DYNAMICS OF THE POLARIZATION OF PARTICLES NEAR SPIN RESONANCES

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The motion of the spin of particles in storage rings (accelerators) is investigated. The methods and the results of articles [2-7] on the investigation of spin resonances are generalized to the case of an arbitrary closed orbit.[1] In addition to resonances of the first approximation, higher-order resonances are considered, for which the selection rules of the resonating harmonics are derived. The major part of the article is devoted to the crossing of the resonances. The concept of the effective and the adiabatic bands is introduced. The complete solution of the problem of a single crossing is given, unifying the special results.[2,5-7] On this basis the problem of the periodic crossing of a resonance is solved by using the general nature of the motion of a spin in a periodic field, which was established in [1].

1. INTRODUCTION

QUESTIONS of the preservation of the coherent polarization of the beam and controlling its direction and degree of polarization are crucial for experiments on the spin dependence of the interaction between charged particles. The smallness of the deviations from the equilibrium orbit is a specific property of the motion of particles in storage rings (accelerators). Because of this the spin trajectories can diverge only during a time interval which appreciably exceeds the period of the cyclic motion of the particles. As far as the dynamics of beam polarization is concerned, the important feature is the existence of a direction of stable polarization, **n**. In the case of a planar equilibrium orbit, **n** is directed along the guiding magnetic field (**n** = const). As shown in [1], a direction of stable polarization exists even in a storage ring with an arbitrary field. In this connection **n** is a periodic function of the azimuth of the particle. In practice it turns out to be possible to create any arbitrary direction **n** at a given point in the orbit.

As is well known,[1-7] the motion of the spin becomes unstable only in narrow resonance regions when the frequency of the spin's precession is close to some combination of the frequencies of the orbital motion. In this connection, a small spread in the trajectories of the particles may substantially decrease the initial degree of polarization of the beam. This effect is dangerous for experiments with polarized beams; on the other hand, it can be used for intentional depolarization.

In practice the situation involving a crossing of the resonances is important, the crossing being produced both by natural causes and by artificial means (acceleration of the particles, modulation of the frequencies of the motion, synchrotron oscillations of the energy, etc.). The case of a single crossing was investigated in articles [2,5-7]. In spite of its practical importance, the problem of a multiple crossing has not been investigated up to now. The reason for this is its mathematical and physical complexity.

The problem of a multiple periodic crossing is solved in the present article. The main idea for its solution is an understanding of the general nature of the motion of a spin in a periodic field, which was established in [1]. The obtained results answer the question about the behavior of the polarization of the beam as a whole associated with periodic crossing of a resonance, in particular, they determine the conditions for depolarization.

In addition, in this work the results of the investigation of steady-state resonances and of a single crossing are generalized and determined more accurately; this is necessary for practical applications and for the investigation of multiple crossing.

2. RESONANCES OF THE SPIN MOTION

The motion of the spin vector is described by the equation[1,8,9]

$$d\boldsymbol{\zeta}/dt = [\mathbf{w}_L\boldsymbol{\zeta}],$$
$$\mathbf{w}_L = \left(1+\gamma\frac{q'}{q_0}\right)\frac{[\mathbf{v}\dot{\mathbf{v}}']}{v^2} - \frac{q}{\gamma}\frac{\mathbf{H}\mathbf{v}}{v^2}\mathbf{v} + \frac{q}{\gamma^2 v^2}[\mathbf{v}\mathbf{E}], \tag{2.1*}$$

where $q = q_0 + q' \equiv (e/m) + q'$ is the gyromagnetic ratio, q' is its anomalous part, $\gamma = (1 - v^2)^{-1/2}$, $c = 1$, $\mathbf{v}$ and $\dot{\mathbf{v}}$ denote the velocity and acceleration of the particle which is moving in the electromagnetic field **E**, **H**. Let us introduce the following notation: $\mathbf{w}_S(\theta) = \mathbf{w}_S(\theta + 2\pi)$ is the value of $\mathbf{w}_L$ on the equilibrium trajectory; θ is the generalized azimuth of the particle which subsequently plays the role of the time; ω_S is the equilibrium frequency of revolution ($d\theta_S/dt = \omega_S$).

As is shown in [1], on the equilibrium trajectory the solution (2.1) has the form (γ_S = const).

$$\boldsymbol{\zeta}(\theta) = \zeta_n \mathbf{n}(\theta) + \operatorname{Re} c\boldsymbol{\eta}(\theta), \quad \zeta_n,\ c = \text{const},$$

where **n**, $\boldsymbol{\eta}$, $\boldsymbol{\eta}^*$ are orthogonal solutions of (2.1) on the equilibrium trajectory, possessing the properties

$$\mathbf{n}(\theta+2\pi) = \mathbf{n}(\theta), \quad \boldsymbol{\eta}(\theta+2\pi) = e^{-2\pi i\nu}\boldsymbol{\eta}(\theta),$$
$$\mathbf{n}^2 = 1, \quad \boldsymbol{\eta}\boldsymbol{\eta}^* = 2, \quad \nu = \text{const}. \tag{2.2}$$

The quantity $2\pi\nu$ has the meaning of the angle of rotation of the spin around **n** during the period of the particle's motion along the equilibrium orbit. In the moving

*$[\mathbf{v}\mathbf{E}] \equiv \mathbf{v}\times\mathbf{E}$; $\mathbf{H}\mathbf{v} \equiv \mathbf{H}\cdot\mathbf{v}$.

periodic system of basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{n}$

$$\mathbf{e}_1 + i\mathbf{e}_2 \equiv \mathbf{e} = \eta e^{i\nu\theta}, \quad (2.3)$$

Eq. (2.1) takes the form

$$\frac{d\boldsymbol{\zeta}}{d\theta} \equiv \dot{\boldsymbol{\zeta}} = [\mathbf{W}\boldsymbol{\zeta}], \quad \mathbf{W} = \nu\mathbf{n} + \mathbf{w} \equiv \nu\mathbf{n} + \frac{\mathbf{w}_L}{\dot{\theta}} - \frac{\mathbf{w}_S}{\omega_0} \quad (2.4)$$

Under steady-state conditions for the motion of the particles in a storage ring, one can represent $\mathbf{w}$ in the form

$$\mathbf{wn} = \sum_l \omega_l e^{i\nu_l\theta}, \qquad \mathbf{we} = \sum_k \omega_k e^{i\nu_k\theta} \quad (2.5)$$

(we shall denote the quantities pertaining to $\mathbf{w}\cdot\mathbf{n}$ and $\mathbf{w}\cdot\mathbf{e}$ by the indices l and k respectively). The frequencies ν_l and ν_k are combinations of the frequencies of motion of the particles. In view of its smallness, taking account of the perturbation $\mathbf{w}$ can essentially change the motion of the spin only near the spin resonances, when ν is close to some frequency ν_c—which is a combination of the frequencies ν_l and ν_k.

The method of averaging is applied in order to investigate the behavior of the spin near a resonance. In articles [2-7] only resonances of the first approximation, $\nu \approx \nu_k$, were investigated for the case $\mathbf{n}(\theta) = \text{const}$ (plane equilibrium orbit). One can easily extend the results of these articles to the general case $\mathbf{n}(\theta) \neq \text{const}$.

The motion of the spin near a resonance, $\nu \approx \nu_k$, takes place in the following manner. In the system which is rotating relative to (2.3) with a frequency $\nu_k\mathbf{n}$, the spin slowly precesses with a frequency $[\,|\omega_k|^2 + (\nu - \nu_k)^2]^{1/2}$ around the direction which makes an angle $\alpha = \tan^{-1}[\,|\omega_k|/(\nu - \nu_k)]$ with $\mathbf{n}$.

Combination resonances $\nu \approx \nu_c$ appear in the higher approximations. It is simplest to obtain the equation describing the averaged spin motion near a resonance of arbitrary order in the "resonance" system, which is rotating with the frequency ν_c around $\mathbf{n}$. In this system the spin is moving in a "field" $\mathbf{w}' = (w'_x, w'_y, w'_z)$,

$$w_x' + iw_y' = \mathbf{we}\exp\{-i\nu_c\theta\}, \quad w_z' = \mathbf{wn} + \nu - \nu_c. \quad (2.6)$$

For small times $T \ll |\mathbf{w}'|^{-1}$ one can determine the solution for $\boldsymbol{\zeta}$ at the instant of time T in the form of a series

$$\zeta_T = (1 + \lambda)\zeta_0, \qquad \lambda = \int_0^T (w' + w'\tilde{w}' + \ldots)\,d\theta, \quad (2.7)$$

where ζ is the column formed out of the components of the vector $\boldsymbol{\zeta}$; w' is the matrix defined by

$$w_{ik}' = \sum_\alpha \varepsilon_{i\alpha k} w_\alpha', \qquad \tilde{w}' = \int_0^\theta w'\,d\theta.$$

In our case the method of averaging means the determination of a constant effective "field" h such that the solution $\zeta_T = e^{hT}\zeta_0$ of the equation $\dot{\zeta} = h\zeta$ averaged over the time interval T will agree with (2.7):

$$e^{hT} = 1 + \lambda, \qquad h = \frac{1}{T}\ln(1+\lambda) = \frac{1}{T}\left(\lambda - \frac{\lambda^2}{2} + \ldots\right). \quad (2.8)$$

Having chosen T with the relations $\theta_{ch} \ll T \ll 1/|\mathbf{w}|$ taken into consideration, where θ_{ch} denotes the characteristic time of variation of $\mathbf{w}'$, we obtain

$$\mathbf{h} = \langle \mathbf{w}' - {}^1/_2[\tilde{\mathbf{w}}'\mathbf{w}'] + \ldots\rangle, \quad (2.9)$$

where the angular brackets indicate averaging over the time θ. Upon taking account of the terms of n-th order, the solution (2.9) will obviously differ very little from the exact solution for

$$\theta \lesssim \frac{1}{|\mathbf{w}'|}|\mathbf{w}'\theta_{ch}|^{1-n}.$$

Strictly speaking, the resonance occurs when the direction $\mathbf{h}(\nu)$ differs substantially from $\mathbf{n}$. For the exact resonance $h_z(\nu_r) = 0$ the difference $\nu_r - \nu_c$ is a correction to the position of the exact resonance, due to perturbation $\mathbf{w}'$. The effective width of the resonance is determined by the quantity $h(\nu_r) = |h_x(\nu_r) + ih_y(\nu_r)| \equiv h_\perp$, which for a resonance of n-th order is proportional to the n-th power of the perturbation.

As is evident, in the second approximation only the resonances $\nu \approx \nu_l + \nu_k$, which arise as a consequence of the correlations between the oscillations of the precession frequencies and the transverse perturbation, are possible. For a constant value of $\mathbf{w}\cdot\mathbf{n}$ ($\nu_l = 0$) the second approximation does not give a new resonance, but is a correction to the first resonance.

Let us indicate the simple selection rule for the combinations of frequencies for a resonance of arbitrary order. In the resonance system, one can write the general condition for a resonance of n-th order in the form[1)]

$$\nu_{l_1} + \ldots + \nu_{l_{n_\parallel}} + s_1(\nu_{k_1} - \nu) + \ldots + s_{n_\perp}(\nu_{k_{n_\perp}} - \nu) \approx 0,$$
$$s_i = \pm 1, \quad n_\parallel + n_\perp = n.$$

(One can include the choice of the sign in front of ν_l in the subscript "l," because the frequencies ν_l form the spectrum of the real quantity $\mathbf{w}\cdot\mathbf{n}$.) Since only linear resonances $\nu \approx \nu_c$ are possible[1] for the spin system, then one must have

$$s_1 + \ldots s_{n_\perp} = 1. \quad (2.10)$$

From here, in particular, it follows that $n_\perp$ is odd. Thus, in the n-th order the resonances

$$\nu \approx \nu_{l_1} + \ldots + \nu_{l_{n_\parallel}} + s_1\nu_{k_1} + \ldots + s_{n_\perp}\nu_{k_{n_\perp}} \quad (2.11)$$

are possible with fulfillment of the condition (2.10). The intensity of the resonance is proportional to the product $\omega_{l_1} \ldots \omega_{l_{n\parallel}}\omega_{k_1} \ldots \omega_{k_{n\perp}}$. The resonances of n-th order reduce to a correction of the lower-order resonances, when out of the total combination (2.11) a part which is small in magnitude is isolated, where the numbers of frequencies ν_k entering into this subcombination with positive ($s_i > 0$) and with negative signs ($s_i < 0$) must coincide. For example, the resonances

$$\nu \approx \nu_l + \nu_{l'} + \nu_k, \quad \nu \approx \nu_k + \nu_{k'} - \nu_{k''}$$

are possible in third order. They reduce to corrections to the resonances of first or second order if $\nu_l + \nu_{l'} \approx 0$, $\nu_{k'} \approx \nu_{k''}$, or $\nu_l \approx 0$.

Let us apply the described method of averaging just for an isolated resonance, i.e., when the condition

$$|\nu - \nu_c| \lesssim |\mathbf{w}| \quad (2.12)$$

is satisfied only for a single combination frequency ν_c. If several combination frequencies ν_c satisfy the con-

1) Identical frequencies may occur among ν_l and ν_k.

dition (2.12), then after averaging with respect to the fast frequencies the effective field **h** will be a function of the time whose frequencies of variation are comparable with the quantity $|\mathbf{h}|$:

$$\mathbf{h}=\mathbf{h}_0+\Delta(\theta),\qquad \Delta(\theta)=\sum_{m\neq 0}\Delta_m e^{i\Omega_m\theta} \tag{2.13}$$
$$\mathbf{h}_0=(h_\perp,\,0,\,\varepsilon),\quad |\Omega_m|\lesssim h,$$

where $h_\perp$ and ϵ are the width and frequency deviation of the selected (fundamental) resonance, and the Δ_m characterize the intensities of the remaining resonances.

Such a situation in general means the presence in the spectrum of a perturbation of small frequencies (synchrotron oscillations of the energy, external modulation, and so forth).

Let us consider the case of overlapping resonances of essentially different intensities. Let $|\Delta_m| \ll |\Omega_m|$. It again turns out to be possible to use the method of averaging in order to solve such a problem. For $\Delta = 0$ a single (fundamental) resonance occurs, and the spin precesses around h_0. Taking Δ into account leads to a substantial distortion of the motion only near the "modulated" resonances. The resonances $\Omega_m \approx h_0$ are possible in the first approximation. Their width is given by $\omega_m = h_0^{-1}|\Delta_m \times h_0|$.

The treatment under discussion gives results, which differ substantially from the theory of an isolated resonance only for $|\epsilon| \lesssim h_\perp$. In the opposite case ($|\epsilon| \gg h_\perp$) the side resonances are actually separated from the main resonance and can be regarded as isolated resonances.

In the case $\Delta \gg |\Omega_m|$, when many resonances of the same intensity (in order of magnitude) overlap, the method of averaging is not applicable. In this connection an approach to the problem, not in terms of an overlapping of individual resonances but rather by means of repeated crossings of the "fundamental" resonance, turns out to be more useful.

3. SINGLE CROSSING OF A RESONANCE

The single crossing was first considered in [2] for a constant rate of crossing ($\dot{h}_z$, $h_\perp$ = const) and for special initial conditions with regard to the polarization of the field. Under arbitrary initial conditions the result is known only for the case of rapid crossing ($|\dot{h}_z| \gg h_\perp^2$).[2,5-7]

The complete answer to the problem of a single crossing is not only of independent interest, but we also need it in order to construct the solution in the case of a periodic crossing. It turns out to be possible to solve such a problem in a somewhat broader formulation.

Let the spin in the resonance system be moving in the "field"

$$\mathbf{h}=(h_\perp,\,0,\,h_z). \tag{3.1}$$

(One can always eliminate the rotation $\mathbf{h}_\perp$ by making a transformation to a system which is rotating with respect to the resonance together with $\mathbf{h}_\perp$.)

The initial conditions for the spin are given for $h_z \to -\infty$. It is required to find the solution $\boldsymbol{\zeta}$ after crossing the resonance for $h_z \to \infty$. In the region of adiabatic variation of $\mathbf{h}(\theta)$ the motion of the spin corresponds to a precession around **h** with a frequency h. In order to fulfill the adiabatic condition, it is necessary that the change in the angular velocity **h** of rotation of the spin should be small during the time $2\pi/h$ required for one revolution:

$$h^2 \gg |\dot{\mathbf{h}}| = \sqrt{\dot{h}_z^2+\dot{h}_\perp^2}. \tag{3.2}$$

Condition (3.2) guarantees the exponential accuracy of the solution in the adiabatic region associated with the monotonic variation of **h**. If **h** undergoes oscillations during the crossing, then the additional condition that the frequency Ω of the oscillations be small is also required:

$$\Omega \ll h. \tag{3.2'}$$

We note that if (3.2) is satisfied for all values of h_z ($h_\perp^2 \gg |\dot{h}|$), then the adiabatic solution is valid everywhere with exponential accuracy in the adiabatic parameter. In this connection, the exponentially small difference from the exact solution piles up in the region where (3.2) is least strongly satisfied, i.e., in the region $|h_z| \lesssim h_\perp$. Therefore we shall understand the boundary of the effective resonance region (band) to be given by

$$h_z^{\text{eff}} \sim \max(h_\perp, \sqrt[4]{\dot{h}_z^2+\dot{h}_\perp^2}). \tag{3.3}$$

Our goal is to match the adiabatic solutions to the right and to the left of the effective band (3.3). The present problem can be completely solved for an arbitrary rate of crossing, $\dot{h}_z$, provided that the relative change of $\dot{h}_z$ and of $h_\perp$ are small in the effective band:

$$|\delta\dot{h}_z/\dot{h}_z| \ll 1,\quad |\delta h_\perp/h_\perp| \ll 1. \tag{3.4}$$

For this it is more convenient to use the equation for the variable χ [2)]

$$i\dot{\chi}={}^1/{}_2\sigma\mathbf{h}\chi,\quad \zeta=\chi^+\sigma\chi, \tag{3.5}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices.

In the system where the 3 axis is directed along the field **h**, the adiabatic solution of Eq. (3.5) has the form

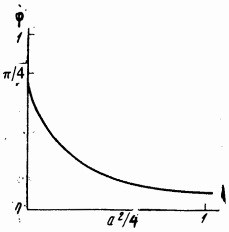
$$\chi=S_{\theta 0}\begin{pmatrix}A\\B\end{pmatrix},\qquad S_{\theta 0}=\exp\left\{-\frac{i}{2}\sigma_z\int_0^\theta h\,d\theta\right\}. \tag{3.6}$$

In order to find the connection between A and B for $\theta < 0$ and $\theta > 0$, we shall use the method of joining the solutions by means of going into the complex plane of the time θ. In our case the singular points, determined from the equation

$$h=(h_z^2+h_\perp^2)^{1/2}=0,$$

lie in the complex plane of θ, and the present problem is mathematically equivalent to the problem of above-barrier reflection in quantum mechanics. If $\dot{h}_z$ and $h_\perp$ are constant, then there are only two singular points—the "turning" points $\theta_t = \pm ih_\perp/\dot{h}_z$. The deviations of $\dot{h}_z$ and $h_\perp$ from constant values lead to a displacement of these points and to the appearance of new singular points, which can be neglected provided (3.4) is satisfied. Following articles [11-13], let us go around the ef-

2)For spin s = 1/2 the variable χ is the spin wave function. In the case of arbitrary spin, one can interpret Eq. (3.5) as the equation for the wave functions χ_a (a = 1, 2, . . . , 2s) of 2s independent particles with spin 1/2, out of which one can formally construct particles with spin s. [10]



Graph of the function $\varphi(a^2/4)$. For a rapid crossing of the resonance ($a^2/4 \ll 1$) one has $\varphi \approx (\pi/4)-(a^2/4)(\ln(4e/a^2)-0.577)+\ldots$ For a slow crossing ($a^2/4 > 1$) one has $\varphi \approx (1/3a^2) + (8/45a^6)+\ldots$

fective band along a circumference of large radius on which the solution has the form (3.6). We obtain the desired relation between A and B:[3)]

$$\begin{pmatrix} A \\ B \end{pmatrix}_{\theta>0} = R \begin{pmatrix} A \\ B \end{pmatrix}_{\theta<0} \qquad (3.7)$$

$$R(\dot{h}_z > 0) = [1-e^{-2\delta}]^{1/2} e^{-i\sigma_z\varphi} + ie^{-\delta}\sigma_y = R^T(\dot{h}_z < 0),$$

$$2\delta = \left| \int_{\theta_t^*}^{\theta_t} h\, d\theta \right|. \qquad (3.8)$$

φ denotes the constant phase which remains undetermined in [11-13]. The instant $\theta = 0$ is chosen such that

$$\operatorname{Re}\theta_t = 0. \qquad (3.9)$$

From (3.6) and (3.7) one can obtain the relation $\chi_{\theta_2} = S_{\theta_2 0} R S_{0\theta_1} \chi_{\theta_1} \equiv U\chi_{\theta_1}$. One can find the phase φ from a comparison with the exact solution for constant $\dot{h}_z$ and $h_\perp$, the exact solution being constructed in the Appendix. In this case

$$\delta = \tfrac{1}{4}\pi h_\perp^2 / |\dot{h}_z| \equiv \tfrac{1}{4}\pi a^2 \qquad (3.10)$$

in agreement with (3.8), and

$$\varphi = -\frac{\pi}{4} + \frac{a^2}{4}\ln\frac{a^2}{4e} - \arg\Gamma\left(i\frac{a^2}{4}\right) \qquad (3.11)$$

where Γ denotes the Gamma function. A graph showing the dependence $\varphi(a^2/4)$ is given in the accompanying figure.[14] For a small deviation of $\dot{h}_z$ and $h_\perp$ from constant values, in the effective band one can always neglect the difference of φ from (3.11). (We note that for a slow crossing ($a^2 \gg 1$) the absolute change of δ and the shift of $\theta = 0$ in this connection may be large.) Thus, under conditions (3.4) the matrix R is given by expression (3.7) with φ from (3.11).

From Eqs. (3.5) and (3.10) one can obtain the relation between the components $\zeta_h = \boldsymbol{\zeta}\cdot\mathbf{h}/h$ and the phase ψ of the rotation around **h**. Let us present the expression for ζ_h:

$$\zeta_h(\theta_2) = (1-e^{-2\delta})\zeta_h(\theta_1) - 2e^{-\delta}(1-e^{-2\delta})^{1/2}(1-\zeta_h^2(\theta_1))^{1/2} \times \cos\left(\varphi + \int_{\theta_1}^{0} h\, d\theta + \psi(\theta_1)\right). \qquad (3.12)$$

In connection with a slow crossing ($|\dot{h}_z| \ll h_\perp^2$) the spin preserves its projection along **h** with exponential accuracy, reversing together with **h**. In the fast case ($|\dot{h}_z| \gg h_\perp^2$) the change in the projection along **n** is small ($\sim\sqrt{\delta}$). Formula (3.12) generalizes the result given in [2].

One can, for example, use the slow crossing of the resonance in a storage ring in order to obtain beams of electrons and positrons with identical directions of polarization (under the effect of the synchrotron radiation the beams should be polarized in opposite directions[15,16]). By switching on a radial electric field E_r one can separate the beams according to energies (for coincident orbits). After this, by slowly passing one of the beams through the spin resonance ($\nu_{\text{positron}} \neq \nu_{\text{electron}}$), one can change its polarization to the opposite state.[4)]

4. CROSSING OF THE RESONANCES $\nu = k$

Hitherto we have not considered the effects which are related to the variation with time of **n** and ν as functions of the equilibrium energy γ_s (or of the other parameters). Far away from the resonance $\nu = k$ the change of **n** and ν is adiabatic and does not lead to any important effects. The problem of crossing these resonances requires special investigation. Let $\mathbf{n}(\gamma_s, \theta)$ and $\nu(\gamma_s)$ be known for each value of γ_s. As the resonance point $\gamma_s = \gamma_r$ we choose the value of the energy at which ν is closest to an integer k:

$$\nu(\gamma_r) - k = \Delta\nu_{min} \qquad (4.1)$$

(in particular, $\Delta\nu_{min}$ may be equal to zero). Let us write down the equations of motion of the spin in the periodic system with $\mathbf{n}_r \equiv \mathbf{n}(\gamma_r, \theta)$ and $\mathbf{e}_r \equiv \mathbf{e}(\gamma_r, \theta)$:

$$\dot{\boldsymbol{\zeta}} = [\mathbf{w}'\boldsymbol{\zeta}], \quad \mathbf{w}' = \left(\Delta\nu_{min} + \mathbf{n}_r\frac{\mathbf{w}_s}{\omega_s}\right)\mathbf{n}_r + \operatorname{Re}\left(\delta\frac{\mathbf{w}_s}{\omega_s}\mathbf{e}_r\right)\mathbf{e}_r,$$

$$\delta\frac{\mathbf{w}_s}{\omega_s} = \frac{\mathbf{w}_s(\gamma_s,\theta)}{\omega_s(\gamma_s)} - \frac{\mathbf{w}_s(\gamma_r,\theta)}{\omega_s(\gamma_r)} \qquad (4.2)$$

(the deviations from the equilibrium orbit are unimportant). After averaging with respect to θ we obtain

$$\langle\mathbf{w}'\rangle = \Delta\nu_{min}\mathbf{n}_r + \operatorname{Re}\left\langle \mathbf{e}_r^*\delta\frac{\mathbf{w}_s}{\omega_s}\right\rangle\mathbf{e}_r \qquad (4.3)$$

($\langle\mathbf{n}_r\cdot\delta(\mathbf{w}_s/\omega_s)\rangle = 0$ according to the definition of the resonance point γ_r: $(\partial\nu/\partial\gamma)_r = 0$). Formula (4.3) gives the following relation between $\mathbf{n}(\gamma_s, \theta)$ and $\nu(\gamma_s)$, and $\mathbf{n}_r$ and $\Delta\nu_{min}$:

$$(\nu(\gamma_s) - k)\mathbf{n}(\gamma_s, \theta) = \langle\mathbf{w}'\rangle. \qquad (4.4)$$

The width of the resonance is determined by the quantity $\Delta\nu_{min}$.

According to the usual behavior of the axis of precession in the region of a spin resonance, for a broad resonance the direction **n** is transverse to $\mathbf{n}_r$.

All of the results obtained in the present work are also applicable to the resonance $\nu \approx k$ with the following substitutions:

$$h_\perp \to \Delta\nu_{min}; \quad h_z \to (\gamma - \gamma_r)\frac{d}{d\gamma}\left|\left\langle\frac{\mathbf{w}_s(\gamma,\theta)\mathbf{e}_r}{\omega_s(\gamma)}\right\rangle\right|_r.$$

5. PERIODIC CROSSINGS

From a general point of view, the problem of periodic crossings of a resonance is the problem about the motion of a spin in a periodic field

$$\mathbf{h}(\theta) = \mathbf{h}(\theta + T) = (h_\perp, 0, h_z), \qquad (5.1)$$

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3) In [11-13] the cut in the complex θ plane is drawn between the turning points. In our definition h is positive everywhere on the real axis.

4) We note that this method is not feasible near points γ_{cr} at which $(\partial\gamma/\partial\gamma)_{el} \approx (\partial\nu/\partial\gamma)_{pos}$. For a planar equilibrium orbit $\gamma_{cr} = (q/q')^{1/2}$. (Here ν does not depend on E_r.)

where $2\pi/T$ is the frequency of the crossings.

The general nature of the motion of a spin during a periodic crossing follows from the results of [1]. A certain periodic solution $\mathbf{m}(\theta)$ exists for the spin, which is repeated after an interval T. During the period, all of the remaining solutions are rotated around $\mathbf{m}$ through one and the same angle $2\pi\mu$. Thus, our problem reduces to finding the periodic solution $\mathbf{m}$ and the frequency of precession μ. Let the matrix Λ be known: $\chi_T = \Lambda\chi_0$; according to the meaning of $\mathbf{m}$ and μ the matrix Λ has the form $(\Lambda^+\Lambda = 1)$

$$\Lambda = e^{-i\pi\mu\sigma\mathbf{m}}, \quad \mathbf{m} \equiv \mathbf{m}(0),$$

from where it follows that

$$\cos\pi\mu = {}^1/{}_2\,\mathrm{Sp}\,\Lambda, \quad \mathbf{m} = \frac{i}{2\sin\pi\mu}\mathrm{Sp}\,\sigma\Lambda. \tag{5.2}$$

Let us consider the case when the condition (3.4) is satisfied in the effective band. (In this connection, it is assumed that the amplitude h_z of the oscillations is sufficiently large.) Then one can construct the matrix Λ by using the results of Section 3:

$$\Lambda = S_{T\theta_2}R_2S_{\theta_2\theta_1}R_1S_{\theta_1 0}. \tag{5.3}$$

Here θ_2 and θ_1 denote the moments of crossing the resonance, which satisfy the condition (3.9). Introducing the notation

$$x = \varphi_1 + \varphi_2 + \frac{\varkappa}{2}, \quad \varkappa = \varkappa_+ + \varkappa_- = \int_0^T h\,d\theta,$$

$$y = \frac{\varkappa_+ - \varkappa_-}{2}, \quad \varkappa_- = \int_0^{\theta_1} h\,d\theta + \int_{\theta_2}^{T} h\,d\theta \equiv \varkappa_-^{(1)} + \varkappa_-^{(2)}, \tag{5.4}$$

$$\varphi_{1,2} = \varphi(a_{1,2}^2/4),$$

from Eq. (5.3) we obtain the following results for the matrix elements:

$$\Lambda_{11} = \sqrt{(1-e^{-2\delta_1})(1-e^{-2\delta_2})}\,e^{-ix} + e^{-\delta_1-\delta_2+iy},$$

$$\Lambda_{12} = \{\sqrt{1-e^{-2\delta_1}}\exp\{-\delta_2 + i(x+y)/2\} - \sqrt{1-e^{-2\delta_2}}\exp\{-\delta_1 - i(x+y)/2\}\exp\{i(\varphi_1 + \varkappa_-^{(1)} - \varphi_2 - \varkappa_-^{(2)})/2\}. \tag{5.5}$$

Thus

$$\cos\pi\mu = \sqrt{(1-e^{-2\delta_1})(1-e^{-2\delta_2})}\cos x + \exp\{-\delta_1-\delta_2\}\cos y, \tag{5.6}$$

$$m_h = \frac{\mathrm{Im}\,\Lambda_{11}^*}{\sin\pi\mu}, \quad m_1 + im_2 = -i\frac{\Lambda_{12}^*}{\sin\pi\mu}.$$

The obtained formulas give all necessary information about the behavior of the spin. The vector $\mathbf{m}$ specifies the direction of the periodic solution at the moments 0, T, ... (The moment $\theta = 0$ is chosen in the adiabatic band, where the spin motion is known.) Formula (5.6) determines the angle $2\pi\mu$ through which the spin rotates around $\mathbf{m}$ during the period T.

In practice it is interesting to trace how the projection along $\mathbf{n}$ varies in the adiabatic zone during a periodic crossing. It is obvious that its variation substantially depends on the orientation of $\mathbf{m}$ with respect to $\mathbf{h}$. For example, for the initial condition $\zeta_h = 1$ the component ζ_h varies within the interval from 1 to $(2m_h^2 - 1)$.

Now let us see how the orientation of $\mathbf{m}$ depends on the parameters of the problem.

a) Fast crossing ($\delta_1 \ll 1$, $\delta_2 \ll 1$). Here, as one can verify without difficulty,

$$\sin\pi\mu \approx (\sin^2 y + 2(\delta_1+\delta_2) - 4\sqrt{\delta_1\delta_2}\cos x\cos y)^{1/2}, \quad m_h \approx \sin y/\sin\pi\mu. \tag{5.8}$$

As is clear, m_h is almost always equal to unity, with the exception of narrow bands in y:

$$|y - k\pi| \lesssim (\delta_1 + \delta_2 - 2\sqrt{\delta_1\delta_2}(-1)^k\cos x)^{1/2}. \tag{5.9}$$

Formula (5.9) determines the "resonance region" in which the periodic solution strongly depends on the parameters. The polarization ζ_h may vary substantially only in this region. Here the spin is rotating slowly around $\mathbf{m}$ with a frequency $\sim\sqrt{\delta}$. For example, for a symmetric crossing $\kappa_+ = \kappa_-$; $\delta_1 = \delta_2 = \delta$:

$$\mu = \frac{2\sqrt{2\delta}}{\pi}\sin\frac{x}{2} = \sqrt{\frac{2}{\pi}}\frac{h_\perp}{(|\dot{h}_z|)^{1/2}}\sin\left(\frac{\pi}{4} + \frac{1}{4}\int_0^T h\,d\theta\right).$$

Even a small violation of the symmetry of the crossing $(1 \gg |\kappa_+ - \kappa_-| \gg \sqrt{\delta})$ displaces the resonance (5.9) and then the component ζ_n is conserved.

b) Slow crossing ($e^{\delta_1} \gg 1$, $e^{\delta_2} \gg 1$). In this case

$$\sin\pi\mu \approx (\sin^2 x + e^{-2\delta_1} + e^{-2\delta_2} - 2e^{-\delta_1-\delta_2}\cos x\cos y)^{1/2}, \quad m_h = \sin x/\sin\pi\mu. \tag{5.10}$$

Just as in the previous case, almost always $m_h = 1$, with the exception of the resonance region:

$$|x - k\pi| \lesssim (e^{-2\delta_1} + e^{-2\delta_2} - 2e^{-\delta_1-\delta_2}(-1)^k\cos y)^{1/2}. \tag{5.11}$$

c) Mixed case ($\delta_1 \ll 1$, $e^{\delta_2} \gg 1$). Here

$$\mu \approx \frac{1}{2} - \frac{\sqrt{2\delta_1}}{\pi}\cos x - \frac{1}{\pi}e^{-\delta_2}\cos y,$$

$$m_h \approx \sqrt{2\delta_1}\sin x - e^{-\delta_2}\sin y, \tag{5.12}$$

$$m_1 + im_2 = i\exp\left\{\frac{i}{2}(\varkappa_+ + \varkappa_-^{(2)} - \varkappa_-^{(1)})\right\}.$$

It is easy to understand the meaning of this solution in the following way. During rapid motion "from below upward" ζ_n isn't able to vary. Then, during the slow crossing "downward" ζ_n changes sign. Therefore the spin undergoes a half-rotation ($\mu = 1/2$) around a certain direction, transverse to the axis $\mathbf{n}$.

In contrast to the previous cases, the smallness of m_h does not indicate a resonance ($\mu \neq k$).

d) Intermediate case ($\delta_1 \sim 1$, $\delta_2 \sim 1$). In contrast to the cases which have been considered, here the direction of $\mathbf{m}$ substantially depends on (x, y) over the entire range of their variation. In this connection m_h takes all possible values

$$0 \leqslant |m_h| \leqslant ((1-e^{-2\delta_1})(1-e^{-2\delta_2}))^{1/2} + e^{-\delta_1-\delta_2}. \tag{5.13}$$

The greatest sensitivity to the position of the point (x, y) is observed near the resonances $\mu = k$, when

$$\delta_1 \approx \delta_2 = \delta, \quad \cos x \approx \cos y \approx \pm 1.$$

Near a resonance the approximate formulas have the form

$$\sin\pi\mu \approx \left[\frac{(\delta_1-\delta_2)^2}{e^{2\delta}-1} + (1-e^{-2\delta})(\Delta x)^2 + e^{-2\delta}(\Delta y)^2\right]^{1/2}, \tag{5.14}$$

$$m_h \approx \frac{1}{\sin\pi\mu}[e^{-2\delta}\Delta y - (1-e^{-2\delta})\Delta x].$$

Here Δx and Δy denote the deviations from the resonance point $\cos x = \cos y = \pm 1$.

6. FAST CROSSINGS WITH AN ARBITRARY PERIODIC DEPENDENCE $h(\theta)$

Formulas (5.6) and (5.7) of the periodic solution are valid under conditions (3.4). In the important case of fast crossings, one can solve the problem without any restrictions on the form of the periodic dependence. During a fast crossing, in the effective band the spin is not able to substantially change, and therefore the solution in this band can be found by using perturbation theory.

For a single crossing, the relation between χ to the right and to the left of the resonance in the system rotating relative to the resonance with a velocity $h_z n$ has the following form in the first approximation:

$$\chi_{\theta_2} = \left(1 - \frac{i}{2}\sigma H_0\right)\chi_{\theta_1}, H_x^0 + iH_y^0 = \int_{-\infty}^{\infty} h_\perp \exp\left\{-i\int_{\theta_0}^{\theta} h_z d\theta\right\} d\theta,$$

$$h_z(\theta_0) = 0, \quad \theta_2 > \theta_0 > \theta_1, \quad H_n^0 = 0. \tag{6.1}$$

If we return to the initial system and match with the solutions in the adiabatic bands, we obtain the following result for the matrix Λ' (in the resonance system):

$$\chi_T = \Lambda'\chi_0, \quad \Lambda' = \exp\{-{}^1\!/_2 i\sigma_z \varkappa_T\}(1 - {}^1\!/_2 i\sigma H),$$

$$H_x + iH_y = \int_0^T h_\perp e^{-i\varkappa_\theta} d\theta, \quad H_n \approx 0,$$

$$\varkappa_\theta = \int_0^\theta \varepsilon^{\text{eff}} d\theta, \quad \varepsilon^{\text{eff}} = \begin{cases} h_z & \text{in the effective band} \\ hh_z/|h_z| & \text{in the adiabatic band.} \end{cases}$$

Hence

$$\sin \pi\mu = (\sin^2 {}^1\!/_2\varkappa_T + {}^1\!/_4 H^2)^{1/2},$$
$$m_n \sim \frac{\sin {}^1\!/_2\varkappa_T}{\sin \pi\mu}, \quad m_x + im_y = \frac{H_x + iH_y}{2\sin \pi\mu}\exp\left\{i\frac{\varkappa_T}{2}\right\}. \tag{6.2}$$

In the presence of adiabatic bands and conditions (3.4), the solution (6.2) goes over into (5.8). The periodic solution, just as previously, is always directed along the axis **n**, with the exception of the resonance region (compare with Eq. (5.9)):

$$|{}^1\!/_2\varkappa_T - k\pi| \lesssim H. \tag{6.3}$$

We note that in contrast to (5.8), the solution (6.2) is applicable even in that case when the adiabatic band is not reached ($\kappa_T = \langle h_z \rangle T \ll 1$).

In this connection, as is evident from (6.2) the spin is moving in the average field:

$$\langle \mathbf{h} \rangle = (\langle h_\perp \rangle, 0, \langle h_z \rangle).$$

By the same token the condition $\kappa_T \ll 1$ indicates the limits of applicability of the method of averaging for the investigation of multiple fast crossings.

7. CONCLUSION

Now let us briefly consider the application of the obtained results to the problem of the behavior of the polarization, averaged over the beam, associated with periodic crossings. The spread in the parameters entering into the problem leads to a spread in **m** and μ. The spread in μ guarantees a mixing of the phases of precession around **m**. Therefore, after multiple crossing the average polarization will be given by (in the absence of intermixing of the particles' trajectories)

$$\bar{\zeta} = \overline{(\zeta \mathbf{m})_{t=0}\mathbf{m}}.$$

Very special conditions are required in order to eliminate the initial polarization along **n** by a multiple periodic crossing. Thus, for either fast or slow crossings the variation of ζ_n takes place only in the narrow resonance bands given by Eqs. (5.9), (6.3), and (5.11). The mixed case of fast-slow crossing deserves attention, when no additional conditions are required in practice for depolarization of the beam.

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APPENDIX

For constant values of $\dot{h}_z$ and $h_\perp$ Eqs. (3.5) can be solved exactly. In the resonance system these equations

$$\chi = \begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix}, \quad \begin{aligned}\dot{\chi}_+ &= -{}^1\!/_2 i\dot{h}_z\theta\chi_+ - {}^1\!/_2 ih_\perp\chi_- \\ \dot{\chi}_- &= {}^1\!/_2 i\dot{h}_z\theta\chi_- - {}^1\!/_2 ih_\perp\chi_+\end{aligned}$$

correspond to the functional relations for parabolic cylinder functions $D_p(z)$.[17]

The solution has the form ($\dot{h}_z > 0$)

$$\chi(\theta) = \begin{pmatrix} D_p(z), & D_p(-z) \\ {}^1\!/_2 ae^{i\pi/4}D_{p-1}(z), & -{}^1\!/_2 ae^{i\pi/4}D_{p-1}(-z)\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \equiv M(\theta)\begin{pmatrix}\alpha \\ \beta\end{pmatrix}, \tag{A.1}$$

where

$$a = h_\perp / \dot{h}_z^{1/2}, \quad p = -ia^2/4, \quad z = e^{i\pi/4}\dot{h}_z^{1/2}\theta,$$

α and β are constants determined by the initial conditions. In order to determine U (see Eq. (3.10)) it is necessary to transform the matrix $M(\theta_2)M^{-1}(\theta_1)$ into the system connected with the direction of **h**, which can be accomplished by making a rotation through the angle $\alpha = \tan^{-1}(h_\perp/h_z)$ around the y axis:

$$U = \exp\{{}^1\!/_2 i\sigma_y\alpha(\theta_2)\}M(\theta_2)M^{-1}(\theta_1)\exp\{-{}^1\!/_2 i\sigma_y\alpha(\theta_1)\}. \tag{A.2}$$

In order to determine the constant phase φ in Eq. (3.7) it is sufficient to compare the matrix elements U_{11} given by Eqs. (3.10) and (A.2) in the limit $|\theta_1|$, $|\theta_2| \to \infty$,

$$U_{11} = (1 - e^{-2\delta})^{1/2}\exp\left\{-i\left(\varphi + \int_{\theta_1}^{\theta_2} h\, d\theta\right)\right\} \approx (M(\theta_2)M^{-1}(\theta_1))_{12}.$$

Substituting $\delta = \pi a^2/4$, $h = (\dot{h}_z^2\theta^2 + h_\perp^2)^{1/2}$ and using the limiting asymptotic expressions for $D_p(z)$,[17] we obtain expression (3.11) for the phase φ.

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