THE QUASI-CLASSICAL APPROXIMATION IN THE INVERSE SCATTERING PROBLEM

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An approximate solution is derived for the inverse scattering problem at fixed energy in a central field, for the case of strongly oscillating cross sections. The imaginary part of the optical-model potential is determined in the quasi-classical approximation from the known real part of the potential and the differential cross section.

THE most direct method for determining an interaction law from scattering data—the inverse problem—either presents, in its exact solution, considerable mathematical difficulties, or requires information that is difficult to obtain. Under certain conditions, one can overcome these difficulties and construct fairly simple approximate solutions that are of interest because of the logical simplicity of the inverse-problem approach to the experimental data. Thus, for potential scattering in a central field with E = const, it is possible to solve the inverse problem in the quasi-classical approximation. In this case, it is necessary to establish only one function $\Theta(\rho)$, the dependence of the scattering angle on the impact parameter. Since the modulus and phase of the amplitude, and also the potential, are expressed in a known way[1] in terms of $\Theta(\rho)$, additional conditions such as the unitarity condition are not required to determine the potential from the differential cross section.

Problems of this type arise, e.g., in nuclear physics in the case when a nuclear scattering experiment is analyzed in the optical model, in optics in light scattering in a refracting (and possibly also absorbing) medium with central symmetry, in atomic collisions, etc. Of course, in the presence of absorption, we are concerned with determining only the refractive or absorptive part of the interaction (the latter is determined in the present paper for nuclear scattering in the optical model). The essential point is that in most of these cases the cross section will be a strongly oscillating function of the angles, and this is used in this paper for an approximate solution of the inverse problem.

For a monotonic cross section, $\Theta(\rho)$ is defined simply as the inverse function to $\rho(\theta)$, where

$$\rho^2(\theta)=\int_\theta^\pi \sigma(\theta)\,d\theta,\qquad \sigma(\theta)=2\sin\theta\frac{d\sigma}{d\Omega}.$$

In the presence of interference, when the cross section has the form

$$\sigma(\theta)=\Big|\sum_n\sqrt{\frac{d}{d\theta}\rho_n^2}\exp\Big\{\frac{i}{\hbar}S_n\Big\}\Big|^2=\Big|\sum_n\sqrt{a_n}\exp\Big\{\frac{i}{\hbar}\operatorname{Re}S_n\Big\}\Big|^2, \quad (1)$$

the problem of deriving $\Theta(\rho)$ becomes more complicated since $\rho(\theta)$ is here a many-valued function. In such cases, the problem has been solved by means of parametrization of the potential (or $\Theta(\rho)$) with an assumption about the number of interfering waves (cf., e.g., [2]).

However, for sufficiently large ka, one can, without making similar assumptions, construct all the monotonic branches $\rho_n(\theta)$, at the cost of some loss of exactness. In this case, the cross section will be a strongly oscillating function of a known type, whereas $\rho_n(\theta)$ is a smooth function which is practically unvarying over the period $\Delta\theta\sim(ka)^{-1}$ of the oscillations. The validity of (1) will be violated only in narrow intervals $\Delta\theta\sim(ka)^{-2/3}$ close to the caustic rays and in regions of order $(ka)^{-1}$ around $\theta=0,\pi$. Therefore, the average of (1) over the region $(ka)^{-1}\ll\Delta\theta\ll 1$ is equal to

$$\overline{\sigma(\theta)}=\sum_n|a_n|.$$

For constant $|a_n|$, this averaging is clearly exact to within $\sim(ka)^{-1}$; if $da_n/d\theta\sim 1$, then averaging over an interval $\Delta\theta\sim(ka)^{-1/2}$ will give an accuracy for the procedure of order $(ka)^{-1/2}$. For the case of two interfering waves, when

$$\sigma(\theta)=|a_1|+|a_2|+2\sqrt{|a_1a_2|}\sin\hbar^{-1}\operatorname{Re}(S_2-S_1),$$

the averaging can be performed with accuracy of order $(ka)^{-1}$, if it is carried out over the range of angles $\Delta\theta\sim(ka)^{-1}$ between the extrema of the cross section, which coincide, within $(ka)^{-1}$, with the extrema of $\sin\hbar^{-1}\times\operatorname{Re}(S_2-S_1)$.

The quantities $\overline{\sigma^k}$, which are obtained by analogous averaging of the corresponding powers of (1), can be expressed in terms of $|a_n|$ equally simply. For example,

$$\overline{\sigma^3}=\sum_n|a_n|^3+9\sum_{n\neq m}|a_n^2a_m|+36\sum_{n<m<l}|a_na_ma_l|. \quad (2)$$

If now, using $\overline{\sigma^k}$, we construct elementary symmetric polynomials c_i from the $|a_n|$:

$$c_1(\theta)=\sum_n|a_n|=\overline{\sigma(\theta)},\qquad c_2(\theta)=\sum_{n<m}|a_na_m|=\frac{1}{2}[\overline{\sigma^2}-(\bar\sigma)^2],$$
$$c_3(\theta)=\sum_{n<m<l}|a_na_ma_l|=\frac{1}{12}[\overline{\sigma^3}-3\overline{\sigma^2}\bar\sigma+2(\bar\sigma)^3], \quad (3)$$

it is clear that in the range of angles where $\Theta(\rho)$ has j branches, the c_i for $i>j$ vanish identically, and the remaining ($i\le j$) relations (3) enable us to calculate $|a_n(\theta)|$. Obviously, a_n ($n=1,\ldots,j$) will be the roots of the polynomial:

$$x^j-c_1x^{j-1}+c_2x^{j-2}-\ldots+(-)^jc_j=0.$$

In the classically unattainable region, where the cross section is small (like $\exp -ka\theta$), $j = 0$ within the error bars of the method.

The averaging procedure gives, in all, $|a_n|$, the number of real branches of $\rho(\theta)$, and the boundaries of the region for each branch. The points $\theta = 0, \pi$ and the extrema θ_{0n} of the function $\Theta(\rho)$, at which $(\rho^2)'$ changes sign, can serve as these boundaries. Finally, $\rho_n(\theta)$ is found from $(\rho_n^2)' = \pm |a_n|$ by simple integration; the unknown signs and constants are determined from the conditions that $\Theta(\rho)$ (for $\rho \geq 0$) be single-valued and be continuous at the boundaries, and from the condition that $\Theta(0)$ be equal to zero or $\pm\pi$.

Thus, for example, in the simplest case of scattering with interference, when $j = 2$ in the region $0 < \theta < \theta_0$ and $j = 0$ for $\pi > \theta > \theta_0$, it is clear that $\Theta(0) = 0$ and θ_0 is an extremum of $\Theta(\rho)$. Then

$$\rho_1^2(\theta) = \frac{1}{2}\int_0^\theta [c_1 - \sqrt{c_1^2 - 4c_2}]\, d\theta,$$

$$\rho_2^2(\theta) = \rho_1^2(\theta_0) + \frac{1}{2}\int_\theta^{\theta_0} [c_1 + \sqrt{c_1^2 - 4c_2}]\, d\theta. \quad (4)$$

Equations (4) give $|\Theta(\rho)|$ for potential scattering in an attractive or repulsive field. Correspondingly, the solution of the inverse problem will be two potentials:

$$\frac{\pi}{2}\ln(1 - \nu_{1,2}) = \pm \int_{\beta_{1,2}}^\infty \frac{|\Theta(\rho)|\, d\rho}{\sqrt{\rho^2 - \beta_{1,2}^2}}, \quad \beta_{1,2}^2 = r^2[1 - \nu_{1,2}(r)],$$

$$\nu(r) = \frac{V(r)}{E}, \quad (5)$$

with $\nu_1^{-1}(r) + \nu_2^{-1}(r) = 1$. In particular, for the depth of the potential well we shall have

$$|\nu(0)| = \exp\left\{\frac{1}{\pi}\int_0^{\theta_0} \ln\frac{\rho_2^2(\theta)}{\rho_1^2(\theta)}\, d\theta\right\} - 1. \quad (6)$$

Another typical case of scattering with interference will be the situation when there exists one branch $|a_1|$ in the region $0 < \theta < \pi$, and two more branches $|a_2|$ and $|a_3|$ for $0 < \theta < \theta_0$. Then $|\Theta(0)| = \pi$ and θ_0 is an extremum of $\Theta(\rho)$; assuming for definiteness that for $\theta = 0$ the value of $|a_1|$ is close to $|a_2|$, we shall have

$$\rho_1^2(\theta) = \int_\theta^\pi |a_1|\, d\theta, \quad \rho_{2,3}^2(\theta) = \rho_0^2 \mp \int_\theta^{\theta_0} |a_{2,3}|\, d\theta, \quad (7)$$

$$\rho_0^2 = \rho_1^2(0) + \int_0^{\theta_0} |a_2|\, d\theta.$$

Equations (7) define a positive function of ρ having a discontinuity at $\rho = \rho_1(0)$. If $|a_{1,2}(0)|^{-1}$ are not small (quasi-classically), the function we have constructed is the modulus of the true function $\Theta(\rho)$, which changes sign at $\rho = \rho_1(0)$.[1)] This case corresponds to scattering in a field consisting of attraction and repulsion.

To conclude, we shall consider potential scattering in the optical model from the point of view of the inverse problem. The complex potential in this model contains a practically arbitrary function, constrained only by the integral condition determining the total reaction cross section. In the quasi-classical approximation, it is impossible, because of the divergence at small angles, to set up this condition, and therefore the inverse problem must here include an assumption that removes this arbitrariness. It is natural to assign the form of the real part of the optical potential and solve the inverse problem for the imaginary part.

The presence of an imaginary part W in the potential leads to the fact that the cross-section will be an exponentially decreasing function with argument of order $|ka\theta\xi|$ (for $\xi = W/E$ small), and the impact parameters $\rho(\theta)$ will be complex, with $\operatorname{Im}\rho \sim \xi$.

For the case under consideration, we shall elucidate the condition for which the scattering is potential scattering, i.e., the condition for which we can neglect the contribution to the amplitude from poles of the scattering matrix (in the complex angular momentum plane[3]). We shall consider a situation which is almost critical (in the sense of the appearance of resonances). Then the conditions for spiral scattering[4]

$$\rho^2 = r^2(1 - \tilde\nu), \quad \tilde\varepsilon(r) \equiv 1 - \tilde\nu - {}^1\!/_2 r\tilde\nu' = 0, \quad \tilde\nu = (V(r) + iW(r))/E \quad (8)$$

will be fulfilled for the almost real values, r_c and ρ_c; more exactly, for small $\operatorname{Im} r_c$ and ξ, we shall have

$$\operatorname{Im}\rho_c \cong b\xi + c\varepsilon^{3/2}|_{r=\operatorname{Re} r_c}; \quad b, c \sim a.$$

In this case there will exist a scattering-matrix pole ρ_S close to the real axis, since the matrix has a pole at a distance of order $\lambda = k^{-1}$ from ρ_c. Then the contribution to the amplitude from the pole ρ_S will be of order $\exp\{-(\theta/\lambda)\operatorname{Im}\rho_c\}$. Hence follows the condition on the real part of the optical potential (for $\xi \ll 1$) for which scattering through finite angles can be regarded as potential scattering: on the real axis we shall have

$$\varepsilon(r) \equiv (1 - \nu(r) - {}^1\!/_2 r\nu'(r)) > \xi^{2/3}, \quad \nu = V/E. \quad (9)$$

For $\xi \ll 1/ka$, in place of (9) we shall have $\epsilon \gg (ka)^{-2/3}$.

If condition (9) is fulfilled, then in the neighborhood of points on the real axis of the complex plane $\rho = x + iy$, we can write an expansion in ξ for the function

$$\tilde\Theta(\rho) = \int_\rho^\infty \frac{\rho\, du}{\sqrt{u^2 - \rho^2}} \frac{d}{du}\ln(1 - \tilde\nu), \quad u^2 = r^2(1 - \tilde\nu). \quad (10)$$

Using the symbol $\Theta(x)$ for the scattering angle for scattering by the real part of the potential V, we can write the expansion in the form:

$$\operatorname{Re}\tilde\Theta(\rho) = \Theta(x) + O(\xi^2), \quad \operatorname{Im}\tilde\Theta(\rho) = y\Theta'(x) - \alpha(x) + O(\xi^3), \quad (11)$$

where

$$\alpha(x) = \int_x^\infty \frac{x\, dt}{\sqrt{t^2 - x^2}} \frac{d}{dt}\frac{W}{E\varepsilon}, \quad t^2 = r^2[1 - \nu(r)].$$

The system $\operatorname{Re}\tilde\Theta(\rho) = \theta$, $\operatorname{Im}\tilde\Theta(\rho) = 0$ determines $\rho_n(\theta)$ and, consequently, the amplitude and number of the interfering waves. A feature of the interference pattern for $W \neq 0$ is the absence of caustics: the branches $\rho_n(\theta)$ for small ξ come together in the neighborhood of the points $y = 0$, $x = x_{0n}$, where $\Theta'(x_{0n}) = 0$, but do not intersect. For $\xi \to 0$, the system of lines $\operatorname{Im}\tilde\Theta(\rho) = 0$ goes over into the real axis $x \geq 0$ and the group of "caustic branches" intersecting the latter at the points x_{0n}.

We shall obtain the desired expression for the imagi-

[1)] If the averaging is insufficiently exact, a further possibility appears: $\Theta(\rho)$ has at this point an extremum equal to zero.

nary part of the optical potential from the equation $\operatorname{Im}\Theta(\rho) = 0$. Transforming the integral $\alpha(x)$ in (11) by a known method,[1] we shall have

$$\frac{\pi}{2}\frac{W}{E - V - {}^1/_2 rV'} = -\int_t^\infty \frac{y\Theta'(x)\,dx}{\sqrt{x^2 - t^2}} \qquad (12)$$

for energies $E > \max(V + \frac{1}{2} rV')$ (cf. [2] and the condition (9)). Hence, to calculate W, in addition to V we must know the quantity $y = \operatorname{Im}\rho(\theta)$. This can be found from experiment, if, by the method described previously, we determine the $|a_n|$, which in the given case have the form

$$|a_n| = \left|\frac{d}{d\theta}\rho_n^2\right| \exp\left\{-\frac{2}{\hbar}\operatorname{Im} S_n\right\}, \quad \operatorname{Im} S_n > 0. \qquad (13)$$

Since

$$\frac{1}{\hbar}\frac{d}{d\theta}\operatorname{Im} S_n = \frac{|\operatorname{Im}\rho_n|}{\lambda\!\!\!^-},$$

from (13), by differentiation with respect to the angle, we obtain for $y \ll x$:

$$\frac{|y_n|}{\lambda\!\!\!^-} = \frac{1}{2}\frac{d}{d\theta}\ln\left|\frac{(x_n^2)'}{a_n}\right|, \qquad (14)$$

where $x_n(\theta)$ is calculated using V(r), and $|a_n(\theta)|$ is found from the experimental cross section.

If V(r) is a monotonic potential well and the number of interfering waves is $j = 2$ in the region $0 < \theta < \theta_0$, then, according to (3), we have

$$|a_{1,2}| = {}^1/_2 (c_1 \mp \sqrt{c_1^2 - 4c_2}).$$

Going over in (12) to integration over the angles, we obtain for $|W(0)|$:

$$\frac{\pi}{2}|W(0)| = |E - V(0)| \int_0^{\theta_0} d\theta \left(\left|\frac{y_2}{x_2}\right| + \left|\frac{y_1}{x_1}\right|\right). \qquad (15)$$

Here, θ_0 is the limiting classical scattering angle in the field V; for $E > V(0)$, we have $\theta_0 < \pi$, i.e., to determine W we do not need to know $\sigma(\theta)$ in the whole range of angles.

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