GRAVITATIONAL POTENTIAL TENSOR AND EQUATIONS OF MOTION IN RELATIVISTIC MECHANICS

I. G. FIKHTENGOL'TS

Leningrad Institute of Precision Mechanics and Optics

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The concept of the gravitational potential tensor is introduced. Equations are established which define the gravitational potential tensor. An approximate solution of the equations is obtained for the astronomical problem of an isolated mass system. The equations of motion in relativistic mechanics are considered. General expressions are derived for the force acting on the body and for the momentum of the body. It is shown that the force and momentum depend not only on the mass tensor but also on the gravitational potential tensor.

THE article consists of two parts. In the first part we introduce a tensor which we call the gravitational potential tensor, and we investigate some of its properties. The purpose of the second part is to establish a connection between the equations of motion of relativistic mechanics (see[1-6]) and the tensor of the gravitational potential.

1. THE TENSOR OF THE GRAVITATIONAL POTENTIAL

We introduce first the vector V_α, which satisfies the equations

$$\rho^* g^{\mu\nu}\nabla_\mu\nabla_\nu V_\alpha = c^2\Pi_{\alpha\nu}{}^\mu T_\mu{}^\nu, \tag{1}$$

where ρ^* is the invariant mass density, $g_{\mu\nu}$ the fundamental tensor, $T_{\mu\nu}$ the mass tensor, c the velocity of light in vacuum, and

$$\Pi_{\mu\nu}{}^\alpha = \Gamma_{\mu\nu}{}^\alpha - (\Gamma_{\mu\nu}{}^\alpha)_0. \tag{2}$$

Here $\Gamma^\alpha_{\mu\nu}$ are Christoffel symbols of the second kind for the considered space-time $(\Gamma^\alpha_{\mu\nu})_0$ are also Christoffel symbols of the second kind, not for the investigated space-time but for the auxiliary space-time against the background of which we consider the real physical space-time. In Eqs. (1) ∇_μ, as usual, denotes the covariant derivative in the investigated space-time. Greek indices take on values 0, 1, 2, 3 and summation from zero to 3 is implied for repeated Greek indices. The aggregate of the values of V_α defined by Eqs. (1) composes a general-covariant vector.

If we choose as the auxiliary space-time against the background of which we consider the investigated space-time a space-time for which

$$(\Gamma_{\mu\nu}{}^\alpha)_0 = 0, \tag{3}$$

then Eqs. (1) take the form

$$\rho^* g^{\mu\nu}\nabla_\mu\nabla_\nu V_\alpha = c^2\Gamma_{\alpha\nu}{}^\mu T_\mu{}^\nu. \tag{4}$$

The aggregate of the quantities V_α defined by Eqs. (4) does not make up a general-covariant vector. This is a vector only with respect to linear transformations; in particular, V_α is a vector in a harmonic coordinate system.

We write down Eqs. (4) as applied to the case when we can put

$$T_{\mu\nu} = \frac{1}{c^2}\rho^* u_\mu u_\nu. \tag{5}$$

Here u_μ is a three-dimensional covariant velocity vector, normalized in accordance with the formula

$$g_{\mu\nu}u^\mu u^\nu = c^2. \tag{6}$$

The conditions

$$\nabla_\mu T^{\mu\nu} = 0 \tag{7}$$

with allowance for the continuity equation

$$\nabla_\mu \rho^* u^\mu = 0 \tag{8}$$

are equivalent, as applied to the assumption (5), to the equations

$$u^\mu \nabla_\mu u_\nu = 0, \tag{9}$$

or, which is the same,

$$du_\alpha / d\tau - \Gamma_{\alpha\nu}{}^\mu u^\nu u_\mu = 0, \tag{10}$$

where τ is the proper time. From (5) and (10) it follows that

$$\rho^* \frac{du_\alpha}{d\tau} = c^2\Gamma_{\alpha\nu}{}^\mu T_\mu{}^\nu. \tag{11}$$

Taking (11) into consideration, we arrive at the conclusion that as applied to the assumption (5) Eqs. (4) take the form

$$g^{\mu\nu}\nabla_\mu\nabla_\nu V_\alpha = du_\alpha / d\tau. \tag{12}$$

Let us consider now Eqs. (4) as applied to the astronomical problem of an isolated mass system. Starting from the solution obtained in Fock's book[6] for the Einstein gravitational equations, we can show that for the astronomical problem of an isolated system of masses the following equations are valid in first approximation:

$$\Gamma_{\alpha\nu}{}^\mu T_\mu{}^\nu = -\frac{1}{c^2}\rho\frac{\partial U}{\partial r}, \tag{13}$$

where U is the Newtonian gravitational potential and ρ is the ordinary mass density.

Taking equality (13) into consideration, we arrive at the conclusion that as applied to the astronomical problem of an isolated system of masses Eqs. (4) assume in first approximation the form

$$\Delta V_\alpha = \partial U / \partial x_\alpha. \tag{14}$$

Here Δ is the usual Laplace operator. The solution of Eqs. (14) is the vector

$$V_\alpha = \partial W / \partial x_\alpha, \tag{15}$$

where W is a function satisfying the Poisson equation

$$\Delta W = U. \tag{16}$$

In[6] it is shown that the solution of (16) is the function

$$W = \frac{1}{2} \gamma \int_{(\infty)} \rho' |\mathbf{r} - \mathbf{r}'| (dx')^3, \tag{17}$$

where γ is the Newtonian gravitational constant, $\mathbf{r}$ as usual is the radius vector of the point with coordinates x_1, x_2, x_3, $(dx')^3 = dx'_1 dx'_2 dx'_3$, $\rho' = \rho(x'_1, x'_2, x'_3, t)$. The integration is carried out over the entire infinite space.

We construct the tensor

$$J_{\mu\nu} = \nabla_\mu V_\nu \tag{18}$$

and assume

$$J_\nu^{\ \mu} = \nabla^\mu V_\nu. \tag{19}$$

From (1) and (19) it follows that

$$\rho^* \nabla_\mu J_\alpha^{\ \mu} = c^2 \Pi_{\alpha\nu}^{\ \ \mu} T_\mu^{\ \nu}. \tag{20}$$

In the approximation corresponding to equalities (13)–(17) we have

$$J_{00} = \frac{\partial^2 W}{\partial t^2} + \frac{\partial U}{\partial x_i}\frac{\partial W}{\partial x_i}, \quad J_{0i} = J_{i0} = \frac{\partial^2 W}{\partial x_i \partial t}, \quad J_{ik} = \partial^2 W / \partial x_i \partial x_k. \tag{21}$$

The Latin indices assume values 1, 2, and 3 and summation from 1 to 3 is applied for repeated Latin indices. In formulas (21) it is assumed that $x_0 = t$.

Starting from the presented components of the tensor $J_{\mu\nu}$, we can easily show that in the approximation in question we have

$$\oint_{(S)} J_0^{\ i} (dx)_i^{\ 2} = -\frac{d}{dt} \int_{(V)} U (dx)^3. \tag{22}$$

Here (S) is an arbitrary closed surface and (V) is the volume bounded by the surface (S). In formula (22) it is understood that $(dx)_1^2 = dx_2 dx_3$, $(dx)_2^2 = dx_1 dx_3$ $(dx)_3^2 = dx_1 dx_2$ and $(dx)^3 = dx_1 dx_2 dx_3$.

The foregoing gives grounds for calling the tensor J the gravitational potential tensor.

2. EQUATIONS OF MOTION OF RELATIVISTIC MECHANICS

Let us establish the connection between the equations of motion of relativistic mechanics and the gravitational potential tensor. For any factor χ and tensor T_μ^ν we have

$$\chi \nabla_\nu T_\mu^{\ \nu} = \frac{\partial(\chi T_\mu^{\ \nu})}{\partial x_\nu} - \sqrt{-g}\frac{\partial(\chi/\sqrt{-g})}{\partial x_\nu} T_\mu^{\ \nu} - \chi \Gamma_{\mu\nu}^{\ \ \alpha} T_\alpha^{\ \nu}, \tag{23}$$

and therefore, assuming that T_μ^ν is the mass tensor, we arrive, by virtue of the conservative nature of the mass tensor, at the following equations:

$$\frac{\partial(\chi T_\mu^{\ \nu})}{\partial x_\nu} - \sqrt{-g}\frac{\partial(\chi/\sqrt{-g})}{\partial x_\nu} T_\mu^{\ \nu} - \chi \Gamma_{\mu\nu}^{\ \ \alpha} T_\alpha^{\ \nu} = 0. \tag{24}$$

In these formulas g is a determinant made up of the components of the fundamental tensor $g_{\mu\nu}$. From (24) it follows that

$$\frac{d}{dt}\int_{(a)} \chi T_i^{\ 0} (dx)^3 = \int_{(a)} \sqrt{-g}\frac{\partial(\chi/\sqrt{-g})}{\partial x_\nu} T_i^{\ \nu} (dx)^3 + \int_{(a)} \chi \Gamma_{i\nu}^{\ \ \mu} T_\mu^{\ \nu} (dx)^3 - \oint_{(S_a)} \chi T_i^{\ k} (dx)_k^{\ 2}. \tag{25}$$

Here $\int_{(a)}$ is the integral over the volume occupied by the mass a, and $\oint_{(S_a)}$ is the integral over the surface enclosing the same mass a. Equations (25) are the equations of motion of the mass a. However, the right-hand sides of (25) cannot be interpreted as the components of the force acting on the mass a. This can be verified by applying Eqs. (25) to an isolated system of masses. Indeed, the sum of the right-hand sides of Eqs. (25), taken (at a fixed index i) over all the bodies of the system considered, differs from zero. Accordingly, the quantities under the sign of the derivative with respect to time in the left-hand sides of Eqs. (25) cannot be interpreted as components of the momentum of the mass a.

We put

$$P_{ai} = -c \int_{(a)} T_i^{\ 0} \sqrt{-g} (dx)^3 - \frac{1}{c} \int_{(a)} \rho^* J_i^{\ 0} \sqrt{-g} (dx)^3, \tag{26}$$

$$F_{ai} = -c \int_{(a)} \Gamma_{i\nu}^{\ \ \mu} T_\mu^{\ \nu} \sqrt{-g} (dx)^3 + c \oint_{(S_a)} \sqrt{-g} T_i^{\ k} (dx)_k^{\ 2} - \frac{1}{c}\frac{d}{dt}\int_{(a)} \rho^* J_i^{\ 0} \sqrt{-g} (dx)^3. \tag{27}$$

It follows from (25), (26), and (27) that the equations of motion of the mass a can be represented in the form (see[6])

$$dP_{ai} / dt = F_{ai}. \tag{28}$$

Let us verify that the quantities P_{ai} and F_{ai} defined by relations (26) and (27) represent the momentum of the mass a and the force acting on this mass.

Let us consider relations (26) and (27) as applied to the astronomical problem of an isolated system of masses.

Starting from the Einstein gravitational equations obtained in[6], it can be shown that for the astronomical problem of an isolated system of masses the following equalities are valid accurate to quantities of the order of U/c^2:

$$\sqrt{-g} T_i^{\ 0} = -\frac{1}{c}\left\{\rho v_i \left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + \Pi + 3U\right)\right] - \frac{1}{c^2} p_{ik} v_k - \frac{4}{c^2}\rho U_i\right\} \tag{29}$$

$$\sqrt{-g}\Gamma_{i\nu}^{\ \ \mu} T_\mu^{\ \nu} = -\frac{1}{c}\left\{\frac{\partial U^*}{\partial x_i}\sigma - \frac{4}{c^2}\rho v_k \frac{\partial U_k}{\partial x_i}\right\}. \tag{30}$$

In these formulas v_i is the component of the velocity of the particle of the medium and $v^2 = v_1^2 + v_2^2 + v_3^2$, p_{ik} is the three-dimensional tensor of the elastic stresses, Π is the elastic potential energy of a unit mass of the medium, and U_i is the vector potential of the gravitational field, i.e.,

$$\Delta U_i = -4\pi\gamma\rho v_i. \tag{31}$$

The potential U^* is determined by the equation

$$\Delta U^* - \frac{1}{c^2}\frac{\partial^2 U^*}{\partial t^2} = -4\pi\gamma\sigma, \tag{32}$$

with

$$\sigma = \rho + \frac{\rho}{c^2}\left(\frac{3}{2}v^2 + \Pi - U\right) - \frac{p_{kk}}{c^2} \tag{33}$$

and, of course $p_{kk} = p_{11} + p_{22} + p_{33}$.

To calculate the quantities P_{ai} and F_{ai} by formulas (26) and (27), it is necessary to find the components of the gravitational potential tensor contained in (26) and (27). It follows from (21) that in the approximation of interest to us

$$J_i^0 = \frac{1}{c^2}\frac{\partial^2 W}{\partial x_i\, \partial t}. \qquad (34)$$

Taking into consideration (29), (30), and (34) and recognizing that in the considered case of the astronomical problem of an isolated system of masses the integral over the surface (S_a) in (27) is equal to zero, we arrive in accordance with (26) and (27) at the conclusion that in the approximation of interest to us

$$P_{ai} = \int_{(a)} \left\{ \rho v_i \left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + \Pi + 3U\right)\right] - \frac{1}{c^2} p_{ik} v_k - \frac{4}{c^2}\rho U_i - \frac{1}{c^2}\rho \frac{\partial^2 W}{\partial x_i\, \partial t}\right\} (dx)^3, \qquad (35)$$

$$F_{ai} = \int_{(a)} \frac{\partial U^*}{\partial x_i}\sigma (dx)^3 - \frac{1}{c^2}\frac{d}{dt}\int_{(a)} \rho \frac{\partial^2 W}{\partial x_i\, \partial t}(dx)^3 - \frac{4}{c^2}\int_{(a)} \rho v_k \frac{\partial U_k}{\partial x_i}(dx)^3. \qquad (36)$$

For convenience in the comparison of formulas (25) and (36) with the corresponding formulas (75.31) and (75.32) from[6], we assume, following the cited book

$$p_{ik} = -p\delta_{ik}, \qquad (37)$$

where p is the isotropic pressure and, as usual, $\delta_{ik} = 1$ when $i = k$ and $\delta_{ik} = 0$ when $i \neq k$. Further, following[6], we divide the potentials U^*, W, and U_k into internal and external, putting

$$U^* = u_a^* + U^{*(a)}, \qquad (38)$$

$$W = w_a + W^{(a)}, \qquad U_k = u_{ak} + U_k^{(a)}.$$

Then, as shown in[6]

$$\int_{(a)} \frac{\partial u_a^*}{\partial x_i}\sigma (dx)^3 = \frac{1}{c^2}\frac{d}{dt}\int_{(a)} \rho \frac{\partial^2 w_a}{\partial x_i\, \partial t}(dx)^3, \qquad (40)$$

$$\int_{(a)} \rho v_k \frac{\partial u_{ak}}{\partial x_i}(dx)^3 = 0. \qquad (41)$$

Taking (37)—(41) into account, we arrive at the conclusion that relations (35) and (36) coincide with relations (75.31) and (75.32) of[6]. Consequently, the quantities P_{ai} and F_{ai}, defined by Eqs. (26) and (27), can indeed be interpreted as the angular momentum of the mass a and the force acting on this mass.

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