CONTRIBUTION TO THE THEORY OF DOMAIN STRUCTURES IN MAGNETS AND FERROELECTRICS

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The conditions for the coexistence of phases in magnetic and ferroelectric substances are found by taking into account the elastic deformation energy (formulas (17) and (17′)). Depth properties of the domain structure are investigated. It is shown that a consequence of the condition of minimum of the volume part of the free energy is that in the interior of an ellipsoidal sample both $E_t(H_t)$ and $E_n(H_n)$ are continuous at the phase interfaces. The number of parameters determining the depth properties of the domain structure exceeds by one the number of relations between the parameters, so that the problem involves a degeneracy. The parameters can be determined only if effects related to the emergence of the domains to the surface are taken into account. Nevertheless, it is possible to calculate for uniaxial ferromagnets the magnetization and the deformation for an external field H_0 arbitrarily oriented with respect to the ellipsoid axes and the crystallographic axes, and averaged over the domain structure (see formulas (5), (37), and (38)). The optical properties of the simplest domain boundary (Bloch wall) are investigated in the low-frequency range. It is shown that for waves of a certain polarization the domain structure is impenetrable (perfectly reflecting). Hence a massive domain bounded by such walls is essentially a waveguide.

1. INTRODUCTION

IN one of the author's papers[1], conditions were obtained for the coexistence of phases in magnetic substances[1)] and in ferroelectrics:

$$\mathbf{H}_t = \text{const},\quad B_n = \text{const},\quad \Phi'(\mathbf{H}_t, B_n) = \text{const},$$

$$\Phi'(\mathbf{H}_t, B_n) = \tilde{\Phi} + \frac{1}{4\pi}H_nB_n = -\frac{1}{4\pi}\int_0^{\mathbf{H}}\mathbf{B}d\mathbf{H} + \frac{1}{4\pi}H_nB_n, \tag{1}$$

$$\mathbf{E}_t = \text{const},\quad D_n = \text{const},\quad \Phi'(\mathbf{E}_t, D_n) = \text{const},$$

$$\Phi'(\mathbf{E}_t, D_n) = \tilde{\Phi} + \frac{1}{4\pi}E_nD_n = -\frac{1}{4\pi}\int_0^{\mathbf{E}}\mathbf{D}d\mathbf{E} + \frac{1}{4\pi}E_nD_n. \tag{1'}$$

The simplest domain structures are found in bulky ellipsoidal bodies, in plane-parallel plates, and in cylindrical samples of ellipsoidal cross section. If the external field is homogeneous, then, neglecting effects connected with the emergence of the domains to the surface, it can be assumed that several homogeneous phases separated by plane-parallel boundaries coexist in the sample. In the simplest cases the number of phases is equal to two. Each of the phases represents a system of domains in contact with the domains of the other phase. The width of the domains a is small compared with the dimensions of the sample l. In the case of an unbranched structure $a \sim l^{1/2}$[5,6], and for a branched structure $a \sim l^{2/3}$[7]. The values of the magnetic field H and of the induction B averaged over the domain structure (or else of the electric field E and of the induction D) inside such samples are homogeneous. These quantities are connected with the external field H_0 (or E_0) by the relations

$$H_{0i} = (\delta_{ik} - n_{ik})\langle\langle H_k\rangle\rangle + n_{ik}\langle\langle B_k\rangle\rangle, \tag{2}$$

$$E_{0i} = (\delta_{ik} - n_{ik})\langle\langle E_k\rangle\rangle + n_{ik}\langle\langle D_k\rangle\rangle. \tag{2'}$$

The double angle brackets denote averaging over the volume of the sample, n_{ik} is the tensor of the demagnetization (depolarization) coefficients. Stratification into domains leads to a decrease of the thermodynamic potential, which is proportional to the volume of the sample.

The depth properties of the domain structures are characterized completely by specifying H_t and B_n (or E_t and D_n), by the two angles determining the orientations of the interfaces, and by the phase concentrations. These quantities satisfy three equations (2) (or (2′)) and the conditions for the coexistence of the phases

$$\Phi_1'(\mathbf{H}_t, B_n) = \Phi_2'(\mathbf{H}_t, B_n)$$

There are two fewer equations than unknowns[2)]. Two parameters must be determined from the condition that the total thermodynamic potential of the body must be a minimum. In magnetic substances it is necessary to minimize the thermodynamic potential

$$\tilde{\Phi}_n = \frac{1}{V}\int d^3\mathbf{x}\left[\tilde{\Phi}(\mathbf{x}) + \frac{H_0^2}{8\pi}\right],$$

and in ferroelectrics the thermodynamic potential

$$\Phi_n = \frac{1}{V}\int d^3x\left[\Phi(x) - \frac{E_0^2}{8\pi}\right],\quad \Phi = \tilde{\Phi} + \frac{1}{4\pi}\mathbf{ED}.$$

Here V is the volume of the sample, and the integra-

[1)]We have in mind ferro- and antiferromagnets, and also diamagnetic metals under conditions of the de Haas–van Alphen effect. The conditions for the existence of the domain structure in antiferromagnets are considered in [2]. Experimental and theoretical investigations of the domain structure of diamagnets are dealt with in [3,4].

[2)]Structures of the "checkerboard" type have a fewer number of degrees of freedom and can be realized only in special cases, for example in the absence of an external field.

tion is carried out over the entire space, including the volume outside the body.

It was shown in[1] that as a result of minimization of the volume part of the thermodynamic potential there is obtained only one additional condition $H_{n1} = H_{n2}$ (or $E_{n1} = E_{n2}$). The complete system of conditions for the minimum can be written in the form

$$\mathbf{H} = \text{const}, \quad B_n = \text{const}, \quad \tilde{\Phi} = \text{const} \tag{3}$$

or

$$\mathbf{E} = \text{const}, \quad D_n = \text{const}, \quad \tilde{\Phi} = \text{const}. \tag{3'}$$

The number of equations is one less than the number of unknowns, so that there is degeneracy in the problem. We emphasize that the condition H_n = const (or E_n = const) is the consequence of the ellipsoidal shape of the sample and is not satisfied in samples of more complicated shape.

In a uniaxial ferromagnet

$$\begin{gathered}\tilde{\Phi} = U_{\text{an}} - \mathbf{MH} - H^2/8\pi, \\ U_{\text{an}} = {}^1/_2\beta M^2 \sin^2\theta, \quad \beta > 0, \quad |\mathbf{M}| = M = \text{const},\end{gathered} \tag{4}$$

where θ is the angle of deflection of the magnetization $\mathbf{M}$ from the easy axis (the z axis), the conditions (3) denote that the interface is parallel to the easy axis[6] and the field $\mathbf{H}$ is perpendicular to this axis. The degeneracy in this model is connected with the fact that at given phase concentrations c_1 and c_2 ($c_1 + c_2 = 1$) and for a given field $\mathbf{H} \perp z$ the volume part of the total thermodynamic potential of the body $\tilde{\Phi}_n$ does not change when the interface is rotated about the z axis. This makes it possible to find the connection between $\langle\langle \mathbf{M} \rangle\rangle$ and $\mathbf{H}_0$ in the region of the existence of the domain structure for all the orientations of $\mathbf{H}_0$ and of the crystallographic axes relative to the axes of the ellipsoids (this was not done earlier).

We take into account the fact that when $H_z = 0$ and $H_x^{2/3} + H_y^{2/3} < (\beta M)^{2/3}$ the "equation of state" of the uniaxial ferromagnet takes the form

$$\beta M_x = H_x, \quad \beta M_y = H_y, \quad M_z = \pm(M^2 - M_x^2 - M_y^2)^{1/2}.$$

Since $\mathbf{H}$ = const, relation (2) can be rewritten in the form

$$H_{0i} = H_i + 4\pi n_{ik} \langle\langle M_k \rangle\rangle,$$

or else

$$H_{0i} = 4\pi \tilde{n}_{ik} \langle\langle M_k \rangle\rangle, \tag{5}$$

where

$$\tilde{n}_{ik} = n_{ik} + (\beta/4\pi)(\delta_{ik} - \delta_{i3}\delta_{k3}).$$

The concentrations of the phases are determined from the conditions

$$\begin{gathered}\langle\langle M_z \rangle\rangle = (c_1 - c_2)(M^2 - M_x^2 - M_y^2)^{1/2}, \\ M_x = M_{x1} = M_{x2} = \langle\langle M_x \rangle\rangle, \\ M_y = M_{y1} = M_{y2} = \langle\langle M_y \rangle\rangle.\end{gathered} \tag{6}$$

All these results were obtained without taking into account the energy of the elastic deformation. In ferromagnets, the magnetostriction energy is in most cases small compared with the magnetostatic energy, but there are substances in which they are comparable (see, e.g.,[8]). In ferroelectrics the electrostriction energy is always appreciable[9]. We note also that the problem of elastic deformation under magnetization (polarization) is of independent interest.

The generally accepted theories of electrostriction and magnetostriction are not fully correct. In Sec. 2 we shall write out the correct thermodynamic relations for the striction phenomena. In the same section we obtain the condition for the existence of phases with allowance for the elastic deformations.

In Sec. 3 we investigate the depth properties of the domain structures and show that the degeneracy mentioned above remains in force also when account is taken of the energy of the elastic deformations, in spite of the fact that the condition for the minimum of the volume part of the total free energy leads to one more additional condition E_n = const. Therefore all the parameters characterizing the properties of the domain structure in the interior of the sample can be obtained only when account is taken of the effects connected with the emergence of the domains to the surface. Nonetheless, we shall calculate the average deformation of uniaxial ferromagnets with allowance for the domain structure.

In Sec. 4 we investigate the optical properties of the simplest domain boundary (of the Bloch wall type) at low frequencies. It is shown that the domain boundary is impenetrable (perfectly reflecting) for waves of definite polarization. Therefore a massive domain bounded by such walls constitutes a waveguide.

2. THERMODYNAMIC THEORY OF ELECTROSTRICTION AND MAGNETOSTRICTION: CONDITIONS FOR THE COEXISTENCE OF THE PHASES

We note first that in an anisotropic body the free energy depends not only on the symmetrical components of the strain tensor

$$u_{ik} = {}^1/_2(\partial u_i/\partial x_k + \partial u_k/\partial x_i),$$

but also on the antisymmetrical ones:

$$v_{ik} = {}^1/_2(\partial u_i/\partial x_k - \partial u_k/\partial x_i).$$

Let us consider, for example, the free energy of a dielectric

$$F\left(\mathbf{E}, \frac{\partial u_i}{\partial x_k}\right) = F\left(\mathbf{E} = 0, \frac{\partial u_i}{\partial x_k}\right) - \frac{1}{4\pi}\int_0^{\mathbf{E}} \mathbf{D}\,d\mathbf{E}.$$

This quantity does not change under rotations*

$$\mathbf{u} = [\boldsymbol{\omega}, \mathbf{r}], \quad v_{ik} = -\varepsilon_{ikl}\omega_l, \quad u_{ik} = 0,$$

if the field $\mathbf{E}$ is rotated simultaneously:

$$\delta\mathbf{E} = [\boldsymbol{\omega}\mathbf{E}],$$

i.e.,

$$\varepsilon_{ikl}\omega_l \partial F/\partial v_{ik} + ({}^1/_4\pi)\,\varepsilon_{ikl} D_i \omega_k E_l = 0.$$

From this we readily find that

$$\left(\frac{\partial F}{\partial v_{ik}}\right)_{\mathbf{E}, u_{ik}} = -\frac{1}{8\pi}(E_i D_k - E_k D_i). \tag{7}$$

The derivatives $(\partial F/\partial u_{ik})_{\mathbf{E}, v_{ik}}$ are determined by the relation

$$\sigma_{ik} = \frac{\partial F}{\partial u_{ik}} + F\delta_{ik} + \frac{1}{8\pi}(E_i D_k + E_k D_i), \tag{8}$$

where $\sigma_{ik} = \sigma_{ki}$ is the stress tensor. This relation is

*$[\omega, \mathbf{r}] \equiv \omega \times \mathbf{r}$.

given in the book of Landau and Lifshitz[6] 3).

Using relations (7) and (8), we can easily show that

$$dF = \left(\sigma_{ik} - \tilde{F}\delta_{ik} - \frac{1}{4\pi}E_iD_k\right) d\frac{\partial u_i}{\partial x_k} - \frac{1}{4\pi}\mathbf{D}d\mathbf{E}. \qquad (9)$$

For the free energy

$$F = \tilde{F} + \frac{1}{4\pi}\mathbf{ED}$$

we obtain the relation

$$dF = \left(\sigma_{ik} - F\delta_{ik} + \frac{1}{4\pi}\mathbf{DE}\delta_{ik} - \frac{1}{4\pi}E_iD_k\right) d\frac{\partial u_i}{\partial x_k} + \frac{1}{4\pi}\mathbf{E}d\mathbf{D}. \qquad (10)$$

We introduce also the free energy F' in terms of the variables E_t, D_n, and $\partial u_i/\partial x_k$, where the indices t and n denote components tangential and normal to the interface:

$$F' = F - \frac{1}{4\pi}\mathbf{E}_t\mathbf{D}_t = \tilde{F} + \frac{1}{4\pi}E_nD_n,$$

$$dF' = \left(\sigma_{ik} - F'\delta_{ik} + \frac{1}{4\pi}E_nD_n\delta_{ik} - \frac{1}{4\pi}E_iD_k\right) d\frac{\partial u_i}{\partial x_k} - \frac{1}{4\pi}\mathbf{D}_td\mathbf{E}_t + \frac{1}{4\pi}E_ndD_n. \qquad (11)$$

In elasticity theory it is convenient to introduce the free energies corresponding to a specified mass, namely, the mass per unit undeformed volume. Such quantities we shall mark with a zero subscript:

$$d\tilde{F}_0 = \left(\sigma_{ik} - \frac{1}{4\pi}E_iD_k\right) d\frac{\partial u_i}{\partial x_k} - \frac{1}{4\pi}(1 + u_{ll})\mathbf{D}d\mathbf{E}, \qquad (12)$$

$$dF_0 = \left(\sigma_{ik} + \frac{1}{4\pi}\mathbf{DE}\delta_{ik} - \frac{1}{4\pi}E_iD_k\right) d\frac{\partial u_i}{\partial x_k} + \frac{1}{4\pi}(1 + u_{ll})\mathbf{E}d\mathbf{D}, \qquad (13)$$

$$dF_0' = \left(\sigma_{ik} + \frac{1}{4\pi}E_nD_n\delta_{ik} - \frac{1}{4\pi}E_iD_k\right) d\frac{\partial u_i}{\partial x_k} + \frac{1}{4\pi}(1 + u_{ll})(-\mathbf{D}_td\mathbf{E}_t + E_ndD_n). \qquad (14)$$

On the interface there are conserved, besides E_t and D_n, also the quantities $\partial u_i/\partial x_\alpha$ and $\sigma_{ik}n_k = \sigma_{in}$, where the index α numbers the components in the plane of the interface. To obtain the condition for the coexistence of the phases, it is necessary to construct the thermodynamic potential in terms of conserved variables. To this end we represent dF_0' in the form

$$dF_0' = \left(\sigma_{i\alpha} + \frac{1}{4\pi}E_nD_n\delta_{i\alpha} - \frac{1}{4\pi}E_iD_\alpha\right) d\frac{\partial u_i}{\partial x_\alpha} + \left(\sigma_{\alpha n} - \frac{1}{4\pi}E_\alpha D_n\right) d\frac{\partial u_\alpha}{\partial n} + \sigma_{nn}d\frac{\partial u_n}{\partial n} + \frac{1}{4\pi}(1 + u_{ll})(-\mathbf{D}_td\mathbf{E}_t + E_ndD_n). \qquad (15)$$

The sought thermodynamic potential, which has a minimum at specified E_α, D_n, $\partial u_i/\partial x_\alpha$, and σ_{in} is the thermodynamic potential

$$\Phi_0'\left(E_\alpha,\ D_n,\ \frac{\partial u_i}{\partial x_\alpha},\ \sigma_{\alpha n} - \frac{1}{4\pi}E_\alpha D_n,\ \sigma_{nn}\right) = F_0' - \left(\sigma_{\alpha n} - \frac{1}{4\pi}E_\alpha D_n\right)\frac{\partial u_\alpha}{\partial n} - \sigma_{nn}\frac{\partial u_n}{\partial n}. \qquad (16)$$

The complete system of boundary conditions is

$$E_\alpha = \text{const},\ D_n = \text{const},\ \partial u_i/\partial x_\alpha = \text{const},\ \sigma_{in} = \text{const}, \qquad (17)$$

$$\Phi_0'\left(E_\alpha,\ D_n,\ \frac{\partial u_i}{\partial x_\alpha},\ \sigma_{\alpha n} - \frac{1}{4\pi}E_\alpha D_n,\ \sigma_{nn}\right) = \text{const}. \qquad (17')$$

Analogous formulas hold also for magnetic substances. They are obtained from (7)–(17′) by making the substitutions $\mathbf{E} \to \mathbf{H}$ and $\mathbf{D} \to \mathbf{B}$.

3. DEPTH PROPERTIES OF DOMAIN STRUCTURES

Let us consider first an ellipsoidal sample, the shape of which is maintained constant, placed in a homogeneous electric field $\mathbf{E}_0$. In such a sample there is possible inhomogeneous deformation (that varies from domain to domain), which on the average is equal to zero. The depth properties of the domain structure are determined completely by specifying nine parameters, which do not change on going through the interface:

$$E_\alpha,\ D_n,\ \sigma_{in},\ c_1,\ \mathbf{n}.$$

Here c_1 is the concentration (by weight) of one of the two phases, and n is a unit vector normal to the interface. Six more such parameters $(\partial u_i/\partial x_\alpha)$ are equal to zero, since the ellipsoid remains undeformed in the mean. These quantities satisfy the three equations (2′), the condition for the coexistence of the phases (17′), and three more equations

$$\langle \partial u_i/\partial n \rangle = 0, \qquad (18)$$

where

$$\langle f \rangle = c_1f_1 + c_2f_2. \qquad (19)$$

Thus, the number of equations is equal to seven, two less than the number of parameters determining the depth properties of the domain structure. In addition, it is necessary to obtain the condition for the minimum of the volume part of the total free energy

$$F_n = \frac{1}{V}\int d^3\mathbf{x}\left(F(\mathbf{x}) - \frac{E_0^2}{8\pi}\right) = \langle F_0\rangle - \frac{E_0^2}{8\pi} + \varphi(\langle\langle\mathbf{P}\rangle\rangle, \mathbf{E}_0), \qquad (20)$$

regarded as a function of the nine parameters determining the depth properties of the domain structure, under seven additional conditions (2′), (17′), and (18). The double angle brackets in formula (20), unlike the single ones, denote averaging over of the volume; $\mathbf{P}$ is the dipole moment per unit volume. The function $\varphi(\langle\langle\mathbf{P}\rangle\rangle, \mathbf{E}_0)$ was calculated in[1]:

$$\varphi(\langle\langle\mathbf{P}\rangle\rangle, \mathbf{E}_0) = n_{ik}E_{0i}\langle\langle P_k\rangle\rangle - \langle\langle\mathbf{P}\rangle\rangle\mathbf{E}_0 + 2\pi(n_{ik} - n_{ci}n_{ek})\langle\langle P_i\rangle\rangle\langle\langle P_k\rangle\rangle. \qquad (21)$$

We shall show now that by minimizing expression (20) we obtain only one additional condition

$$E_n = \text{const}. \qquad (22)$$

Together with the electrodynamic condition E_t = const, the condition (22) denotes that $\mathbf{E}$ = const.

Let us assume that the following conditions are satisfied on the interfaces

$$\mathbf{E} = \text{const},\quad D_n = \text{const},$$
$$\Phi_0'\left(E_\alpha, D_n, \frac{\partial u_i}{\partial x_\alpha}, \sigma_{\alpha n} - \frac{1}{4\pi}E_\alpha D_n, \sigma_{nn}\right) = \text{const},$$
$$\partial u_i/\partial x_\alpha = 0,\quad \sigma_{in} = \text{const} \qquad (23)$$

and let us show that under infinitesimally small changes of the parameters determining the depth properties of

3) It is frequently stated in the literature that in an anisotropic body the stress tensor σ_{ik} is not symmetrical (see, e.g., [10,12]). This statement is incorrect. The correct result (see formula (8)) is obtained when account is taken of the asymmetrical terms (proportional to v_{ik}) in the free energy.

the domain structure, the variation δF_n is equal to zero. This variation can be represented in the form

$$\delta F_n = (F_0^{(1)} - F_0^{(2)})\delta c_1 + \langle \delta F_0 \rangle + \delta\varphi. \quad (24)$$

In formula (24), the averaging is carried out with the zeroth-approximation phase concentrations (c_1 and $c_2 = 1 - c_1$).

Representing F_0 in the form

$$F_0 = F_0 - F_0' + F_0' - \Phi_0' + \Phi_0'; \quad F_0 - F_0' = (4\pi)^{-1}(1+u_{ll})D_l E_l,$$

$$F_0 - \Phi_0' = \left(\sigma_{\alpha n} - \frac{1}{4\pi}E_\alpha D_n\right)\frac{\partial u_\alpha}{\partial n} + \sigma_{nn}\frac{\partial u_n}{\partial n} \quad (25)$$

and recognizing that

$$E_{n1} = E_{n2}, \quad \Phi_{01}' = \Phi_{02}',$$

we obtain

$$F_0^{(1)} - F_0^{(2)} = \frac{1}{4\pi}\mathbf{E}(\mathbf{D}_1 - \mathbf{D}_2) + \frac{1}{4\pi}\mathbf{E}_l(\mathbf{D}_{1l}u_{ll}^{(1)} - \mathbf{D}_{2l}u_{ll}^{(2)})$$

$$+ \left(\sigma_{\alpha n} - \frac{1}{4\pi}E_\alpha D_n\right)\left(\frac{\partial u_\alpha^{(1)}}{\partial n} - \frac{\partial u_\alpha^{(2)}}{\partial n}\right) + \sigma_{nn}\left(\frac{\partial u_n^{(1)}}{\partial n} - \frac{\partial u_n^{(2)}}{\partial n}\right). \quad (26)$$

We further take into account the fact that

$$\langle \delta F_0 \rangle = \left\langle \left(\sigma_{ik} - \frac{1}{4\pi}E_i D_k\right)\delta\frac{\partial u_i}{\partial x_k}\right\rangle + \frac{1}{4\pi}\langle \mathbf{D}\mathbf{E}\delta u_{ll}\rangle + \frac{1}{4\pi}\langle (1+u_{ll})\mathbf{E}\delta\mathbf{D}\rangle. \quad (27)$$

In order to calculate the first term in expression (27) for $\langle \delta F_0 \rangle$, we use the relation

$$\left\langle \left(\sigma_{ik} - \frac{1}{4\pi}E_i D_k\right)\delta\frac{\partial u_i}{\partial x_k}\right\rangle + \left(\sigma_{in} - \frac{1}{4\pi}E_i D_n\right)$$

$$\times \left(\frac{\partial u_i^{(1)}}{\partial n} - \frac{\partial u_i^{(2)}}{\partial n}\right)\delta c_1 = \int dV\left(\sigma_{ik} - \frac{1}{4\pi}E_i D_k\right)\delta\frac{\partial u_i}{\partial x_k} = 0.$$

The integration is carried out here over the "undeformed" volume and account is taken of the fact that $\mathbf{E} = \text{const}$, $\sigma_{in} - E_i D_n/4\pi = \text{const}$, $\partial u_i/\partial x_\alpha = 0$. Since

$$(\partial u_i^{(1)}/\partial x_k - \partial u_i^{(2)}/\partial x_k)\delta c_1 + \langle \delta\partial u_i/\partial x_k\rangle = \delta\langle \partial u_i/\partial x_k\rangle = 0, \quad (28)$$

it is easily seen that

$$\left\langle \left(\sigma_{ik} - \frac{1}{4\pi}E_i D_k\right)\delta\frac{\partial u_i}{\partial x_k}\right\rangle = -\left(\sigma_{in} - \frac{1}{4\pi}E_i D_n\right)$$

$$\times \left(\frac{\partial u_i^{(1)}}{\partial n} - \frac{\partial u_i^{(2)}}{\partial n}\right)\delta c_1 \quad (29)$$

and

$$\delta F_n = \frac{1}{4\pi}[\mathbf{E}(\mathbf{D}_1 - \mathbf{D}_2) + \mathbf{E}_l(\mathbf{D}_{1l}u_{ll}^{(1)} - \mathbf{D}_{2l}u_{ll}^{(2)})]\delta c_1$$

$$- \frac{1}{4\pi}E_n D_n\left\langle \delta\frac{\partial u_n}{\partial n}\right\rangle + \frac{1}{4\pi}\langle \mathbf{D}\mathbf{E}\delta u_{ll}\rangle$$

$$+ \frac{1}{4\pi}\langle (1+u_{ll})\mathbf{E}\delta\mathbf{D}\rangle + \delta\varphi.$$

After simple transformations we can represent the expression for δF_n in the form

$$\delta F_n = (4\pi)^{-1}\mathbf{E}\delta\langle (1+u_{ll})\mathbf{D}\rangle + \delta\varphi. \quad (30)$$

It was shown in[1] that

$$\delta\varphi = -(4\pi)^{-1}\mathbf{E}\delta\langle\langle \mathbf{D}\rangle\rangle, \quad (31)$$

where, as already mentioned, the double angle brackets denote averaging over the volume:

$$\langle\langle f\rangle\rangle = c_1' f_1 + c_2' f_2,$$
$$c_1' = c_1(1+u_{ll}^{(1)}), \quad c_2' = c_2(1+u_{ll}^{(2)}). \quad (32)$$

It is easily seen that

$$\langle (1+u_{ll})\mathbf{D}\rangle = \langle\langle \mathbf{D}\rangle\rangle.$$

Thus, δF_n vanishes if the conditions (23) are satisfied on the interfaces. The number of relations remains one less than the number of unknowns, i.e., as already noted, there is degeneracy in the problem: the volume part of the total free energy does not change under continuous variation of the parameters in accordance with the conditions (23). It should be emphasized that in the variation we did not assume that the phase-coexistance condition (17′) is satisfied on the changed interfaces. Thus, this condition itself can be obtained from the condition of the minimum total free energy F_n.

In magnetic substances it is necessary to minimize the free energy F_n:

$$F_n = \frac{1}{V}\int d^3\mathbf{x}\left(F(\mathbf{x}) + \frac{H_0^2}{8\pi}\right).$$

Since F_n in dielectrics and F_n in magnetic substances can be represented in the form [6]

$$F_n = \frac{1}{V}\int dV\left(F - \frac{1}{8\pi}\mathbf{ED} - \frac{1}{2}\mathbf{PE}_0\right)$$

$$= \frac{1}{V}\int dV\left(F + \frac{1}{8\pi}\mathbf{ED} - \frac{1}{2}\mathbf{PE}_0\right),$$

$$F_n = \frac{1}{V}\int d^3\mathbf{x}\left(F + \frac{H_0^2}{8\pi}\right) = \frac{1}{V}\int dV\left(F + \frac{1}{8\pi}\mathbf{HB} - \frac{1}{2}\mathbf{MH}_0\right),$$

it is obvious that for magnetic substances we obtain a system of conditions analogous to (23):

$$\mathbf{H} = \text{const}, \quad B_n = \text{const}, \quad \Phi_0' = \text{const}, \quad \partial u_i/\partial x_\alpha = 0,$$
$$\sigma_{in} = \text{const}. \quad (33)$$

So far we have dealt with an ellipsoid whose shape was assumed invariant. This means that mechanical forces are applied to the body and prevent a change in its shape. If there are no such forces, then a body having the shape of an ellipsoid in the demagnetized (unpolarized) state will become deformed upon application of an external field and, in particular, can rotate in the external field under the influence of the purely electrostatic (magnetostatic) Maxwellian stresses

$$\sigma_{ik} = \frac{1}{4\pi}\left(E_i E_k - \frac{1}{2}E^2\delta_{ik}\right), \sigma_{ik} = \frac{1}{4\pi}\left(H_i H_k - \frac{1}{2}H^2\delta_{ik}\right).$$

The torque may be zero only in definite cases, for example in the case of a long cylinder (wire) or else a plane-parallel plate in an external field $E_0(H_0)$ parallel to it. It is precisely such cases that are meaningful and should be considered. The ellipsoidal shape of the sample is conserved in such cases accurate to effects connected with the emergence of the domains to the surface, which are insignificant in the calculation of the volume energies. This is connected, with the fact that the average deformation $\langle \partial u_i/\partial x_k\rangle$ will be homogeneous. The same can occur also in cases when one applies to the body stresses that produce no torque. Such a situation is realized, for example, in experiments with stretched wires. In order to investigate the depth properties of domain structure in these cases, we indicate that in the preceding derivation the concept of undeformed state was arbitrary, since it was not assumed that in the absence of a deformation ($\partial u_i/\partial x_k = 0$) and in the absence of a field ($E = 0$ or $H = 0$) the stresses σ_{ik} are also equal to zero. We shall henceforth interpret the underformed state in precisely this manner. It is then obvious that in those cases when the sample remains ellipsoidal (although the ratio of the semiaxes may change) and the sample on the average remains homogeneous, we obtained in place of (23) and (33) the following system of equations:

$$\mathbf{E}(\mathbf{H}) = \text{const}, \quad D_n(B_n) = \text{const}, \quad \Phi_0' = \text{const}, \qquad \partial u_i/\partial x_\alpha = \text{const}, \quad \sigma_{in} = \text{const}, \quad \langle v_{ik}\rangle = 0. \tag{34}$$

The last relation is the condition that the body as a whole does not rotate in the external field.

As an illustration let us calculate the magnetostriction of a uniaxial ferromagnet with allowance for the domain structure. We first obtain an expression for the free energy of the uniaxial ferromagnet with allowance for the magnetostriction. In the absence of deformation the anisotropy energy in the uniaxial ferromagnet is given by

$$U_{\text{an}}^{(0)} = {}^1/_2\beta M^2 \sin^2\theta = {}^1/_2\beta(M^2 - (\mathbf{Ml})^2),$$

where $\mathbf{l}$ is the unit vector of the easy magnetization axis.

In the case of an infinitesimally small rotation of the body as a whole ($\delta\mathbf{l} = \boldsymbol{\omega}\times\mathbf{l}$) the form of this energy, written relative to the old axes, changes:

$$U_{\text{an}}^{(0)} = {}^1/_2\beta[M^2 - (\mathbf{Ml})^2 + (\mathbf{Ml})(l_iM_k - l_kM_i)v_{ik}]. \tag{35}$$

The free energy F_0 is given by

$$F_0 = U_{\text{an}}^{(0)} + \gamma_{iklm}u_{ik}M_lM_m - \mathbf{MH} - \frac{H^2}{8\pi} + \frac{1}{2}\lambda_{iklm}u_{ik}u_{lm}. \tag{36}$$

The second term here is the change of the anisotropy energy, connected with the deformation, and the last term is the usual elastic energy. The quantities γ_{iklm} are of the order of β. The contribution made by the magnetostriction to the total energy is small in a uniaxial ferromagnet. In cubic ferromagnets this, generally speaking, is not the case, because the term

$$\gamma_{iklm}u_{ik}M_lM_m$$

can be comparable with the anisotropy energy in the absence of deformation, which is proportional to M^4. The dependence of $\mathbf{M}$ on $\mathbf{H}$ and $\partial u_i/\partial x_k$ should be obtained from the condition of the minimum of the free energy F_0, regarded as a function of $\mathbf{M}$ at specified $\mathbf{H}$ and $\partial u_i/\partial x_k$. For σ_{ik} we obtain the relation

$$\sigma_{ik} - (8\pi)^{-1}(H_iB_k + H_kB_i) = \gamma_{iklm}M_lM_m + \lambda_{iklm}u_{lm}.$$

Averaging this relation over the domain structure, we obtain

$$\langle\sigma_{ik}\rangle - (8\pi)^{-1}(H_i\langle B_k\rangle + H_k\langle B_i\rangle) = \gamma_{iklm}\langle M_lM_m\rangle + \lambda_{iklm}\langle u_{lm}\rangle. \tag{37}$$

We recall that

$$H_x = \beta\langle M_x\rangle, \quad H_y = \beta\langle M_y\rangle, \quad H_z = 0,$$
$$\langle M_xM_y\rangle = \langle M_x\rangle\langle M_y\rangle, \quad \langle M_xM_z\rangle = \langle M_x\rangle\langle M_z\rangle,$$
$$\langle M_yM_z\rangle = \langle M_y\rangle\langle M_z\rangle, \quad \langle M_z^2\rangle = M^2 - \langle M_x\rangle^2 - \langle M_y\rangle^2, \tag{38}$$

and the quantities $\langle M_i\rangle$ are equal in this approximation to $\langle\langle M_i\rangle\rangle$, which are determined from relations (5). Thus, we have obtained a connection between the mean values $\langle\sigma_{ik}\rangle$ and $\langle u_{ik}\rangle$ in a homogeneous external field $\mathbf{H}_0$.

4. OPTICAL PROPERTIES OF DOMAIN BOUNDARIES

Let us consider here the simplest 180-degree domain boundary in a ferroelectric:

$$\mathbf{E} = 0, \quad \partial u_i/\partial x_k = 0, \quad \sigma_{ik} = 0, \quad \mathbf{P}_1 = -\mathbf{P}_2; \tag{39}$$

the plane of the boundary is parallel to the vectors $\mathbf{P}_1$ and $\mathbf{P}_2$. We choose the z axis along the direction of $\mathbf{P}_1$.

We write the phase coexistence condition (17′) for the case when the conditions on the boundary deviate little from the conditions (39) and, in particular, the slope of the boundary relative to the z axis is small. Since in the zeroth approximation the condition $D_n = 0$ is satisfied in addition to (39), we can readily see that in the first approximation

$$\Phi_0' = F_0 = -\frac{1}{4\pi}\mathbf{DE} = -\mathbf{PE}.$$

Therefore in the first approximation the condition for the coexistence of the phases takes the form

$$E_{z1} + E_{z2} = 0. \tag{40}$$

We emphasize that this conclusion is not connected with any model.

Let us consider now the problem of the reflection of an electromagnetic wave from the 180-degree boundary (39). We shall assume here that this boundary separates two half-spaces. In addition, we assume that the frequency ω is small compared with the reciprocal relaxation time $1/\tau$ ($\omega\tau \ll 1$), so that there is time for thermodynamic equilibrium to become established on the boundary. We shall assume the plane of the boundary to be the zy plane, and the axes x, y, and z to be the principal axes of the tensor $\epsilon_{ik} = \partial D_i/\partial E_k$. The magnetic properties of the medium will be neglected: $\mu_{ik} = \delta_{ik}$. For simplicity, in addition, we shall assume that the crystal is uniaxial: $\epsilon_{xx} = \epsilon_{yy} = \epsilon_\perp$. In this case in an unbounded homogeneous medium there can propagate waves of two types:

a) ordinary wave

$$E_z = 0, \quad H_z \neq 0, \quad \omega = ck/\sqrt{\varepsilon_\perp};$$

b) extraordinary wave

$$H_z = 0, \quad E_z \neq 0, \quad \omega^2 = c^2(k_z^2/\varepsilon_\perp + k_\perp^2/\varepsilon_{zz}).$$

This distinction becomes meaningless at $k_\perp = 0$.

The thermodynamic boundary condition (40) is unusual for electrodynamics. The presence of such a condition is connected with the fact that the problem contains an additional variable, namely the displacement of the boundary $\zeta(y, z, t)$. The usual electrodynamic conditions E_t = const and H_t = const should be satisfied in a coordinate system moving together with the boundary. In a resting coordinate system, these conditions are (see[6])

$$[\mathbf{n}, \mathbf{E}_2 - \mathbf{E}_1] = (\mathbf{B}_2 - \mathbf{B}_1)v_n/c,$$
$$[\mathbf{n}, \mathbf{H}_2 - \mathbf{H}_1] = -(\mathbf{D}_2 - \mathbf{D}_1)v_n/c,$$

where $\mathbf{n}$ is the normal to the boundary and $v_n = \partial\zeta/\partial t$ is the rate of displacement of the boundary. In the linear approximation, these boundary conditions can be written in the following form:

$$H_{z1}=H_{z2},\quad E_{y1}=E_{y2},\quad E_{z1}=E_{z2},\tag{41}$$

$$H_{y2}-H_{y1}=8\pi P v_n/c.\tag{42}$$

The complete system of conditions determining the reflection problem is the set of equations (40) and (41). It can be rewritten in the form

$$H_{z1}=H_{z2},\ E_{y1}=E_{y2},\ E_{z1}=E_{z2}=0,\tag{43}$$

and Eq. (42) determines the rate of displacement of the boundary v_n.

It is easy to see that the ordinary wave is insensitive to the boundary condition $E_{Z1} = E_{Z2} = 0$; this wave passes through the domain boundary without "noticing" it (the reflection coefficient is equal to zero), whereas the extraordinary wave is completely reflected from the boundary (the reflection coefficient is equal to unity), leading to oscillations of the domain boundary[4)].

Let us consider now a cylindrical domain of an arbitrary cross section that is constant along the cylinder. The side walls of the cylinder are parallel to the z axis and are 180-degree boundaries. We shall show that such a domain represents a wave guide for waves with $E_Z \neq 0$, $H_Z = 0$, i.e., there exists a solution of Maxwell's equations

$$\operatorname{rot}\mathbf{H}=-i(\omega/c)\hat{\varepsilon}\mathbf{E},\quad \operatorname{rot}\mathbf{E}=i(\omega/c)\mathbf{H},$$

different from zero only inside the domain and satisfying the boundary conditions

$$(\mathbf{E}_t)_b=0.\tag{44}$$

Eliminating from Maxwell's equations the vector H, we obtain

$$\Delta\mathbf{E}+\frac{\omega^2}{c^2}\hat{\varepsilon}\mathbf{E}=\nabla\operatorname{div}\mathbf{E}=\left(1-\frac{\varepsilon_{zz}}{\varepsilon_\perp}\right)\nabla\frac{\partial E_z}{\partial z}.$$

Further, putting

$$\mathbf{E}=\mathbf{E}_0(\mathbf{x}_\perp)\exp(ik_z z),$$

we obtain the boundary-value problem

$$\Delta_2 E_z+\varkappa^2 E_z=0,\quad (E_z)_{\text{гр}}=0,$$
$$0<\varkappa^2=\varepsilon_{zz}\left(\frac{\omega^2}{c^2}-\frac{k_z^2}{\varepsilon_\perp}\right),\tag{45}$$

where Δ_2 is the two-dimensional Laplacian. From Maxwell's equations we can easily find $\mathbf{E}_\perp$ and $\mathbf{H}_\perp$:

$$\mathbf{E}_\perp=\frac{ik_z\varepsilon_{zz}}{\varepsilon_\perp\varkappa^2}\nabla_2 E_z,$$
$$H_y=\frac{\omega\varepsilon_\perp}{ck_z}E_x,\quad H_x=-\frac{\omega\varepsilon_\perp}{ck_z}E_y,\tag{46}$$

where $\mathbf{E}_\perp$ satisfies the necessary boundary condition. When such a wave propagates along the domain, the domain boundary will oscillate.

In a flat domain of width a we have

$$\varkappa=n\pi/a,\quad n=1,2,3,\ldots\tag{47}$$

In a flat domain the wave can propagate at an angle to z axis ($k_y \neq 0$). In this case

$$\varkappa^2=k_y^2+(n\pi/a)^2=\varepsilon_{zz}(\omega^2/c^2-k_z^2/\varepsilon_\perp).\tag{48}$$

In a flat domain, just as in a domain that is not singly connected, there can propagate the so-called "principal" wave:

$$E_z=H_z=0,\quad \omega=\frac{ck_z}{\sqrt{\varepsilon_\perp}},\quad k_y=0.\tag{49}$$

The field $\mathbf{E}_\perp$ can be found by introducing the scalar potential φ:

$$\mathbf{E}_\perp=-\nabla_2\varphi,$$

satisfying the Laplace equation $\Delta_2\varphi = 0$ with the boundary conditions $\varphi = \varphi_1$ and $\varphi = \varphi_2$ on the domain boundaries, where φ_1 and φ_2 are specified constants ($\varphi_1 \neq \varphi_2$). The principal wave cannot propagate at an angle to the z axis.

All the results obtained in this section are applicable also to the case of ferrodielectrics in which, however, the satisfaction of the condition $\omega\tau \ll 1$ is difficult, since the relaxation times are usually quite large.

4)We are considering the case of low frequencies. In the opposite limiting case ($\omega\tau \gg 1$) there is no time for thermodynamic equilibrium to become established, the condition (40) is not satisfied, and the wall is not displaced, i.e., the problem can be regarded within the framework of ordinary electrodynamics.

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