"PLASMON" MECHANISM OF SUPERCONDUCTIVITY IN DEGENERATE SEMICONDUCTORS AND SEMIMETALS. II

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A detailed qualitative and quantitative analysis of the nonlinear integral equation for the "gap" in the superconductor quasiparticle spectrum is carried out within the framework of the "jellium" model in the weak and strong coupling approximations (at both absolute temperature $T = 0$ and $T \neq 0$). The results of the numerical solution of the problem by an electron computer (BESM-6) are presented. The results are used to ascertain whether the "plasmon" mechanism previously proposed (in the first half of this investigation) for superconductivity in degenerate semiconductors and semimetals possessing two or more free carrier groups (conduction electrons, holes) with strongly differing effective masses and concentrations is possible. It is shown that for a sufficiently high concentration of the majority ("light") carriers ($N \sim 10^{21}$ cm^{-3}) and for a small crystal dielectric constant ($\epsilon_\infty \sim 1$, but $\epsilon_0 \gg \epsilon_\infty$) quite high critical superconductivity transition temperatures ($T_c \sim 10^2$ °K) should be attainable in principle.

IN the first part of this investigation[1] we advanced the hypothesis of a possible specific (non-phonon) mechanism of superconductivity in degenerate semiconductors and semimetals with essentially different effective masses of the conduction electrons and holes ($m_n \ll m_p$) and relatively large mobilities of the free carriers, when the conditions $E_{Fn} \gg \Omega_p \gg \tau_p^{-1}$, are satisfied, where E_{Fn} is the Fermi energy of the electrons, reckoned from the bottom of the conduction band, Ω_p is the plasma frequency (energy) of the relatively "heavy" holes (we assume $\hbar = 1$), and τ_p is the characteristic relaxation time of the hole momentum. In this case the formation of bound electron (Cooper) pairs occurs as a result of attraction, due principally to exchange of virtual quanta of the plasma-acoustic oscillations of degenerate electron-hole plasma[1-3]—the so-called "acoustic plasmons,"[1)] since the maximum energy of the latter Ω_p at sufficiently high carrier concentration can be much larger than the limiting (Debye) energy ω_D of the acoustic phonons, and the corresponding dimensionless parameter of the "electron-plasmon" (Coulomb) interaction ρ can greatly exceed the constant of the electron-phonon interaction λ_0 (the Frohlich parameter) in nonpolar semiconductors and semimetals. In polar semiconductors with large ionicity, when $\epsilon_0 \gg \epsilon_\infty$ (where ϵ_0 is the static and ϵ_∞ the high-frequency dielectric constant of the crystal), the interaction of the conduction electrons with the longitudinal optical phonons with energy ω_l, as shown in [7], leads to expansion of the region of the effective attraction (up to $\omega_{max} \sim \sqrt{\omega_l^2 + \Omega_p^2}$), and consequently to an enhance-

pressed matter[14, 15]). The interaction between the conduction electrons in semiconductors (semimetals) in the high-density approximation[14] (i.e., in the case of an almost ideal Fermi gas) is described by the following vertex part (see [1]):

[1] to describe the "electron-plasmon" interaction, and present results of a numerical solution with the BÉSM-6 computer (counting rate up to 10^6 operations per second).

1. THE "JELLIUM" MODEL. THE WEAK-COUPLING APPROXIMATION

One of the simplest models in the theory of superconductivity, together with the models of BCS[9] and Frohlich,[10] is the so-called "jellium" model proposed by Pines,[11] which makes it possible to consider in a unified manner, on the basis of a generalized (dynamic) dielectric constant $\epsilon(q, \omega)$, the electron-phonon and Coulomb interactions in metals.[12, 13]

Of course, this model cannot pretend to give a detailed quantitative description of the properties of real superconductors, since no account is taken in it of the singularities of the crystal and band structures of the metal, the anisotropy of the lattice and of the electron spectrum, the presence of transverse acoustic modes and Umklapp processes, etc. (See [12, 13].)

At the same time, the "jellium" model apparently serves as a sufficiently good approximation for the study of superconductivity in such systems with Coulomb interaction as an electron-hole plasma of isotropic degenerate semiconductors and semimetals[1, 7] or a two-component plasma of s- and d-electrons in transition metals and alloys[4-6] (and also in strongly comment of the "plasmon" mechanism of superconductivity.[2)]

In [1], however, only an approximate (qualitative) in-

[1)]A superconductivity mechanism of similar physical nature was independently proposed by Frohlich [1] for transition metals with "heavy" d-electrons, $m_d \gg m_s$ (i.e., with a narrow d-band). The question of the influence of the collective oscillations of the d-electrons (and of the associated dynamic "rescreening" of the Coulomb repulsion between the s-electrons) on the superconducting properties of transition metals was investigated earlier by Garland [5], who used numerical methods (see also [6].

[2)]The possibility of appearance of superconductivity in degenerate (strongly doped) polar semiconductors because of the interaction of electrons with longitudinal optical phonons was considered earlier in [8].

vestigation was made of the vertex part (four-pole) of the electron-electron interaction and of the nonlinear integral equation for the gap in the spectrum of the conduction electrons. In the present article we present a more detailed analysis of the "jellium" model used in

$$\Gamma_{nn}(q,\omega)\equiv\frac{4\pi e^2}{q^2\varepsilon(q,\omega)}=\frac{4\pi e^2}{\varepsilon_i[q^2+\varkappa_n^2(q)]}\frac{\omega^2}{\omega^2-\Omega_q^2},\qquad(1.1)$$

where

$$\Omega_q=\left[\frac{4\pi e^2N_p}{\varepsilon_i m_p}\frac{q^2}{q^2+\varkappa_n^2(q)}\right]^{1/2};\quad \varkappa_n(q)\cong\left[\frac{6\pi e^2N_n}{\varepsilon_i E_{Fn}}\left(1-\frac{q^2}{8p_{Fn}^2}\right)\right]^{1/2};\qquad(1.2)$$

N_n and N_p are the concentrations of the electrons and holes (generally speaking, $N_n \neq N_p$), $p_{Fn} = (3\pi^2N_n)^{1/3}$ is the Fermi momentum of the conduction electrons, and ϵ_i is the dielectric constant of the crystal (for simplicity we assume $\epsilon_i \approx \epsilon_\infty = \text{const}$ in the frequency region $\omega \sim \Omega_p$).

In this case the equation for the gap in the electron spectrum at the absolute zero of temperature (T = 0) in the weak-coupling approximation takes the form (compare with [15]):

$$C(\xi)=\int_0^E d\xi'\,K(\xi,\xi')\frac{C(\xi')}{\sqrt{\xi'^2+C^2(\xi')}},\qquad(1.3)$$

where

$$K(\xi,\xi')=\tfrac{1}{2}[Q(\xi-\xi')+Q(\xi+\xi)];\qquad(1.4)$$

$$Q(\omega)\equiv-\frac{1}{(2\pi)^2v_{Fn}}\int_0^{2p_{Fn}}qdq\,\Gamma_{nn}(q,\omega)=\frac{1}{2}\beta(\omega)\ln\left|1-\frac{1}{\beta(\omega)}\right|,\qquad(1.5)$$

$$\beta(\omega)=\frac{\alpha\omega^2}{\Omega_p^2-\omega^2(1-\alpha/2)},\quad \alpha=\frac{\varkappa_n^2(0)}{4p_{Fn}^2}\equiv\frac{e^2}{\pi\varepsilon_i v_{Fn}};\qquad(1.6)$$

$v_{Fn} = p_{Fn}/m_n$ is the Fermi velocity of the conduction electrons (see below concerning the choice of the upper limit of integration E in (1.3)).

The kernel $Q(\omega) \equiv K(\omega, 0)$ is shown by the solid line of Fig. 1.

We note that for the "jellium" model, the condition of large density $p_{Fn}a_n \gg 1$ (where $a_n = \epsilon_i/e^2m_n$ is the effective Bohr radius of the conduction electron) is simultaneously the condition for the applicability of the weak-coupling approximation $\alpha \equiv 1/\pi p_{Fn}a_n \ll 1$ (for more details see Sec. 2).

Let us find the approximate (asymptotic) solution of Eq. (1.3) with allowance for the exponential smallness of the gap (see [18, 19]). Replacing for simplicity the function $C(\xi)$ under the square root in the right-hand side of (1.3) by its value on the Fermi surface $C(0) \equiv \Delta$, we will reduce, by integrating by parts, Eq. (1.3) to the form

$$C(\xi)=-K(\xi,0)\Delta\ln\frac{\Delta}{2E}-\int_0^E\frac{d}{d\xi'}[K(\xi,\xi')C(\xi')]\ln\frac{\xi'}{E}d\xi'.\qquad(1.7)$$

Equation (1.7) coincides in fact with the asymptotic expansion of Eq. (1.3), obtained in [18, 19] accurate to terms $\sim \Delta \ln \Delta$ and Δ. Introducing, by analogy with [18, 19], a new function $\varphi(\xi)$, connected with $C(\xi)$ by the relation

$$C(\xi)=2E\frac{\varphi(\xi)}{\varphi(0)}e^{-1/\varphi(0)}\equiv\Delta\frac{\varphi(\xi)}{\varphi(0)},\qquad(1.8)$$

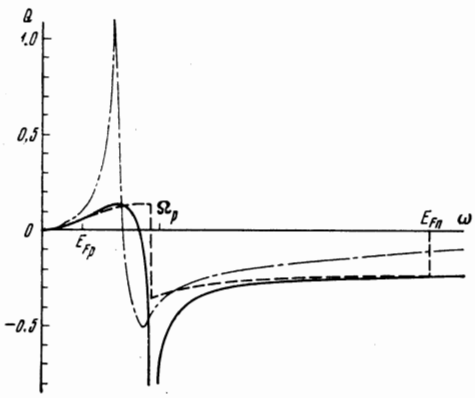

FIG. 1. The kernel $Q(\omega) \equiv K(\omega, 0)$ for different models ($\alpha = 0.3$, $\mu = 20$, $\nu = 1/E_{Fn}\tau_p = 10^{-3}$): solid curve—"jellium" model [11,16] (see Sec. 1); dashed—McMillan approximation [17] (see Sec. 3) dash-dot—"random phase" approximation [13] (see Sec. 5).

we obtain for it, according to (1.7), the following linear integral equation:

$$\varphi(\xi)=K(\xi,0)-\int_0^E\frac{d}{d\xi'}[K(\xi,\xi')\varphi(\xi')]\ln\frac{\xi'}{E}d\xi'.\qquad(1.9)$$

As shown in [18, 19], the nontrivial solution of Eq. (1.3), which branches away smoothly from the trivial solution $C(\xi) \equiv 0$ when the interaction $K(\xi, \xi')$ is turned on and corresponds to the appearance of superconductivity, exists only under the condition $\varphi(0) > 0$ (the latter, generally speaking, can be satisfied even when $K(0, 0) < 0$). However, the method used in [18, 19] to solve Eq. (1.9) is not suitable in the case of the "jellium" model, for here, according to (1.6)–(1.4), the static interaction on the Fermi surface $K(0, 0) = 0$. (In this connection, an attempt was made in [15] to solve an equation of the type of (1.9) by the method of iteration with $\alpha \to 0$.)

Nonetheless, in this case, too, it is easy to obtain an approximate solution of Eq. (1.9) (and consequently also of (1.3)) if it is recognized that the kernel $K(\xi, 0) = K(0, \xi) = Q(\xi)$ has a logarithmic singularity at $\beta(\xi) = 1$, i.e., at the point $\xi = \Omega \equiv \Omega_p(1 + \alpha/2)^{-1/2}$. A similar singularity according to (1.9), should also be possessed by the function $\varphi(\xi)$, so that derivative $d[K(\xi, \xi')\varphi(\xi')]/d\xi'$ experiences a discontinuity ($\mp\infty$) at the point $\xi' = \Omega$ at any value of ξ. Taking the slowly-varying function $\ln(\xi'/E)$ outside the integral sign at the discontinuity point, we obtain for $\varphi(\xi)$ an approximate functional equation

$$\varphi(\xi)=K(\xi,0)-\ln\frac{\Omega}{E}[K(\xi,E)\varphi(E)-K(\xi,0)\varphi(0)],\qquad(1.10)$$

from which follow the simple recurrence relations

$$\varphi(0)\left[1-K(0,0)\ln\frac{\Omega}{E}\right]=K(0,0)-\varphi(E)K(0,E)\ln\frac{\Omega}{E};$$
$$\varphi(E)\left[1+K(E,E)\ln\frac{\Omega}{E}\right]=k(E,0)\left[1+\varphi(0)\ln\frac{\Omega}{E}\right].\qquad(1.11)$$

Solving the system (1.11) relative to $\varphi(0)$, we obtain

$$\frac{1}{\varphi(0)}=\frac{1-K(E,E)\ln(E/\Omega)}{K(0,0)[1-K(E,E)\ln(E/\Omega)]+K(0,E)K(E,0)\ln(E/\Omega)}-\ln\frac{\Omega}{E}.\qquad(1.12)$$

We note that from (1.12) with allowance for (1.8) there follow immediately the results of BCS[9] and of Bogolyubov–Tolmachev[18, 19] for the gap

$$\Delta \equiv 2E \exp\{-1/\varphi(0)\}.$$

In the case of the "jellium" model (according to (1.4)–(1.6), we have

$$K(0,0) \equiv Q(0) = 0, \quad K(0,E) \equiv K(E,0) = Q(E), \quad K(E,E) = {}^1/_2 Q(2E). \tag{1.13}$$

Under the condition $E \gg \Omega$, the kernel $Q(\omega)$ in the energy region $\omega \gtrsim E$ is practically independent of ω and is equal to its asymptotic value as $\omega \to \infty$:

$$Q(\infty) \equiv -\rho(\alpha) = -\frac{\alpha}{2-\alpha}\ln\left(\frac{2+\alpha}{2\alpha}\right), \tag{1.14}$$

which corresponds to screened Coulomb repulsion.

As a result, with allowance for (1.12)–(1.14), we obtain the following approximate formula for the gap:

$$\Delta = 2\Omega \exp\left\{-\frac{2+\rho(\alpha)\ln(E/\Omega)}{2\rho^2(\alpha)\ln(E/\Omega)}\right\}. \tag{1.15}$$

It follows, therefore, in particular, that within the framework of the "jellium" model there always exists a nontrivial solution of Eq. (1.3), corresponding to the superconducting state. In other words, on the basis of this model it is impossible to obtain the criterion for the superconductivity of metals that is frequently discussed in the literature.[11, 18-20]

Formally, in the limit $E \to \infty$ and $\alpha \to 0$ we obtain from (1.15) the simpler expression:

$$\Delta = 2\Omega_p \exp\left\{-\frac{1}{\alpha\ln(1/\alpha)}\right\}, \tag{1.16}$$

which agrees qualitatively with the general conclusion of Abrikosov[14] concerning the almost exponential decrease of the gap with increasing density.

However, owing to the strong damping of the elementary excitations,[11, 12, 19] there is a "cutoff" of the Coulomb interaction in the energy region $\omega \gtrsim \Omega_n$ (where $\Omega_n = (4\pi e^2 N_n/\epsilon_i m_n)^{1/2}$ is the electron plasma frequency), so that in (1.15) it is natural to choose as the upper limit $E = \min\{E_{Fn}, \Omega_n\}$. When $\alpha < 0.2$, we have $\Omega_n/E_{Fn} \equiv (16\alpha/3)^{1/2} < 1$).

It is interesting to note that from (1.8) and (1.12) there follows, as a particular case, the formula obtained for the gap in [21] (see the review [22]). Indeed, putting in (1.13) $K(E, E) = Q(E) \simeq -\rho(\alpha)$, with $E = E_F$, $\Omega = \omega_i$, and $\alpha \ll 1$, we obtain

$$\Delta = 2\omega_i e^{-1/g}; \quad g = \frac{[(\alpha/2)\ln(1/\alpha)]^2 \ln(E_F/\omega_i)}{1+(\alpha/2)\ln(1/\alpha)\ln(E_F/\omega_i)}. \tag{1.17}$$

It is easy to show that expression (1.17) corresponds to the approximate solution of Eq. (1.3) with kernel $K(\xi, \xi')$ having the form of a "square well" (compare with [16]):

$$K(\xi,\xi') = \begin{cases} -(\alpha/2)\ln(1/\alpha) & \text{for } \omega_i \leqslant |\xi|,\ |\xi'| \leqslant E_F; \\ 0 & \text{for } |\xi|,\ |\xi'| < \omega_i \text{ and } |\xi|,\ |\xi'| > E_F. \end{cases} \tag{1.18}$$

As will be shown below, in the region of the parameters of real semiconductors (semimetals) which is of greatest interest from the practical point of view, formulas (1.15) and (1.17) considerably underestimate the gap compared with the numerical solution (which, incidentally, depends little on the upper limit E).

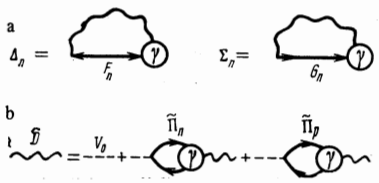

FIG. 2. Self-energy electron parts (a) and the Dyson equation for the "plasmon" Green's function (b).

2. STRONG-COUPLING APPROXIMATION. VERTEX PART

At not too high concentrations of the free carriers in degenerate semiconductors and semimetals, i.e., at not very small values of the parameter $\alpha \equiv e^2/\pi\epsilon_i v_{Fn}$), but such that $\alpha\Omega_p/E_{Fn} \ll 1$,[3)] the interaction in the "jellium" model can no longer be regarded as weak, and the superconductivity phenomenon must be considered on the basis of the so-called "strong coupling" approximation.

For superconductors with electron-phonon interaction, the theory of strong coupling with allowance for retardation was developed by Éliashberg,[23] who generalized Migdal's method[24] for normal metals. In the case of systems with predominant Coulomb interaction,[1, 4-6] in analogy with [23], the equations of the strong-coupling approximation at T = 0 can be represented in the form (compare with [1]):

$$\Delta_n(p) = \frac{i}{(2\pi)^4}\gamma_0 \int dp' F_n(p') \cdot \tilde{D}(p-p')\gamma(p,p'), \tag{2.1}$$

Here

$$\Sigma_n(p) = \frac{i}{(2\pi)^4}\gamma_0 \int dp' G_n(p') \cdot \tilde{D}(p-p')\gamma(p,p'). \tag{2.2}$$

$$F_n(p) = -\Delta_n(p)/\Omega_n(p) \quad p \equiv (\mathbf{k}, \omega); \tag{2.3}$$

$$G_n(p) = [\omega + \xi_n(\mathbf{k}) + \Sigma_n(-p)]/\Omega_n(p); \tag{2.4}$$

$$\Omega_n(p) = [\omega - \xi_n(\mathbf{k}) - \Sigma_n(p)] \cdot [\omega + \xi_n(\mathbf{k}) + \Sigma_n(-p)] - |\Delta_n(p)|^2; \tag{2.5}$$

γ_0 is a simple Coulomb three-pole, equal to the charge of the electron e; $\gamma(p, p')$ is a three-pole corresponding to the (effective) charge of the quasiparticles, renormalized as a result of the interaction (see below), and $\tilde{D}(p - p')$ is the Fourier component of the time-dependent Green's function of the longitudinal (Coulomb) field, satisfying the Dyson equation (see Fig. 2) and equal to

$$\tilde{D}(\mathbf{q},\omega) = V_0(\mathbf{q})\left[1 - V_0(\mathbf{q})\sum_j \tilde{\Pi}_j(\mathbf{q},\omega)\right]^{-1} \equiv \frac{4\pi}{\mathbf{q}^2\tilde{\varepsilon}(\mathbf{q},\omega)}, \tag{2.6}$$

where $V_0(\mathbf{q}) = 4\pi/\mathbf{q}^2\epsilon_i$, and $\tilde{\Pi}_j$ is the polarization operator of the particles of type j:

$$\tilde{\Pi}_j(p) = -\frac{2i}{(2\pi)^4}\gamma_0 \int dp' G_j(p') G_j(p-p')\gamma(p,p'). \tag{2.7}$$

[3)] To satisfy the "adiabaticity" condition $\Omega_p \ll E_{Fn}$ with sufficient margin (at least by a factor of several times), it is necessary to satisfy, besides the condition $m_p \gg m_n$, also the inequality $N_p \ll N_n$ (which is possible only in impurity semiconductors and in uncompensated multiband metals).

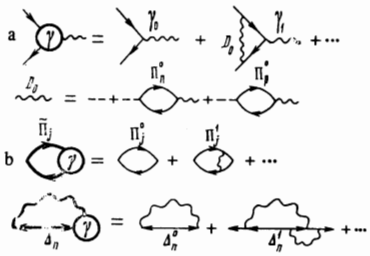

FIG. 3. Approximations used in the calculation of the Coulomb vertex part (a) and the self-energy of the plasmons and electrons (b).

We see that the function $\tilde{D}$ coincides with the longitudinal component D_{00} of the causal Green's function of an electromagnetic field in a homogeneous isotropic medium [25] (with allowance for spatial dispersion).

Comparing (2.1), (2.2), (2.6), and (2.7) with the corresponding expressions of [1], we readily see that the transition to the approximation of large density is equivalent to replacement of the complete vertex $\gamma(p, p')$ by the simple vertex $\gamma_0 \equiv e$. Let us estimate the error resulting from such a replacement.[4)] To this end we calculate the first-order correction to γ_0 (see Fig. 3):

$$\gamma_1(p,p') = \frac{ie^3}{(2\pi)^4}\int dp_1 D(p-p_1) G_n\left(p_1+\frac{p'}{2}\right) G_n\left(p_1-\frac{p'}{2}\right). \quad (2.8)$$

The function D differs from $\tilde{D}$ in (2.6) in that $\tilde{\Pi}_j$ is replaced by $e^2\Pi_j$, where

$$\Pi_j(p) = -\frac{2i}{(2\pi)^4}\int dp' G_j(p') G_j(p-p'). \quad (2.9)$$

In addition, in the calculation of Π_j and γ_1 we can replace G_j with high accuracy by the unperturbed Green's function of the free carriers G_j^0 (see [24]).

Just as in [24], the most "dangerous" region for γ_1 is the region of small $k' \ll p_{Fn}$ and $\omega' \ll E_{Fn}$. In particular, in the limit as $k' \to 0$, $\omega'/k' \to 0$ (but $k \sim p_{Fn}$) for the "jellium" model, we obtain in accord with (2.8) (see (1.5)):

$$\gamma_1(p,0) \cong \frac{e^3}{(2\pi)^2 v_{Fn}}\int_0^{2p_{Fn}} q\,dq\, D(q,\omega) = -eQ(\omega). \quad (2.10)$$

The dependence of γ_1 on ω in this case is significant only in the energy region $\omega \lesssim \Omega_p$ (with $\gamma_1 \to 0$ as $\omega \to 0$), whereas in the region $\omega \gg \Omega_p$ we have $\gamma_1 \simeq e\rho(\alpha) = \text{const}$ (see (1.14)). Therefore at $k' \sim p_{Fn}$ (but $\omega' \ll k' v_{Fn}$), accurate to terms of the order of $\Omega_p/E_{Fn} \equiv (16\alpha/3\mu)^{1/2}$, where $\mu \equiv m_p N_n / m_n N_p \gg 1$ and $\rho^2(\alpha) < \alpha^2$ (for $\alpha > 0.1$) the vertex $\gamma(p, p')$ can be approximately replaced by $\gamma \simeq e[1+\rho(\alpha)]$, and the interaction $\gamma_0 \tilde{D}\gamma$ in (2.1) and (2.2) can in this case be represented in the form of a renormalized Coulomb (irreducible) four-pole (compare with [1]):

$$\tilde{\Gamma}_{nn}(q,\omega) = \frac{4\pi\tilde{e}^2}{q^2\varepsilon_1 - 4\pi\tilde{e}^2[\Pi_n(q,\omega)+\Pi_p(q,\omega)]}, \quad \tilde{e}^2 \cong e^2[1+\rho(\alpha)]. \quad (2.11)$$

As a result, the system (2.1) and (2.2) takes in the isotropic case the form (compare with [13, 23]):

$$\Delta_n(\omega) = -\frac{1}{2(2\pi)^2 v_{Fn}}\int_0^{2p_{Fn}} q\,dq \int_L dz\, \tilde{\Gamma}_{nn}(q, |z-\omega|)\frac{\Delta_n(z)}{\Omega(z)}\,\text{sign Im}\,\Omega(z), \quad (2.12)$$

$$f_n(\omega) = \frac{1}{2(2\pi)^2 v_{Fn}}\int_0^{2p_{Fn}} q\,dq \int_L dz\, \tilde{\Gamma}_{nn}(q, |z-\omega|)\frac{z-f_n(z)}{\Omega(z)}\,\text{sign Im}\,\Omega(z), \quad (2.13)$$

where

$$\Omega^2(z) = [z-f_n(z)]^2 - |\Delta_n(z)|^2, \quad \Delta_n(\omega) \equiv \Delta_n(p_{Fn}, \omega), \quad f_n(\omega) \equiv f_n(p_{Fn}, \omega). \quad (2.14)$$

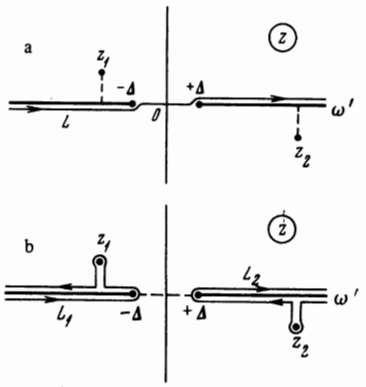

FIG. 4

The integration in (2.12) and (2.13) is carried out in the complex z plane along the contour L shown in Fig. 4a (see [23]). Within the framework of the "jellium" model, the function $\tilde{\Gamma}_{nn}(q, |z-\omega|)$ is analytic everywhere except for the poles at the points $z_{1,2} = \omega \pm (\Omega_q - i\delta_q)$ corresponding to the plasma-acoustic oscillations, with $\delta_q \ll \Omega_q$ determining the weak damping of the acoustic plasmons (see [1]). Therefore the integration contour L can be deformed into a contour L_1 or L_2 (Fig. 4b) passing along the edges of the cuts $(-\infty, -\Delta)$ and (Δ, ∞) and going around the corresponding poles. It can readily be shown that the residues at the points z_1 and z_2 contribute only to the imaginary part of $\Delta_n(\omega)$ in a narrow energy region $\omega \sim \Omega_q$, and can therefore be discarded.

As a result, carrying out the integration in (2.12) and (2.13), and symmetrizing the kernels with respect to the signs of the arguments ω and ω', we obtain the following system of equations for the functions $C(\omega) \equiv \text{Re}\{\Delta_n(\omega) \cdot [1 - f_n(\omega)/\omega]^{-1}\}$ and $f_0(\omega) \equiv \text{Re}\{f_n(\omega)\}$ (compare with [1, 23]):

$$C(\omega) = \frac{\tilde{\alpha}}{2}\left[1-\frac{f_0(\omega)}{\omega}\right]^{-1}\int_\Delta^\infty d\omega'\,\text{Re}\left\{\frac{C(\omega')}{\sqrt{\omega'^2 - C^2(\omega')}}\right\} \cdot \times \int_0^{2p_{Fn}} \frac{q\,dq}{q^2+\tilde{\varkappa}_n^2(q)}\left\{\frac{\tilde{\Omega}_q}{2}\left[\left(\frac{1}{\omega'-\omega+\tilde{\Omega}_q}+\frac{1}{\omega'+\omega+\tilde{\Omega}_q}\right) - \left(\frac{1}{\omega'-\omega-\tilde{\Omega}_q}+\frac{1}{\omega'+\omega-\tilde{\Omega}_q}\right)\right]-2\right\}, \quad \Delta \equiv C(\Delta), \quad (2.15)$$

$$f_0(\omega) = -\frac{\tilde{\alpha}}{2}\int_\Delta^\infty d\omega'\,\text{Re}\left\{\frac{\omega'}{\sqrt{\omega'^2 - C^2(\omega')}}\right\}\int_0^{2p_{Fn}} \frac{q\,dq}{q^2+\tilde{\varkappa}_n^2(q)}\frac{\tilde{\Omega}_q}{2} \cdot \times\left[\left(\frac{1}{\omega'-\omega+\tilde{\Omega}_q}-\frac{1}{\omega'+\omega+\tilde{\Omega}_q}\right)-\left(\frac{1}{\omega'-\omega-\tilde{\Omega}_q}-\frac{1}{\omega'+\omega-\tilde{\Omega}_q}\right)\right], \quad (2.16)$$

where

[4)] As shown in [24], in the case of electron-phonon interaction the error connected with the substitution $\Gamma_0 = 1$ in lieu of Γ does not exceed the ratio $\omega_D/E_F \sim \sqrt{m_0/m_i}$ (where m_0 and M_i are the masses of the electron and the ion).

$$\tilde{\alpha}\cong\alpha[1+\rho(\alpha)],\ \tilde{\Omega}_q=\tilde{\Omega}_p\left(\frac{q^2}{q^2+\tilde{\varkappa}_n^2}\right)^{1/2},$$
$$\tilde{\Omega}_p^2=\Omega_p^2\frac{\tilde{\alpha}}{\alpha},\quad \tilde{\varkappa}_n^2=\varkappa_n^2\frac{\tilde{\alpha}}{\alpha}. \tag{2.17}$$

Grouping corresponding terms in the kernels of Eqs. (2.15) and (2.16) and integrating with respect to q, we represent these equations in more compact form:

$$C(\omega)=\left[1-\frac{f_0(\omega)}{\omega}\right]^{-1}\int_\Delta^\infty d\omega' K^+(\omega,\omega')\,\mathrm{Re}\left\{\frac{C(\omega')}{\sqrt{\omega'^2-C^2(\omega')}}\right\}, \tag{2.18}$$

$$f_0(\omega)=-\int_\Delta^\infty d\omega' K^-(\omega,\omega')\,\mathrm{Re}\left\{\frac{\omega'}{\sqrt{\omega'^2-C^2(\omega')}}\right\}, \tag{2.19}$$

where (compare with (1.4) and (1.5):

$$K^\pm(\omega,\omega')=\tfrac{1}{2}[\tilde{Q}(\omega-\omega')\pm\tilde{Q}(\omega+\omega')], \tag{2.20}$$

$$\tilde{Q}(\omega)=\frac{\tilde{\alpha}\omega^2}{2[\tilde{\Omega}_p^2-\omega^2(1-\tilde{\alpha}/2)]}\ln\left|\frac{\tilde{\Omega}_p^2-\omega^2(1+\tilde{\alpha}/2)}{\tilde{\alpha}\omega^2}\right|. \tag{2.21}$$

We call attention to one interesting circumstance. Inasmuch as in the calculation of f_0 we can neglect the pairing of the electrons (see [23]), we obtain, according to (2.16), for $\omega\ll\tilde{\Omega}_p$ (see [1])

$$f_0(\omega)=-\frac{\tilde{\alpha}}{2}\int_0^{2p_{Fn}}\frac{q\,dq}{q^2+\tilde{\varkappa}_n^2(q)}\frac{\tilde{\Omega}_q}{2}\ln\left|\frac{\tilde{\Omega}_q+\omega}{\tilde{\Omega}_q-\omega}\right|\approx-\omega\rho(\tilde{\alpha}). \tag{2.22}$$

(In the region $\omega\gg\tilde{\Omega}_p$ we have $f_0(\omega)\sim 1/\omega$.)

Thus, accurate to terms $\sim\rho^2(\alpha)$, the renormalization of the "electron-plasmon" interaction in the strong-coupling approximation, characterized in the most significant region of the energies $\omega<\Omega_p$ by the quantity $[1-f_0(\omega)/\omega]^{-1}\approx[1+\rho(\alpha)]^{-1}$, compensates for the renormalization of the charge in the Coulomb vertex (2.11) (or (2.21)), which is connected with the violation of the condition for the applicability of the high-density approximation $\alpha\ll 1$.[5)] In other words, owing to the almost accidental cancellation of the corresponding terms, the region of applicability of the so-called "gas approximation" in the "jellium" model broadens greatly and is determined by the condition

$$\rho^2(\alpha)\equiv\left[\frac{\alpha}{2-\alpha}\ln\left(\frac{2+\alpha}{2\alpha}\right)\right]^2\ll 1, \tag{2.23}$$

which is much weaker than the condition $\alpha\ll 1$, and can be relatively easily satisfied in real systems (in particular, the inequality $\rho^2\ll 1$ is satisfied for practically all superconductors[11, 12, 20]).

As a result, the equation (2.18) for the gap takes, with good accuracy, the form (we have replaced $\tilde{\alpha}$ by α throughout):

$$C(\omega)=\int_\Delta^\infty d\omega' K^+(\omega,\omega')\,\mathrm{Re}\left\{\frac{C(\omega')}{\sqrt{\omega'^2-C^2(\omega')}}\right\}. \tag{2.24}$$

Changing over in (2.24) from the energy of the quasiparticles ω to the energy of the electrons ξ, i.e., making the substitution $\omega\to\epsilon(\xi)=\sqrt{\xi^2+C^2(\xi)}$, where $C(\xi)\equiv C(\epsilon(\xi))$, we obtain

$$C(\xi)=\int_0^\infty d\xi' K^+(\varepsilon(\xi),\varepsilon(\xi'))\left[1+\frac{C(\xi')}{\xi'}\frac{dC(\xi')}{d\xi'}\right]\frac{C(\xi')}{\varepsilon(\xi')}. \tag{2.25}$$

As $C(\xi)\to 0$, Eq. (2.25) goes over into Eq. (1.3) of the weak-coupling approximation.

3. CRITICAL TEMPERATURE

To determine the critical temperature T_c of the superconducting transition it is necessary to generalize the equation for the gap (2.15) (or (2.18)) to the case of finite temperatures $T\neq 0$. To this end, in accordance with the method developed in [26, 27], we use the following procedure: the denominators of the type $[\omega'\mp\omega\pm\Omega_q]^{-1}$ in the kernel of Eq. (2.15) are multiplied by the sum of the distribution functions of the electrons $f(\mp\omega')=[\exp\{\mp\omega'/T\}+1]^{-1}$ and of the plasmons $N(\pm\Omega_q)=[\exp\{\pm\Omega_q/T\}-1]^{-1}$, and the term corresponding to the Coulomb repulsion is multiplied by $[1-2f(\omega')]$. As a result, the expression in the curly brackets in (2.15) reduces to

$$\begin{aligned}&\left\{\frac{\Omega_q}{2}\left[(N(\Omega_q)+f(-\omega'))\left(\frac{1}{\omega'-\omega+\Omega_q}+\frac{1}{\omega'+\omega+\Omega_q}\right)\right.\right.\\&-(N(\Omega_q)+f(\omega'))\left(\frac{1}{-\omega'-\omega+\Omega_q}+\frac{1}{-\omega'+\omega+\Omega_q}\right)\Big]\\&-\frac{\Omega_q}{2}\left[(N(-\Omega_q)+f(-\omega'))\left(\frac{1}{\omega'-\omega-\Omega_q}+\frac{1}{\omega'+\omega-\Omega_q}\right)\right.\\&\left.\left.-(N(-\Omega_q)+f(\omega'))\left(\frac{1}{-\omega'-\omega-\Omega_q}+\frac{1}{-\omega'+\omega-\Omega_q}\right)\right]\right.\\&\left.-2[1-2f(\omega')]\right\}=\left\{\mathrm{th}\frac{\omega'}{2T}\left[\frac{\Omega_q}{2}\left(\frac{1}{\omega'-\omega+\Omega_q}+\frac{1}{\omega'+\omega+\Omega_q}\right)\right.\right.\\&-\frac{\Omega_q}{2}\left(\frac{1}{\omega'-\omega-\Omega_q}+\frac{1}{\omega'+\omega-\Omega_q}\right)-2\Big]\\&+\mathrm{cth}\frac{\Omega_q}{2T}\left[\frac{\Omega_q}{2}\left(\frac{1}{\omega'-\omega+\Omega_q}+\frac{1}{\omega'+\omega+\Omega_q}\right)\right.\\&\left.\left.+\frac{\Omega_q}{2}\left(\frac{1}{\omega'-\omega-\Omega_q}+\frac{1}{\omega'+\omega-\Omega_q}\right)\right]\right\}.\end{aligned} \tag{3.1}$$

Here we have taken into account the relations

$$[f(-\omega')-f(\omega')]=[1-2f(\omega')]=\mathrm{th}\frac{\omega'}{2T};\ [N(\Omega_q)-N(-\Omega_q)]=\mathrm{cth}\frac{\Omega_q}{2T}.$$

It is also possible to transform in similar fashion the kernel of Eq. (2.19) for the function $f_0(\omega)$. However, as shown above, accurate to terms $\sim\rho^2(\alpha)$, the renormalization of the interaction can be neglected, and therefore we shall henceforth omit throughout the terms containing $f_0(\omega)$. At the same time, to determine the damping of the excitations, it is necessary to take into account the imaginary part $f_1(\omega)\equiv\mathrm{Im}\{f_n(\omega)\}$ (see [23, 28]).

Thus, the equation for the gap at $T\neq 0$ takes the form

$$\begin{aligned}C(\omega,T)=&\frac{\alpha}{2}\int_0^{2p_{Fn}}\frac{q\,dq}{q^2+\varkappa_n^2(q)}\int_{\Delta(T)}^\infty d\omega'\,\mathrm{Re}\left\{\frac{C(\omega',T)}{\sqrt{\omega'^2-C^2(\omega',T)}}\right\}\\&\times\left\{\frac{\Omega_q}{2}\left[\left(\mathrm{th}\frac{\omega'}{2T}+\mathrm{cth}\frac{\Omega_q}{2T}\right)\left(\frac{1}{\omega'-\omega+\Omega_q}+\frac{1}{\omega'+\omega+\Omega_q}\right)-\right.\right.\\&\left.\left.-\left(\mathrm{th}\frac{\omega'}{2T}-\mathrm{cth}\frac{\Omega_q}{2T}\right)\left(\frac{1}{\omega'-\omega-\Omega_q}+\frac{1}{\omega'+\omega-\Omega_q}\right)\right]-2\,\mathrm{th}\frac{\omega'}{2T}\right\}\end{aligned} \tag{3.2}$$

We note that (3.2) differs from the equation for the gap in the case of the electron-photon interaction, obtained in [28] by the method of the temperature Green's functions[25, 29] only in that ω_q is replaced by Ω_q and α_q^2

[5)] Strictly speaking, all that is cancelled is the renormalization of the effective parameter of the "electron-plasma" interaction $\tilde{\rho}$, which stands in the argument of the exponential (see (1.15)), whereas the renormalization of the plasmon energy $\tilde{\Omega}_p$ leads to an insignificant change (increase) of the pre-exponential factor, and consequently of the gap parameter Δ, by an amount $\sim p(\alpha)$.

by $2\pi e^2\Omega_q/\epsilon_i(q^2+\kappa^2)$, and also in that the Coulomb repulsion is taken into account. Accurate to terms $\sim (T_c/\Omega_p)^2$, Eq. (3.2) reduces as $T \to T_c$ to the form

$$C(\omega, T_c) = \int_0^\infty \frac{d\omega'}{\omega'} K^+(\omega, \omega') C(\omega', T_c) \operatorname{th}\frac{\omega'}{2T_c}. \qquad (3.3)$$

From a comparison of (3.3) and (2.24) it follows that in the "plasmon" mechanism of superconductivity, the usual BCS relation[9] between the gap $\Delta(0)$ at $T = 0$ and the critical temperature T_c (see [28, 30]) is retained, with high accuracy.

By virtue of the linearity of Eq. (3.3), it is easy to obtain with its aid an approximate expression for T_c (see, for example, [17]). Approximating in the first approximation the function $C(\omega, T_c)$ by the sign-alternating step function

$$C(\omega, T_c) = \begin{cases} \Delta_0 & \text{for } \omega < \Omega, \\ -\Delta_\infty & \text{for } \omega > \Omega, \end{cases} \qquad (3.4)$$

where $\Omega = \Omega_p(1+\alpha/2)^{-1/2}$ is the limiting energy of the virtual acoustic plasmons (for $q = 2p_{Fn}$), we obtain under the condition $T_c \ll \Omega_p \ll E_{Fn}$ the following system of equations (compare with [17]):

$$C(0,T_c) \equiv \Delta_0 = \Delta_0 \int_0^\Omega \frac{d\omega'}{\omega'} Q(\omega') \operatorname{th}\frac{\omega'}{2T_c} - \Delta_\infty \int_\Omega^{E_{Fn}} Q(\omega')\frac{d\omega'}{\omega'}, \qquad (3.5)$$

and

$$C(\infty, T_c) \equiv -\Delta_\infty = \Delta_0 Q(\infty) \int_0^\Omega \frac{d\omega'}{\omega'} \operatorname{th}\frac{\omega'}{2T_c} - \Delta_\infty Q(\infty) \ln\frac{E_{Fn}}{\Omega}. \qquad (3.6)$$

If we neglect in (3.5) the terms $\sim(\omega'/\Omega)^2$ in the region $\omega' < \Omega$ and the terms $\sim(\Omega/\omega')^2$ in the region $\omega' > \Omega$, then, as can readily be seen, we arrive at the expression $T_c = 1.14\,\Omega \exp\{-1/g\}$, corresponding to the approximation (1.17) and (1.18). However, this approximation can be improved somewhat by taking into account in (3.5) the quadratic terms ($\sim\omega^2$ and $\sim 1/\omega^2$), so that the kernel $Q(\omega)$ takes the form shown dashed in Fig. 1. We then obtain

$$\Delta_0[1-\rho_0(\alpha)] = \Delta_\infty[\rho(\alpha)\ln(E_{Fn}/\Omega) + \rho_1(\alpha)], \qquad (3.7)$$

$$\Delta_\infty[1+\rho(\alpha)\ln(E_{Fn}/\Omega)] = \Delta_0\rho(\alpha)\ln\left(\frac{1{,}14\Omega}{T_c}\right), \qquad (3.8)$$

where

$$\rho_0(\alpha) = \frac{2-\alpha}{2+\alpha}\left[1+\ln\left|\frac{2+\alpha}{2-\alpha}\right|\right] + \frac{2\alpha}{2+\alpha}\left[1+\ln\left(\frac{2+\alpha}{2\alpha}\right)\right] + 1, \qquad (3.9)$$

$$\rho_1(\alpha) = \frac{\alpha(2+\alpha)}{2(2-\alpha)^2}\left[\ln\left(\frac{2+\alpha}{2\alpha}\right) + \frac{2\alpha}{2+\alpha} - 1\right]. \qquad (3.10)$$

The condition for the solvability (self-consistency) of the system (3.7) and (3.8), which plays the role of an equation for the determination of the critical temperature T_c is (compare with (1.17))

$$1 = \frac{\rho(\alpha)[\rho(\alpha)\ln(E_{Fn}/\Omega) + \rho_1(\alpha)]}{[1-\rho_0(\alpha)][1+\rho(\alpha)\ln(E_{Fn}/\Omega)]} \ln\left(\frac{1{,}14\Omega}{T_c}\right). \qquad (3.11)$$

In the absence of cutoff ($E_{Fn} \to \infty$), the solution of (3.11) takes the form

$$T_c = 1.14\Omega \exp\left\{-\frac{1-\rho_0(\alpha)}{\rho(\alpha)}\right\}. \qquad (3.12)$$

We know that expression (3.11) gives the correct order of magnitude for T_c in the case of superconducting metals with electron-phonon interaction, when $\mu \equiv M_i/z^* m_0 \sim 10^4$–$10^5$, $\alpha \approx 0.3$–0.4, and $T_c \lesssim 0.1\,\omega_D \lesssim 10^{-3} E_F$ (z^* is the effective valence, i.e., the number of free electrons per atom). Thus, for example, for Al with $z^* = 3$, $\mu = 1.65\times10^4$, and $\alpha = 0.34$, we obtain in accordance with (3.11) the estimate $T_c \approx 2.4\,^\circ K$ instead of $T_{c.exp} = 1.2\,^\circ K$, and for Pb with $z^* = 4$, $\mu = 0.95 \times 10^5$, and $\alpha = 0.38$ we have $T_c \approx 3\,^\circ K$ instead of $T_{c.exp} = 7.2\,^\circ K$. In the case of the "plasmon" mechanism of superconductivity in degenerate semiconductors (semimetals), when in practice $\mu \equiv m_p N_n/m_n N_p \sim 10^1$–$10^2$, formulas (3.11) and (1.15) (or (1.17)) lead to values of T_c and Δ which are too low compared with the exact solution by almost two orders of magnitude (see Sec. 5).

4. METHOD OF NUMERICAL SOLUTION OF THE EQUATION FOR THE GAP

In this section we describe briefly the main idea of the method used in the present paper for the numerical solution of the nonlinear Fredholm integral equation of the first kind, such as is the equation (1.3) for the gap:

$$y(x) = \int_0^L dx' K(x,x') \frac{y(x')}{\sqrt{x'^2 + y^2(x')}} \qquad (4.1)$$

(we have introduced here the dimensionless quantities $x = \xi/E_{Fn}$, $y(x) = C(\xi)/E_{Fn}$ and $L = E/E_{Fn}$).

The mathematical difficulties arising in the numerical solution of an equation of the type (4.1) are connected principally with the fact that it is, first, homogeneous (and therefore additional complications arise with separation of the nontrivial solution) and, second, it is essentially nonlinear.

Zubarev[31] and Garland[5] proposed independently a method for "quasilinearization" of Eq. (4.1), which made it possible simultaneously to get around both of these difficulties and to ensure sufficiently rapid convergence. With the aid of suitable normalization of the kernel $I(x, x') = K(x, x')/K(0, 0)$ and of the gap $\Phi(x) = y(x)/y(0)$, with allowance for the boundary conditions on the Fermi surface $I(0, 0) = \Phi(0) = 1$, Eq. (4.1) was reduced in [5, 31] to an inhomogeneous integral equation (Fredholm equation of the second kind) with a weak nonlinearity, the solution of which could easily be obtained by the iteration method.

However, in the case of the "jellium" model, the Zubarev–Garland method is not applicable, for here $K(0, 0) \equiv 0$. We therefore proceed in the following manner. We break up the region of integration $[0, L]$ in (4.1) into intervals (x_i, x_{i+1}) with constant spacing δ (usually $L = 1$ and $\delta = 0.05$). Approximating $y(x)$ by a step function, we obtain in place of the integral equation (4.1) a system of homogeneous nonlinear algebraic equations of n-th order:

$$y_i = \sum_{j=1}^{n} K_{ij} \frac{y_j}{\sqrt{y_j^2 + x_j^2}} \qquad i = 1, \ldots, n, \qquad (4.2)$$

where

$$K_{ij} = \int_{x_j}^{x_{j+1}} dx' K(x_i, x'), \qquad y_i \equiv y(x_i). \qquad (4.3)$$

Since the kernel $K(x, x')$ has logarithmic singularities (see (1.4) and (1.5)), to increase the accuracy of the calculation of the coefficients we broke up the interval

(x_j, x_{j+1}) additionally into small segments with spacing $h \ll \delta$.

Together with (4.2), we consider the system of inhomogeneous equations

$$f_i = y_i - \sum_{j=1}^{n} K_{ij} y_j (y_j^2 + x_j^2)^{-1/2}, \qquad (4.4)$$

which is equivalent to the system (4.2) at $f_i \equiv 0$ ($i = 1, \ldots, n$). Thus, if f_i is regarded as a function of the vector $Y = \{y_1, \ldots, y_n\}$ in n-dimensional Euclidean space, then the finding of the solution of the homogeneous system of equations (4.2) reduces to a selection of such a set of values y_i^* for which all the $f_i(Y^*)$ in (4.4) tend to zero. To this end we introduce the function

$$\mathcal{F}(Y) = \sum_{i=1}^{n} f_i^2(Y) \Big/ \sum_{i=1}^{n} y_i^2 \equiv \sum_{i=1}^{n} \Big\{ y_i - \sum_{j=1}^{n} \frac{K_{ij} y_j}{\sqrt{y_j^2 + x_j^2}} \Big\}^2 \Big/ \sum_{i=1}^{n} y_i^2. \qquad (4.5)$$

The denominator here is introduced in order to get rid of the trivial solution $y_i \equiv 0$ of the system (4.2). In the case when there exists at least one nontrivial solution $y_i^* \neq 0$ of the system (4.2), the corresponding value is $\mathcal{F}(Y^*) \equiv 0$. As a result, we arrive at the problem of minimizing the non-negative function $\mathcal{F}(Y)$ in the space of the vectors Y.

As is well known, the main difficulty of any minimization method lies in the choice of the step. In this respect, it is convenient to use methods in which the step is not an external parameter of the algorithm of the problem, but is chosen automatically during the process of the calculations. Such methods include, for example, the method of steepest descent. However, the rate of convergence of this method depends strongly on the relief of the minimized function and, in particular, for a function of the "ravine" type[6] it can be very small. The performed analysis shows that near the minimum the function $\mathcal{F}(Y)$ has strongly elongated level lines, recalling a "ravine" in shape. This causes the angle between the direction of the maximum gradient and the direction of the bottom (bed) of the "ravine" to be close to $\pi/2$, and the method of steepest descent becomes ineffective. Therefore to increase the rate of convergence of the process of minimization of the function $\mathcal{F}(Y)$ (by the method of steepest descent) the space Y was preliminarily "stretched" in the direction of the maximum gradients in order to decrease the local "ravine" coefficients (i.e., the relative curvature of the level lines), leading to a considerable decrease in the computation time (from 16 hours to 20–40 minutes of BESM-6 computer time per variant). Values $\min\{\mathcal{F}(Y)\} = 10^{-15} - 10^{-18}$ were attained thereby.

In concluding this section we note that the initial equation (4.1) corresponds to the weak-coupling approximation (see (1.3)–(1.5)). It is possible to obtain similarly a solution of the equation (2.25) for the gap in the strong-coupling approximation, and at sufficiently small $|C(\xi)|$ the additional nonlinearity of the kernel, which can be taken into account by successive approximations, leads only to slight corrections.

[6] A function of the "ravine" type is a function that decreases rapidly in one direction (i.e., it has a large gradient), and slowly in other directions (small gradients).

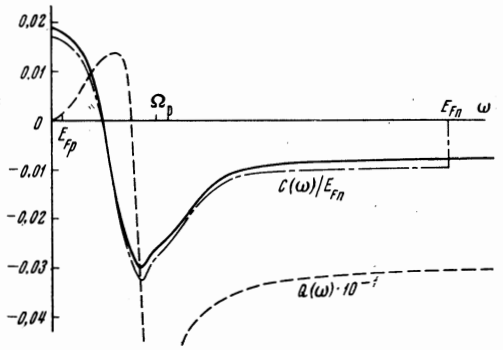

FIG. 5. Dispersion of the gap $C(\omega)$ in the "jellium" model ($\alpha = 0.5$; $\mu = 40$).

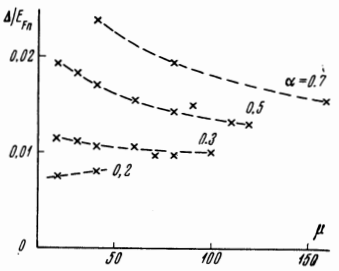

FIG. 6. Dependence of the gap parameter $\Delta \equiv C(0)$ on μ for different values of α.

5. DISCUSSION OF RESULTS

The main results of the calculations are shown in Figs. 5–7. Figure 5 shows a typical form of the dispersion of the gap in the "jellium" model, i.e., the dependence of the function $C(\xi)$, normalized to the Fermi energy of the electrons E_{Fn}, on ξ at values of the parameters $\alpha \equiv (\pi p_{Fn} a_n)^{-1} = 0.5$ and $\mu \equiv m_p N_n / m_n N_p = 40$. The solid curve corresponds to the solution of Eq. (1.3) with an upper limit of integration $E = 3E_{Fn}$ (i.e., $L = 3$ and $n = 60$), whereas the dash-dot curve shows the solution with $E = E_{Fn}$ (the dashed curves represent the kernel $K(\omega, 0) \equiv Q(\omega)$, decreased for convenience by a factor of 10). We see that $C(\xi)$ depends quite little on the upper limit (when E changes by three times the gap $\Delta \equiv C(0)$ changes by approximately 10%). Figure 6 shows the dependence of the normalized gap Δ/E_{Fn} on μ at different values of the parameter α (the crosses show the results of the numerical calculation). We see that in this case the "isotopic" effect turns out to be quite weakened (compared with the $\sim 1/\sqrt{\mu}$ law) and depends on α.[7] We note that in order of magnitude $\Delta/E_{Fn} \sim 10^{-2}$ and $\Delta/\Omega_p \lesssim 10^{-1}$, and the weak-coupling approximation is fully justified. At the same time, the values of Δ obtained with the aid of numerical calculation exceed by almost two orders of magnitude the gap calculated from formulas (1.15) or (1.17) at the same values of α and μ. This discrepancy is connected with the very rough approximation of the kernels and with the approximate character of the solutions (1.15) and (1.17) (see also (3.11)). The latter, being quite sensitive to the form of the kernel of the inte-

[7] In particular, at $\alpha = 0.2$ there is noted a negative "isotopic effect" (compare with the results of Swihart [16] for Ru). In general, within the framework of the "jellium" model, the normal isotopic effect ($T_c \sim \mu^{-1/2}$) takes place only in the limit as $\mu \to \infty$ (see (1.16) and (3.12)).

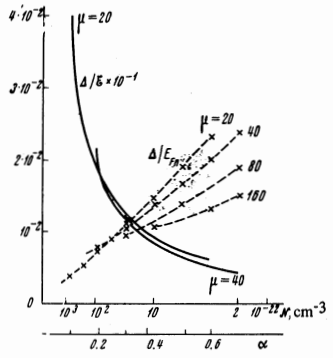

FIG. 7. Dependence of Δ/E_{Fn} and $\Delta/\mathscr{E}$ on α' and $\mathcal{N}$ for different values of μ.

gral equation (1.3), were obtained in essence without allowance for the strong dispersion ("oscillations") of the gap $C(\omega)$ in the "resonant" region of energies $\omega \sim \Omega_p$ (see Fig. 5), the contribution of which increases rapidly with decreasing μ (for $\alpha = \text{const}$). It follows, therefore, in particular, that it is necessary to treat with caution the approximate estimates of Δ (or T_c) obtained on the basis of various simplified models (see below).

Finally, Fig. 7 shows a plot of Δ/E_{Fn} against α for different values of the parameter μ (dashed curves), and also the dependence of the quantity $\Delta/E_{Fn}\alpha^2 \equiv \Delta/\mathscr{E}$, where $\mathscr{E} \equiv e^2/2\pi^2\epsilon_i a_n \simeq 1.38\, m_n/m_0\epsilon_i^2$ (eV), on the reduced electron concentration $\mathcal{N} \equiv N_n(\epsilon_i m_0/m_n)^3 \simeq 7.4 \times 10^{21}\alpha^{-3}$ [cm^{-3}] at $\mu = 20$ and 40 (solid curves). The most interesting dependence is the last one, from which it follows that the gap Δ, first, is directly proportional to the effective mass of the conduction electron m_n, second, is inversely proportional to the square of the dielectric constant ϵ_i^2, and third, increases monotonically with increasing electron concentration N_n, i.e., with decreasing α.[8)]

On the other hand, in the interval $0.1 \leq \alpha \leq 0.7$, at a fixed concentration $N_n = \text{const}$, the product $\epsilon_i\Delta$ depends very little on $\alpha \cong (m_n/\epsilon_i m_0)(7.4 \times 10^{21}/N_n)^{1/3}$, i.e., on the type of the crystal. Indeed, as seen from Fig. 7, when $\mu \leq 80$ the dependence of the quantity $\Delta/\mathscr{E} \equiv f(\alpha)$ on α can be represented with good accuracy ($\lesssim 10\%$) in the form $f(\alpha) = \text{const}/\alpha$. Thus, for example, at $\mu = 20$, the product $\alpha f(\alpha) \cong 0.04$ in the interval $0.1 \leq \alpha \leq 0.6$, and the gap is equal to

$$\Delta \equiv \frac{1{,}38}{\varepsilon_i}\left(\frac{N_n}{7.4\cdot 10^{21}}\right)^{1/3}\alpha f(\alpha) \cong 2.83\cdot 10^{-9}\frac{N_n^{1/3}}{\varepsilon_i}\ [\text{eV}]. \quad (5.1)$$

Putting $\epsilon_i = 1$ and $N_n = 10^{21}$ cm^{-3} ($\mu = 20$) we obtain, according to (5.1), the estimate $\Delta \approx 3 \times 10^{-2}$ eV, corresponding to a critical temperature $T_c \approx 150°K$. At a lower concentration $N_n \approx 3 \times 10^{19}$ cm^{-3} we have accordingly $\Delta \approx 10^{-2}$ eV and $T_c \approx 50°K$.[9)] However, the presented estimates are too high, since $\epsilon_i > 1$ always, and T_c and Δ decrease with increasing ϵ_i like $\sim 1/\epsilon_i$ (in the region of the weak dependence on α) or $\sim 1/\epsilon_i^2$. In addition, in the analysis of the numerical results it should be borne in mind that the "jellium" model does not take into account a number of features of real crystals, for example the anisotropy of the electron and plasmon spectra,[3] the non-parabolicity of the bands and the dependence of the effective mass of the carriers on the concentration under degeneracy conditions, the presence of several bands (or extrema in the bands), the interaction of the electrons with acoustic and optical phonons,[7,8] the possibility of formation of bound electron-hole pairs (excitons),[7,32,33] etc., which can exert a noticeable influence on the value of T_c. Finally, we note one more circumstance pertaining to the choice of the model of the "electron-plasmon" interaction. In the case of an ideal electron-hole plasma, the generalized Coulomb interaction, calculated in the so-called "random phase" approximation,[12,13] differs essentially from the interaction in the "jellium" model (see Fig. 1). But since the random-phase approximation is valid only at small momentum transfers $q \to 0$, and for the effects of the Cooper pairing all the q up to $q_{max} = 2p_{Fn}$ are important, one can hope that the numerical results obtained on the basis of the "jellium" model present qualitatively correctly (and in some cases even quantitatively) the main tendencies and regularities in the behavior of Δ and T_c in the "plasmon" mechanism of superconductivity.

Thus, summarizing the foregoing, we arrive at the conclusion that to obtain high critical temperatures of the superconducting transition as high as are possible on the basis of the "electron-plasma" interaction, it is necessary to have materials (substances, compounds, alloys, etc.) with two or more filled bands (valleys) and the spectrum of the electrons or the holes (the sign of the charge is of no importance for collective, plasma effects) with the maximum possible concentrations of the free carriers ($N_n \gtrsim 10^{21}$ cm^{-3}, but $N_p \ll N_n$), minimal dielectric constant ($\epsilon_i \approx \epsilon_\infty \sim 1$, but $\epsilon_0 \gg \epsilon_\infty$), and essentially different ($m_n \ll m_p$, $\mu > 10$), not too small ($m_n > 0.1m_0$) effective masses.

In conclusion, we are sincerely grateful to G. M. Éliashberg for numerous fruitful discussions and useful critical remarks.

8) It should be emphasized that the growth of Δ with decreasing α occurs only so long as the condition $\Omega_p \gg E_{Fp}$ is satisfied (E_{Fp} is the Fermi energy of the "heavy" holes). At the same time, the conclusion of almost exponential decrease of the gap in the region $\alpha \ll 1$, which follows from (1.16), is valid only for $\mu \to \infty$.

9) We note that the closely similar value $T_c \sim 10^2$ °K was obtained in [1] at about the same parameters with a very rough model, but on the whole the dependence of Δ on α is radically different from the relation shown in Fig. 7.

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