\frac{a_1}{2}[\Sigma_{11}^c(z') - \Sigma_{11}^c(z_1)] \tag{3.8}$$
$$- a_2\Big[\frac{1}{2}(\Sigma_{11}^c(z') + \Sigma_{11}^c(z_1)) - \Psi_{11}(\infty) + \frac{\omega_p^2}{c^2}T_{1,-1}(z')\Big]$$
$$- a_3\Big[\frac{1}{2}(\Sigma_{11}^c(z') - \Sigma_{11}^c(z_1)) + \frac{\omega_p^2}{c^2}(T_{1,-1}(z') - T_{1,-1}(z_1))\Big]\Big\},$$

$$E_z = E_0\Big\{-\frac{a_1}{2\gamma^2}[\Gamma_{02}'(z') - \Gamma_{02}'(z_1)]$$
$$+ a_2\Big[\frac{1}{2\gamma^2}(\Gamma_{02}'(z') + \Gamma_{02}'(z_1)) - \Psi_{00}(u)F'(z') - \frac{u^2\omega_p^2}{c^4}T_{0,-2}'(z')\Big]$$
$$+ a_3\Big[\frac{1}{2\gamma^2}(\Gamma_{02}'(z') - \Gamma_{02}'(z_1)) - \Psi_{00}(u)(F'(z') - F'(z_1))$$
$$- \frac{u^2\omega_p^2}{c^4}(T_{0,-2}(z') - T_{0,-2}(z_1))\Big]\Big\},$$

$$E_r = E_0\Big\{\frac{a_1}{2}[\Sigma_{11}^u(z') - \Sigma_{11}^u(z_1)]$$
$$+ a_2\Big[-\frac{1}{2}(\Sigma_{11}^u(z') + \Sigma_{11}^u(z_1)) + \Psi_{11}(u)F(z') + \frac{u^2\omega_p^2}{c^4}T_{1,-3}''(z')\Big]$$
$$+ a_3\Big[-\frac{1}{2}(\Sigma_{11}^u(z') - \Sigma_{11}^u(z_1))$$
$$+ \Psi_{11}(u)(F(z') - F(z_1)) + \frac{u^2\omega_p^2}{c^4}(T_{1,-3}''(z') - T_{1,-3}''(z_1))\Big]\Big\}.$$

We have introduced here the notation:

$$F(z) = \cos\frac{\omega_p z}{u}\exp\Big\{-\frac{\nu|z|}{2u}\Big\},$$

$$\Psi_{nm}(u) = \int_0 dk_\perp k_\perp^m \frac{J_n(k_\perp r)J_1(k_\perp r_0)}{k_\perp^2 + \omega_p^2/u^2},$$

$$\Psi_{11}(\infty) = \int_0^\infty \frac{dk_\perp}{k_\perp}J_1(k_\perp r)J_1(k_\perp r_0) = \frac{1}{2}\begin{cases} r/r_0, & r < r_0, \\ r_0/r, & r > r_0, \end{cases} \tag{3.9}$$

$$\Sigma_{nm}^{u,c}(z) = \int_0^\infty dk_\perp k_\perp^m \frac{J_n(k_\perp r)J_1(k_\perp r_0)}{k_\perp^2 + \omega_p^2/(u^2, c^2)}\exp(-|z|\gamma\sqrt{k_\perp^2 + \omega_p^2/c^2}),$$

$$\Gamma_{nm}(z) = \frac{\gamma^2 u^2}{\omega_p^2}[\Sigma_{nm}^c(z) - \Sigma_{nm}^u(z)],$$

$$T_{nm}(z) = \int_0^\infty dk_\perp k_\perp^m \frac{J_n(k_\perp r)J_1(k_\perp r_0)}{k_\perp^2 + \omega_p^2/c^2}\exp\Big(-|z|\frac{\nu}{u}\frac{k_\perp^2}{k_\perp^2 + \omega_p^2/c^2}\Big),$$

$$a_1 = \eta(z'), \quad a_2 = \eta(z_1) - \eta(z'), \quad a_3 = \eta(-z_1).$$

The functions $F(z)$ and $T_{nm}(z)$ are defined only for a negative value of the argument z. The primes in the functions of z in (3.8) denote differentiation with respect to z. The Appendix gives analytic expressions for the integrals (3.9) and their derivatives in different limiting case.

4. ANALYSIS OF THE ASYMPTOTIC EXPRESSIONS FOR THE FIELDS AND CURRENTS INDUCED IN THE PLASMA

As already noted above, the problem of beam injection into a plasma was investigated by Hammer and Rostoker[4] who, however, disregarded certain circumstances; namely, no account was taken of the finite beam injection time, and in the calculation of the quantities induced in the plasma the diffusion of the transverse field into the plasma was neglected (the terms due to the fifth pole of (3.7)). In the analysis of the formulas derived by us, we shall pay special attention to these circumstances.

Formulas (3.8) make it possible to investigate plasma perturbations by injection of an electron beam under a great variety of conditions of beam and plasma variation. We confine ourselves here to an analysis of only the most interesting case of a relativistic beam ($u \approx c$) injected into a dense plasma ($\omega_p > c/r_0$) for a sufficiently long time ($\tau\nu \gg 1$). As can be easily seen from (3.8). It is precisely in this case that the reaction of the plasma to the beam injection is strongest and is almost completely concentrated in the region of the beam itself.

a) Region ahead of the beam ($z' > 0$). In this region all the plasma perturbations are purely transverse (nonpotential); there are no potential perturbations (such as charge perturbations) at all. It is important to note here that with increasing distance from the leading front of the beam, the perturbed quantities decrease exponentially[4] with a characteristic scale $c/\gamma\omega_p$. On the other hand, in the radial direction the perturbations have a tubular character and are localized near $r \approx r_0$, and the characteristic scale of the fall-off is equal to c/ω_p. To illustrate the foregoing, we present expressions for j_z, B_φ, and E_z at $z' < c/\gamma\omega_p$ and $r > c/\omega_p$[3)]:

$$j_z = \frac{j_0}{4}\sqrt{\frac{r_0}{r}}\,\mathrm{sign}(r_0 - r)(e^{-\varkappa_c} - e^{-\varkappa_u}), \quad B_\varphi = \frac{B_0 c}{4\omega_p\sqrt{rr_0}}e^{-\varkappa_c},$$

$$E_z = \frac{E_0 c}{2\pi\gamma\omega_p\sqrt{rr_0}}\begin{cases} \mathrm{sign}(r_0 - r)\sqrt{\dfrac{\pi}{2}\varkappa_c}\,e^{-\varkappa_c}, & \varkappa_c \gg 1, \\ \dfrac{c}{2\omega_p r_0} + \mathrm{sign}(r_0 - r)\varkappa_c\ln\dfrac{2}{\varkappa_c}, & \varkappa_c \ll 1, \end{cases} \tag{4.1}$$

where

$$\varkappa_{c,u} = \frac{\omega_p}{c, u}|r - r_0|.$$

It follows therefore that the current induced in the plasma immediately in front of the beam can be of the order of the current j_0 in the beam, whereas the fields

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3) We note that the region ahead of the beam was not investigated quantitatively in [4] at all.

E and B are always smaller by a factor $\omega_p r_0/c$ than the vacuum fields E_0 and B_0 of the beam.

b) Region of the beam itself ($-u\tau < z' < 0$). In this region of the plasma, besides the purely transverse perturbations considered above, which attenuate exponentially from the leading and the trailing edges of the beam over distances on the order of $c/\gamma\omega_p$, there exist also potential perturbations that experience oscillations along the z axis, with a period $2\pi u/\omega_p$, and attenuate over a distance on the order of u/ν from the leading front of the beam[4], and also transverse perturbations of the diffuse type with an attenuation length $\sim z_0 = (u/\nu)(\omega_p r_0/c)^2$. It is the last two perturbations which were not taken into account in[4], since it was assumed that they are important only at very large distances. It will be clear from the following that allowance for diffusion of the transverse field into the plasma leads to quantitative changes in the results of[4] already at distances $|z'| < c/\gamma\omega_p$ from the leading front of the beam, and when $|z'| \gtrsim z_0$ these results turn out to be qualitatively incorrect.

The quantities j_z and B_φ are continuous on the leading front of the beam $z' = 0$, whereas E_z, j_r, and E_r experience discontinuities due to the diffusion of the transverse field into the plasma. For example, the field E_z in the region $z' < 0$, under the condition $|z'| \ll c/\gamma\omega_p$, is made up of the field described by (4.1) and of the quantity ΔE_z, given by (at $r > c/\omega_p$)

$$\Delta E_z = -\frac{E_0 \nu u}{\omega_p^2 r_0}\left[\eta(r_0 - r) - \frac{1}{2}\sqrt{\frac{r_0}{r}}\,\mathrm{sign}(r_0 - r)\,\frac{2+\varkappa_c}{2}e^{-\varkappa_c}\right]. \tag{4.2}$$

It follows therefore that the field E_z changes from a tubular form ahead of the beam into a three-dimensional one immediately behind the leading front, and within the volume of the beam this fields tends to accelerate the plasma electron in a direction opposite to that of the beam motion. Besides the field E_z, the total longitudinal current $j_z + j_0$ is also three-dimensional behind the leading front of the beam. On the other hand j_r, E_r, and B retain the tubular structure they have ahead of the beam, and are concentrated in a narrow layer of thickness c/ω_p near the surface of the beam.

It follows from the foregoing that the diffusion of the transverse field into the plasma becomes strongly manifest (particularly for the field E_z already at small distances $|z'| \ll c/\gamma\omega_p$ from the leading front of the beam.

With further increase from the leading front, $|z'| > c/\gamma\omega_p$, up to distances $|z'| \lesssim u/\nu$, the most appreciable role is assumed by the potential part of the perturbation, which has here an oscillatory character. An exception is the field B_φ, which is purely transverse. We present here formulas for $j_z + j_0$, B_φ, and E_z in the considered region of variation of z' when $r > c/\omega_p$:

$$j_z + j_0 = j_0\left\{\eta(r_0 - r)\cos\frac{\omega_p z'}{u} + \frac{1}{2}\sqrt{\frac{r_0}{r}}\,\mathrm{sign}(r_0 - r)\left[\left(1 + \frac{\nu|z'|\varkappa_c}{2u}\right)\right.\right.$$
$$\left.\left.\times e^{-\varkappa_c} - \cos\frac{\omega_p z'}{u}e^{-\varkappa_u}\right]\right\},$$

$$B_\varphi = \frac{B_0 c}{2\omega_p\sqrt{r r_0}}e^{-\varkappa_c}\left(1 + \frac{\nu|z'|\varkappa_c}{2u}\right),$$

$$E_z = \frac{E_0 u}{\omega_p r_0}\left\{\eta(r_0 - r)\left(\sin\frac{\omega_p z'}{u} - \frac{\nu}{\omega_p}\right) + \frac{1}{2}\sqrt{\frac{r_0}{r}}\,\mathrm{sign}(r_0 - r)\right.$$
$$\left.\times\left[\frac{\nu e^{-\varkappa_c}}{\omega_p}\left(1 + \frac{\varkappa_c}{2}\right) - \sin\frac{\omega_p z'}{u}e^{-\varkappa_u}\right]\right\}. \tag{4.3}$$

We see thus that in the entire region under consideration the "magnetic neutralization" is conserved[4], i.e., the field B_φ has a tubular character and is concentrated in a layer of thickness $\sim c/\omega_p$ near $r = r_0$. The "diffusion" part of the perturbations (the terms proportional to ν in (4.3)) have a qualitatively small effect in this region, but can lead to noticeable quantitative changes in the results of[4].

At still larger distances $|z'| > u/\nu$ from the leading front of the beam, the character of the perturbation of the plasma by the beam changes qualitatively. In this region the potential part of the perturbations attenuates exponentially, and the diffusion part becomes decisive; the "magnetic neutralization" of the beam becomes weaker. Formulas (3.8) differ appreciably in this region from the results of[4], where the diffusion of the transverse field in the plasma was not taken into account at all. The expressions for the fields and currents induced in the plasma then take the form

$$j_z + j_0 = j_0\begin{cases}\mathrm{sign}(r_0 - r)\,{}^1\!/_2\sqrt{r_0/r}\begin{cases}1, & b_1\\ \exp(-\zeta^2)/\sqrt{\pi}\,\zeta, & b_2\end{cases}, & b,\\ 2r^2/r_0^2 + \exp(-z_0/4|z'|), & a\end{cases}$$

$$B_\varphi = B_0\begin{cases}\sqrt{\dfrac{r_0|z'|}{\pi r z_0}}\begin{cases}1, & b_1\\ \exp(-\zeta^2)/2\zeta^2, & b_2\end{cases}, & b\\ (r/2r_0)\left[r^2/r_0^2 + \exp(-z_0/4|z'|)\right], & a\end{cases} \tag{4.4}$$

$$E_z = -E_0\frac{\nu u}{r_0\omega_p^2}\begin{cases}\eta(r_0 - r) + \mathrm{sign}(r_0 - r)\dfrac{1}{2}\sqrt{\dfrac{r_0}{r}}\begin{cases}1, & b_1\\ \dfrac{\exp(-\zeta^2)}{\sqrt{\pi}\,\zeta} & b_2\end{cases}, & b\\ 1, & a\end{cases}$$

$$\zeta = {}^1\!/_2\sqrt{z_0/|z'|}\,|r/r_0 - 1|.$$

Formulas (4.4) were written out for $|z'| \ll z_0$. In the opposite limit $|z'| \ll z_0$ (this is possible only when $u\tau \gg z_0$) we have

$$j_z + j_0 = j_0\begin{cases}-\dfrac{r_0}{r}\sqrt{\dfrac{r_0|z'|}{\pi r z_0}}\exp\left(-\dfrac{r^2 z_0}{4r_0^2|z'|}\right), & b\\ -\dfrac{r_0^4}{r^4} - \dfrac{z_0}{4|z'|}\exp\left(-\dfrac{r^2 z_0}{4r_0^2|z'|}\right), & c,\\ \eta(r_0 - r) - \dfrac{z_0}{4|z'|}, & a\end{cases}$$

$$B_\varphi = \frac{B_0}{2}\begin{cases}\dfrac{4r_0}{\sqrt{\pi}\,r}\left(\dfrac{r_0|z'|}{r z_0}\right)^{3/2}\exp\left(-\dfrac{r^2 z_0}{4r_0^2|z'|}\right), & b\\ \dfrac{r_0}{r}\left[\dfrac{r_0^2}{r^2} + \exp\left(-\dfrac{r^2 z_0}{4r_0^2|z'|}\right)\right], & c,\\ -\dfrac{r z_0}{4r_0|z'|} + \begin{cases}r_0/r, & r > r_0\\ r/r_0, & r < r_0\end{cases}, & a\end{cases} \tag{4.5}$$

$$E_z = -E_0\frac{\nu u}{r_0\omega_p^2}\begin{cases}\dfrac{r_0}{r}\sqrt{\dfrac{r_0|z'|}{\pi r z_0}}\exp\left(-\dfrac{r^2 z_0}{4r_0^2|z'|}\right), & b\\ \dfrac{r_0^4}{r^4} + \dfrac{z_0}{4|z'|}\exp\left(-\dfrac{r^2 z_0}{4r_0^2|z'|}\right), & c.\\ z_0/4|z'|, & a\end{cases}$$

In (4.4) and (4.5) we have introduced the following notation for the regions under consideration in (r, z') space:

$a \to r \ll r_0|z'|/z_0$, $b \to r \gg r_0|z'|/z_0$,
$c \to r_0\sqrt{|z'|/z_0} \ll r \ll r_0|z'|/z_0$,
$b_1 \to |r - r_0| \ll r_0\sqrt{|z'|/z_0}$, $b_2 \to |r - r_0| \gg r_0\sqrt{|z'|/z_0}$.

It is seen from (4.4) and (4.5) that the region of localization of the fields and of the currents in the plasma weakens with increasing z'; the thickness of the localization layer at $|z'| \ll z_0$ is of the order of $r_0(|z'|/z_0)^{1/2}$ near $r = r_0$, and at $|z'| \gtrsim z_0$ the structure of the fields and of the currents ceases to have a tubular character. The magnetic neutralization of the beam then disappears, and the magnetic field and the current in the plasma practically coincide with the field and current of the beam itself.

The lifting of the "magnetic and current neutralization" of the beam at distances $|z'| \gtrsim z_0$ is due entirely to the diffusion of the transverse field into the plasma, which was not taken into account by Hammer and Rostoker[4]. We therefore emphasize once more that this effect is significant only in sufficiently long beams, when $u\tau \gtrsim z_0$.

Finally, near the trailing edge of the beam, at distances $|z_1| < c/\gamma\omega_p$, there appear additional nondissipative perturbations similar to those that appeared ahead of the beam (see formulas (4.1)). Therefore to obtain analytic formulas describing the plasma perturbations near the trailing edge of the beam, it is necessary to add to formulas (4.4) or (4.5) (depending on the beam length) expressions of the type (4.1) with reversed sign and with the substitution $|z'| \to z_1$. It follows therefore that the character of the perturbations in this region depends essentially on the beam length. At $u\tau \ll z_0$ (short beam) the perturbations of the plasma have a tubular character, and at $u\tau \gtrsim z_0$ (long beam) the perturbations near the trailing edge of the beam or three-dimensional, but differ greatly in magnitude from the perturbations far from the trailing edge, owing to the aforementioned transverse perturbations that proceed from the trailing edge of the beam.

c) Region behind the beam ($z_1 = u + z' < 0$). This plasma region, just as the region near the trailing edge of the beam, $z_1 \approx 0$, could not be analyzed at all in[4], since the beam was assumed to be unbounded. The magnetic field B_φ and the induced current j_z (but not the total current $j_z + j_0$) near the trailing edge of the beam vary in a continuous fashion, whereas E_z, E_r, and j_r experiences a discontinuity because of the diffusion of the transverse beam into the plasma. The values of the jumps are the same as on the leading edge of the beam, $z' = 0$, but are of opposite sign.

The plasma perturbations behind the beam, at distances $|z_1| \lesssim c/\gamma\omega_p$, just as directly ahead of its trailing edge, depend strongly on the beam length. In the case of a short beam $u\tau \ll z_0$, the quantities B_φ, j_r, and E_r behind the beam retain the tubular structure they possessed in the region of the beam itself, whereas the total current j_z becomes distributed instead of concentrated in the skin layer, with $j_z \approx -j_0$ in the region $r \lesssim r_0$. The field E_z, to the contrary, ceases to be distributed over the volume and is concentrated in a skin layer $\sim r_0(|z'|/z_0)^{1/2}$ thick near $r = r_0$. In the case of a long beam $u\tau \gtrsim z_0$, the values of B_φ, j_r, and E_r are distributed in the volume behind the beam, just as directly ahead of the trailing edge of the beam, i.e., their spatial distribution is unchanged by passage through the trailing edge. Nor is the distributed character of the total current j_z and of the field E_z changed, but $j_z \approx -j_0 z_0/4|z'|$ decreases in magnitude, while $E_z \approx E_0 \nu u/r_0\omega_p^2$ increases.

With increasing distance from the trailing edge of the beam $|z_1| > c\gamma\omega_p$, up to distances $|z_1| \lesssim u/\nu$, the decisive role in the expressions for the current induced in the plasma, the charge, and the electric field is played by potential perturbations proceeding from the trailing edge of the beam. On the other hand, the magnetic field in this region is determined by perturbations of the diffusion time. At distances $|z_1| \gtrsim u/\nu$ the potential perturbations become attenuated, and the major role in all the quantities induced in the plasma is assumed by the diffusion of the transverse field into the plasma. If the beam is short, $u\tau \ll z_0$, and the quantities induced in the plasma have a tubular distribution (with the exception of E_z, which has a uniform volume distribution), then such a distribution is retained also behind the beam (including E_z) up to $|z_1| \sim z_0$. On the other hand if the beam is long, $u\tau \gtrsim z_0$, then the distribution of the perturbed quantities becomes three-dimensional in space in the region of the beam, at distances $|z'| > u/\nu$ from its leading front, and the distribution remains the same also behind the beam, up to $|z_1| \sim z_0$.

To illustrate the foregoing, we note that in the case of a short beam ($u\tau \ll z_0$) at distances $u/\nu < |z_1|$, $|z'| < z_0$, the currents and fields induced in the plasma are determined by the difference of expressions of the type (4.4) for $|z'|$ and $|z_1|$,[4)] from which it follows clearly that the perturbations have a tubular character. In the case of a long beam ($u\tau > z_0$) with $u/\nu < |z_1| < z_0$, to determine the quantities induced in the plasma, it is necessary to determine the difference of expressions of the type (4.5) and (4.4) for $|z'|$ and $|z_1|$ respectively. It is easily seen that in this case the field B_φ and the current j_z practically coincide with the magnetic field and with the current of the beam itself, i.e., the plasma perturbations have a volume-distributed character.

Finally, at very large distances from the trailing edge of the beam, when $|z_1| \gg z_0$, the plasma perturbations encompass, regardless of the beam length, the entire volume of the plasma, and are very small. This can be easily verified by determining the difference of expressions of the type (4.5) for $|z'|$ and $|z_1|$.

As already emphasized earlier, the theory developed here describes the injection of an electron beam into an unbounded space filled with plasma. The results can be used, with slight stipulations, in the case when the beam is injected into a semi-infinite plasma $z \geq 0$. In this case it is necessary to include in the problem the vacuum field equations in the region $z < 0$ and to make their solutions continuous with the solutions in the region $z > 0$. As a result, in addition to the volume-

[4)] Expressions for the plasma perturbations behind the beam, propagating from the rear boundary of the beam, are obtained from the corresponding formulas of the type (4.4) and (4.5) by making in them the substitution $|z'| \to |z_1|$.

distributed induced fields obtained above, there arise surface waves due to perturbation of the plasma boundary $z = 0$ by the passage of the electron beam. These fields are concentrated in a narrow boundary region $|z| \lesssim c/\gamma\omega_p$ and do not influence the character of the perturbation of the plasma by the beam in the most interesting region $|z| > c/\gamma\omega_p$. At such distances, the picture described above remains therefore in force also for injection of an electron beam into a semi-infinite plasma.

In conclusion, the authors are grateful to S. E. Rosinskiĭ for numerous discussions and valuable remarks.

APPENDIX

The integrals Ψ_{nm} encountered in (3.8) are tabulated in[6]:

$$\Psi_{11}(u)=\begin{cases} I_1\left(\frac{\omega_p r}{u}\right)K_1\left(\frac{\omega_p r_0}{u}\right), & r<r_0,\\ I_1\left(\frac{\omega_p r_0}{u}\right)K_1\left(\frac{\omega_p r}{u}\right), & r>r_0,\end{cases}$$

$$\Psi_{02}(u)=\frac{\omega_p}{u}\begin{cases} I_0\left(\frac{\omega_p r}{u}\right)K_1\left(\frac{\omega_p r_0}{u}\right), & r<r_0,\\ -I_1\left(\frac{\omega_p r_0}{u}\right)K_0\left(\frac{\omega_p r}{u}\right), & r>r_0;\end{cases} \tag{A.1}$$

$$\Psi_{00}(u)=\frac{u^2}{\omega_p^2 r_0}\left[\eta(r_0-r)-r_0\Psi_{02}(u)\right].$$

The integrals $\Sigma_{nm}^{u,c}(z)$ (in terms of which the functions $\Gamma_{nm}(z)$ are expressed) can be calculated in the limiting cases of small and large $|z|$:

$$\Sigma_{11}^{u,c}(z)\approx\begin{cases} \frac{(u^2,c^2)}{\gamma|z|\omega_p c}I_1\left(\frac{\omega_p r_0 r}{\gamma c|z|}\right)\cdot\\ \cdot\exp\left[-\frac{|z|\gamma\omega_p}{c}-\frac{\omega_p(r^2+r_0^2)}{2\gamma c|z|}\right], & |z|\gg\frac{c}{\gamma\omega_p};\\ \Psi_{11}(u,c), & |z|\ll c/\gamma\omega_p,\end{cases}$$

$$\Sigma_{0,2}^{u,c}(z)\approx\begin{cases} -\frac{(u^2,c^2)}{\gamma|z|\omega_p c}\frac{\partial}{\partial r_0}I_0\left(\frac{\omega_p r_0 r}{\gamma|z|c}\right)\cdot\\ \cdot\exp\left[-\frac{\gamma|z|\omega_p}{c}-\frac{\omega_p(r^2+r_0^2)}{2\gamma c|z|}\right], & |z|\gg\frac{c}{\gamma\omega_p};\\ \Psi_{02}(u,c), & |z|\ll c/\gamma\omega_p.\end{cases} \tag{A.2}$$

In calculating these integrals, we took into account the fact that the main contribution is made by the region $k_\perp \ll \omega_p/c$ when $|z| \gg c/\gamma\omega_p$ and $k_\perp \sim \omega_p/c$ when $|z| \ll c/\gamma\omega_p$.

When $|z| \ll c/\gamma\omega_p$, formulas (A.2) are not suitable for differentiation with respect to z. It is more convenient in this case to start directly from the expressions for $\Gamma'_{nn}(z)$. Recognizing that the decisive quantity is the integration region $k_\perp \sim \omega_p/c$, we obtain the following formulas:

$$\Gamma_{11}'(z)\approx-\gamma\,\mathrm{sign}\,z\int_0^\infty dk_\perp\frac{k_\perp J_1(k_\perp r_0)J_1(k_\perp r)}{(k_\perp^2+\omega_p^2/u^2)\sqrt{k_\perp^2+\omega_p^2/c^2}}$$
$$\approx-\mathrm{sign}\,z\frac{\gamma c|r-r_0|}{\pi\omega_p\sqrt{r_0 r}}K_1\left(\frac{\omega_p}{c}|r-r_0|\right),$$
$$\Gamma_{02}'(z)\approx\gamma\,\mathrm{sign}\,z\frac{\partial}{\partial r_0}\int_0^\infty dk_\perp\frac{k_\perp J_0(k_\perp r_0)J_0(k_\perp r)}{(k_\perp^2+\omega_p^2/u^2)\sqrt{k_\perp^2+\omega_p^2/c^2}} \tag{A.3}$$
$$\approx\mathrm{sign}\,z\frac{\partial}{\partial r_0}\frac{\gamma c|r-r_0|}{\pi\omega_p\sqrt{r_0 r}}K_1\left(\frac{\omega_p}{c}|r-r_0|\right).$$

The last relations have been written out for the most interesting case of a dense plasma $(r_0; r) > c/\omega_p$ and of a relativistic beam $u \approx c$.

Finally, in formulas (3.8) we encounter the integrals $T_{nm}(z)$, which can be calculated analytically in the opposite limiting cases of small and large $|z|$. Thus, for $|z| \ll u/\nu$ we have

$$T_{00}(z)\approx\Psi_{00}(c),\quad T_{1,-1}(z)\approx\Psi_{1,-1}(c)=\frac{c^2}{\omega_p^2}[\Psi_{11}(\infty)-\Psi_{11}(c)],$$
$$T_{1,-3}''(z)\approx\left(\frac{\nu c^2}{2u}\right)^2\frac{1}{2\omega_p}\frac{\partial}{\partial\omega_p}\frac{1}{\omega_p}\frac{\partial}{\partial\omega_p}\Psi_{11}(c), \tag{A.4}$$
$$T_{1,-1}'(z)\approx-\frac{\nu c^2}{2u\omega_p}\frac{\partial}{\partial\omega_p}\Psi_{11}(c),\quad T_{0,-2}'(z)\approx-\frac{\nu c^2}{2\omega_p u}\frac{\partial}{\partial\omega_p}\Psi_{00}(c),$$

Concerning $\Psi_{11}(\infty)$ see formula (3.9).

In the case of large $|z| \gg u/\nu$, when the main contribution to the integrals is made by the region $k_\perp \ll \omega_p/c$, the following expressions are obtained for $T_{nm}(z)$:

$$T_{1,-1}'(z)\approx\frac{\nu c^2}{u\omega_p^2}T_{11}(z),\quad T_{1,-3}(z)\approx\left(\frac{\nu c^2}{\omega_p^2 u}\right)^2T_{11}(z),$$
$$T_{0,-2}'(z)\approx\frac{\nu c^2}{\omega_p^2 u}T_{00}(z),\quad T_{11}(z)\approx\frac{u}{2\nu|z|}e^{-\mu}I_1\left(\frac{\lambda}{2}\right),$$

$$T_{00}(z)\approx\frac{c^2}{\omega_p^2 r_0}\begin{cases}(1+r^2/r_0^2)^{-2}\left[1+\left(\frac{\mu r^2}{r_0^2}-1\right)e^{-\mu}\right], & \lambda\ll 1,\\ \eta(r_0-r)-{}^1/_2\sqrt{r_0/r}[1-\Phi(\xi)]\,\mathrm{sign}(r_0-r), & \lambda\gg 1,\end{cases}$$
$$T_{1,-1}(z)\approx\frac{c^2}{2\omega_p^2}\begin{cases}\frac{r}{r_0}(1+r^2/r_0^2)^{-1}(1-e^{-\mu}), & \lambda\ll 1,\\ 2\Psi_{11}(\infty)+\frac{|r-r_0|}{\sqrt{r_0 r}}\left[1-\Phi(\xi)-\frac{e^{-\xi^2}}{\sqrt{\pi}\,\xi}\right], & \lambda\gg 1.\end{cases} \tag{A.5}$$

Here

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x dt\,e^{-t^2},\quad \lambda=\frac{z_0 r}{|z|r_0},\quad z_0=\frac{u}{\nu}\left(\frac{\omega_p r_0}{c}\right)^2,$$
$$\mu=\frac{\lambda}{4}\left(\frac{r}{r_0}+\frac{r_0}{r}\right),\quad \xi=\frac{\sqrt{\lambda}}{2}\left|\sqrt{\frac{r}{r_0}}-\sqrt{\frac{r_0}{r}}\right|.$$

In formulas (A.4) and (A.5) it was taken into account that the functions $T_{mn}(z)$ are defined only for $z < 0$.

Formulas (A.1)–(A.5) cover all possible limiting cases considered in Sec. 4 in the analysis of asymptotic expressions for the fields and currents induced in the plasma.

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