

SOME PROPERTIES OF SURFACE ELECTRONS IN LIQUID HELIUM

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It is proposed that surface electrons can be used to study thermal surface excitations in liquid helium. For this purpose the interaction between surface electrons and the liquid-vapor interface is considered. A description of the self-consistent deformation of the free surface of liquid helium by an electron is obtained. A necessary condition for such deformation to occur in the approximation linear in the deformations is the presence of an external electric field E_{\perp} which exerts an additional pull on the electron towards the helium surface. The temperature ranges in which the effect of a static deformation on the mobility of surface electrons is or is not important are indicated. The electron mobility is calculated in the temperature region in which the static deformation can be neglected.

It is known that charged particles above a liquid-vapor interface are attracted to the free surface of the liquid by electrostatic image forces. After the charge has crossed the interface, the attraction is replaced by repulsion, and charged particles in the image-force field should therefore always go down into the bulk of the liquid phase. However, this statement is fully applicable only to positive ions. If we are speaking of electrons, and if the liquid medium is helium, the situation is somewhat different. For free electrons the surface of liquid helium is a potential barrier of height 1.4 eV. Therefore, electrons which have energy less than 1.4 eV and which are attracted to the liquid helium surface by the image forces cannot pass through the surface into the liquid phase and must be localized above the boundary surface.

A preliminary description of such surface electron states above a plane free surface of liquid helium was given by Cole and Cohen,^[1] and also, independently, by the author of this paper.^[2] In particular, the wavefunction ψ_0 and energy λ_0 of the ground state have the following form

$$\begin{aligned} \psi_0(xyz) &= \text{const} \cdot z \exp[-\gamma z + i(k_x x + k_y y)], \\ \lambda_0 - \hbar^2 k^2 / 2m &= -m\alpha^2 / 2\hbar^2, \\ \gamma &= m\alpha / \hbar^2, \quad \alpha = e^2 |\epsilon_1 - \epsilon_2| / 4\epsilon_1 (\epsilon_1 + \epsilon_2). \end{aligned} \tag{1}$$

Here the helium surface lies in the xy-plane. $z \geq 0$ corresponds to the gas phase. m and e are the mass and charge of a free electron, \mathbf{k} is the wave-vector in the xy-plane, ϵ_1 and ϵ_2 are the dielectric constants of the liquid and vapor and $|\epsilon_1 - \epsilon_2| = 0.06$. With this value of $|\epsilon_1 - \epsilon_2|$ γ is found to be of the order of $\approx 2 \times 10^6 \text{ cm}^{-1}$, and $\lambda_0 \approx 10^{-3} \text{ eV}$. Numerical estimates of the constants γ and λ_0 confirm the correctness of the basic assumptions made in^[2] in the course of the derivation. One of these assumptions is the requirement that the inequality $\lambda_0 \ll 1.4 \text{ eV}$ be fulfilled,^[1] which enables us

¹⁾We note that although the localization energy in the surface levels is small compared with the barrier height of 1.4 eV, it is nevertheless of order $\approx 10^{-3} \text{ eV} \approx 10^\circ \text{K}$, which for typical helium temperatures $T \sim 1^\circ \text{K}$ appreciably exceeds the possible thermal electron energy values in the gas phase. Consequently, at temperatures $\lesssim 1^\circ \text{K}$, thermal electrons in the gas phase must, with high probability, occupy surface levels above the liquid-vapor interface.

to regard the potential barrier at the liquid-vapor interface as infinitely high and to take account of it by the boundary condition $\psi|_{z=0} = 0$. The other important assumption about the possibility of using, for the potential $W(z)$ of the image force, its macroscopic value $W(z) = \alpha/z$ is legitimate if distances large compared with the interatomic spacings a_0 contribute in forming the wavefunction, i.e., $\gamma a_0 \ll 1$. Both these inequalities are found to be fulfilled by an ample margin, by virtue of the smallness of the difference $|\epsilon_1 - \epsilon_2| \ll 1$.

The practical value of surface electrons is determined by the possibility of using them to study the properties of thermal surface excitations in liquid helium. The point is that thermal surface excitations are the chief cause of drag on the surface electrons as they move along the interface. Therefore, the mobility of surface electrons along the free boundary contains valuable information on the nature of these excitations. In order to extract the corresponding information from the experimental data, it is necessary to calculate the concrete form of the interaction between the electrons and the thermal surface excitations. The study of this interaction forms the content of the present paper.

INTERACTION OF ELECTRONS WITH FREE-SURFACE VIBRATIONS

The wave equation for an electron above a free surface in the presence of small deformations $\delta(\mathbf{r})$ of the boundary shape from the equilibrium plane state has the following form (in writing this equation, the assumptions used are the same as in^[2]):

$$\begin{aligned} \frac{\hbar^2}{2m} \Delta \psi + \left[\lambda + \frac{\alpha}{z + \delta(\mathbf{r})} - eE_{\perp} z \right] \psi &= 0, \\ \psi|_{z=-\delta(\mathbf{r})} &= 0, \quad \psi|_{z \rightarrow +\infty} \rightarrow 0, \end{aligned} \tag{2}$$

Δ is the three-dimensional Laplace operator, m and e are the mass and charge of a free electron, α is as in (1), and λ is the eigenvalue of the equation. The plane $z = 0$ coincides with the surface of the unperturbed helium, $z > 0$ corresponds to the gas phase, and \mathbf{r} is a two dimensional radius vector in the xy-plane. The boundary conditions for $\psi(\mathbf{z}, \mathbf{r})$ as a function of \mathbf{r} depend on the concrete formulation of the different problems and will be specified separately below. The potential

$W(\mathbf{z}, \mathbf{r})$ of the image force above a deformed surface has the form used in (2), $W = \alpha/(z + \delta)$, with the condition that the average distance of an electron from the surface is less than the characteristic dimensions of the change in $\delta(\mathbf{r})$ along \mathbf{r} . E_{\perp} is the intensity of the external electric field exerting an additional pull on the electron towards the surface.²⁾ This field turns out to be very convenient and, in practice, a necessary ingredient in forming the electron-phonon interaction.

We make a change of variables in (2) ($\xi = z + \delta(\mathbf{r})$, $r = r$), which allows us to get rid of the perturbation $\delta(\mathbf{r})$ in the boundary conditions (r is a scalar quantity everywhere below). In terms of ξ and r , the system (2) is rewritten as follows:

$$\begin{aligned} \frac{\partial^2 \psi}{\partial \xi^2} (1 + \delta'^2) + \Delta_{\mathbf{r}} \psi + 2\delta' \frac{\partial^2 \psi}{\partial r \partial \xi} + \Delta_{\mathbf{r}} \delta \frac{\partial \psi}{\partial \xi} \\ + \frac{2m}{\hbar^2} \left[\lambda + \frac{\alpha}{\xi} - eE_{\perp}(\xi - \delta) \right] \psi = 0, \quad (2a) \\ \psi|_{\xi=0} = 0, \quad \psi|_{\xi \rightarrow \infty} \rightarrow 0. \end{aligned}$$

$\Delta_{\mathbf{r}}$ is the two-dimensional Laplace operator, and $\delta' = d\delta/dr$. The perturbing terms in (2a) formally have a different order of smallness in δ . However, as will be clear from the following, the terms $\delta' \partial^2 \psi / \partial r \partial \xi$ and $\Delta_{\mathbf{r}} \delta \partial \psi / \partial \xi$ do not contribute to the first order of perturbation theory in δ .

Equation (2a) contains different deformational effects, and we turn now to the description of these.

STATIC SELF-CONSISTENT DEFORMATION

On a smooth surface, the solution of (2) has the form of a surface wave, propagating along the boundary surface. If we take a possible deformation of the shape of the surface into account, Eq. (2a) admits solutions which are localized and in the xy -plane.

To obtain this solution, we supplement (2a) with boundary conditions for ψ with respect to r :

$$\psi|_{r \rightarrow \infty} \rightarrow 0, \quad \partial \psi / \partial r|_{r=0} = 0 \quad (2b)$$

and with the equation of mechanical equilibrium

$$\sigma \Delta_{\mathbf{r}} \delta - \rho g \delta = p_{el}. \quad (3)$$

Here, σ is the surface tension and p_{el} is the pressure on the free surface from the electron side. The magnitude of this pressure is determined by the formulas

$$\begin{aligned} (p_{el})_i = P_{ik} n_k, \\ P_{ik} = \frac{\hbar^2}{4m} \left[\frac{\partial \psi}{\partial x_i} \frac{\partial \psi^*}{\partial x_k} - \psi \frac{\partial^2 \psi^*}{\partial x_i \partial x_k} + c.c. \right], \quad (4) \end{aligned}$$

n is the direction vector, and P_{ik} is the momentum flux tensor for an electron with wavefunction ψ . In expression (4), ψ is understood to be normalized to unity: $\int |\psi|^2 dV = 1$.

The term $\rho g \delta(\mathbf{r})$ in (3), where ρ is the density of liquid helium and g is the acceleration due to gravity, has to be included in the equilibrium equation to avoid

²⁾The potential energy of an electron in a field E_{\perp} above a deformed helium surface actually has the form $eE_{\perp}z$ (plus corrections associated with the deformation of the surface). These corrections, however, are proportional not only to $\delta(\mathbf{r})$, but also to the small difference $|\epsilon_1 - \epsilon_2| \ll 1$. For this reason they can be omitted.

divergences in the dependence of δ on r at large distances. This divergence arises in the solution of the two-dimensional Poisson equation, but is absent in the two-dimensional equation which includes the term $\rho g \delta$.

The system of equations (2)–(4) is closed and permits us to determine the parameters of a localized state of surface electrons. It is convenient to start the solution of this system from an examination of Eq. (2a). Assuming that the function $\psi(\xi, r)$ is equal to

$$\psi(\xi, r) = f(\xi)\varphi(r) + \chi(\xi, r), \quad f\varphi \gg \chi \quad (5)$$

we substitute this expression into (2a) and determine the functions $f(\xi)$ and $\chi(\xi, r)$ from the following equations

$$\begin{aligned} \frac{d^2 f}{d\xi^2} + \frac{2m}{\hbar^2} \left[\lambda_0 + \lambda_1 + \frac{\alpha}{\xi} - eE_{\perp} \xi \right] f = 0, \\ f|_{\xi=0} = 0, \quad f|_{\xi \rightarrow \infty} \rightarrow 0, \quad \int_0^{\infty} f^2 d\xi = 1, \quad (5a) \\ \Delta \chi + \frac{2m}{\hbar^2} \left[\lambda_0 + \frac{\alpha}{\xi} - eE_{\perp} \xi \right] \chi \\ = -f(\xi) \Delta_{\mathbf{r}} \varphi - (\delta')^2 \varphi - 2\delta' \frac{d\varphi}{d\xi} \frac{d\varphi}{dr} - (\Delta_{\mathbf{r}} \delta) \varphi \frac{d\varphi}{d\xi} - (\lambda_1 + eE_{\perp} \delta) f \varphi. \quad (5b) \end{aligned}$$

In Eq. (5b), we include all terms from (2a) which do not enter into (5a). The condition for the existence of a solution of this inhomogeneous equation is the requirement that the right-hand side of (5b) be orthogonal to the solution of the corresponding homogeneous equation, i.e., to the function $f(\xi)\varphi(r)$. Writing out this orthogonality condition explicitly, it is easy to see that the terms on the right-hand side of (5b) containing the first derivative $df/d\xi$ vanish on integration over ξ .³⁾ Omitting also terms with $(\delta')^2$ and bearing in mind that

$$\int_0^{\infty} f^2 d\xi = 1,$$

we reduce the orthogonality condition to the form

$$\int_{-\infty}^{+\infty} \varphi(r) \left[\Delta_{\mathbf{r}} \varphi + \frac{2m}{\hbar^2} (\lambda_1 + eE_{\perp} \delta) \varphi \right] r dr = 0.$$

This condition is always fulfilled if the square-bracketed expression in the integrand is equal to zero. We thus obtain an equation for $\varphi(r)$

$$\begin{aligned} \Delta_{\mathbf{r}} \varphi + \frac{2m}{\hbar^2} [\lambda_1 + eE_{\perp} \delta] \varphi = 0, \\ \varphi|_{r \rightarrow \infty} \rightarrow 0, \quad \partial \varphi / \partial r|_{r=0} = 0. \quad (6) \end{aligned}$$

We can arrive at the same equation for φ by direct separation of the variables in (2a), if we know beforehand that the terms with first derivatives with respect to ξ can be neglected.

It should be noted that in the effective equation (6) for $\varphi(r)$, the interaction with the surface which is linear in δ is completely determined by the intensity of the field exerting the pull. Therefore, confining ourselves in the present paper to a description of the linear deformational interaction, we shall assume that the field E_{\perp} is large enough ($E_{\perp} \gg E_{\perp}^0$) to allow us to leave out of account the terms in the deformational interaction

³⁾In fact

$$\int_0^{\infty} f \frac{df}{d\xi} d\xi = \frac{1}{2} [f^2(\infty) - f^2(0)].$$

This expression is zero by virtue of the boundary conditions (5a).

that are nonlinear in δ . The quantity E_1^0 is determined by the order-of-magnitude relation

$$eE_{\perp}^0 \delta \sim \frac{\hbar^2}{2ma^2} (\delta')^2,$$

where $\hbar^2/2ma^2$ is the transverse kinetic energy of the electron, and $a \sim \gamma^{-1}$ is the average distance of the electron from the free surface. In order of magnitude, $\delta \sim 10^{-8}$ cm, $a \sim 10^{-8}$ cm, $\delta' \sim \delta/L$, where $L \gg a$. Thus $E_1^0 \sim 10^{-3}$ esu.

The next steps in the determination of the function $\varphi(r)$, which determines the localization of an electron in the xy-plane, are as follows. Using expression (5) for $\psi(\xi, r)$, we put this into (4). As a result, the general formula for p_{el} takes the more concrete form

$$p_{el} \approx \frac{\hbar^2}{2m} \left(\frac{df}{d\xi} \Big|_{\xi=0} \right)^2 \varphi^2(r),$$

where $f(\xi)$ is the solution of Eq. (5a).

Substituting p_{el} written in this way into (3) and solving this equation, we find that

$$\delta(r) = A \int_0^{\infty} K_0(\kappa R) \varphi^2(s) s ds, \tag{7}$$

$$A = \frac{\hbar^2}{2m\sigma} \left(\frac{df}{d\xi} \Big|_{\xi=0} \right)^2,$$

where $K_0(\kappa R)$ is a Hankel function of zero order, $\kappa^2 = \rho g/\sigma$ and $R = |\mathbf{r} - \mathbf{s}|$. The final equation for $\varphi(r)$ is now obtained by replacing the quantity $\delta(r)$ in Eq. (6) by its expression in terms of φ from (7).

The nonlinear equation in $\varphi(r)$ arising in this way can hardly be solved completely in general form. But there is no special necessity to do this. In fact, by virtue of the singularities of the two-dimensional equation (3) noted above, the deformation $\delta(r)$ of the free surface, even when induced by a delta-function force, extends over a region with characteristic dimensions $\kappa^{-1} \sim 10^{-1} - 10^{-2}$ cm. And if the wavefunction of an electron captured by a potential well $eE_{\perp} \delta(r)$ is also smeared over distances $L \sim \kappa^{-1}$, the binding energy of this complicated complex turns out to be negligibly small: $\hbar^2/2mL^2 \sim 10^{-23}$ erg $\sim 10^{-7}$ °K. A self-consistent deformation is therefore of interest only in the case when the electron is localized in the xy-plane over distances $L \ll \kappa^{-1}$.

Assuming this inequality to be fulfilled, we can expand the potential energy of the electron from (6) in powers of r : $eE_{\perp} \delta(r) \approx eE_{\perp} [\delta(0) + \frac{1}{2} \delta''(0) r^2 + \dots]$, after which Eq. (6) reduces to an oscillator equation. The ground-state wavefunction satisfying the approximate Eq. (6) has the usual form $\varphi_0(r) = \exp(-r^2/L^2)$, and L and $\delta''(0)$ are found to be connected by the relation

$$L^{-4} = \frac{m}{4\hbar^2} eE_{\perp} \delta''(0). \tag{8}$$

Furthermore, from Eq. (3) we have

$$\Delta \delta|_{z=0} = \sigma^{-1} [p_{el}(0) + \rho g \delta(0)],$$

or, using the expression for p_{el} and $\delta(r)$ from (7), we find

$$\delta''(0) = \frac{4\hbar^2}{m} \frac{\gamma^3}{\sigma} L^{-2}. \tag{8a}$$

In the derivation of (8a), the derivative $df/d\xi|_{\xi=0}$, which occurs in both p_{el} and $\delta(r)$, is calculated in explicit form

for the case when the external field E_{\perp} can be neglected in comparison with the image force in Eq. (5a) for $f(\xi)$. In practice, this can be done for fields $E_{\perp} < 300$ V/cm.

It follows from the equalities (8) and (8a) that the quantity L is determined by the following relation:

$$L^{-2} = eE_{\perp} \gamma^3 / \sigma. \tag{9}$$

For $\sigma = 0.36$ erg/cm², $\gamma \approx 2 \times 10^6$ cm⁻¹ and $E_{\perp} \lesssim 300$ V/cm, the numerical value of L is found to be $L \sim 10^{-4} - 10^{-5}$ cm, i.e., the inequality $\kappa L \ll 1$ used in obtaining (9) is indeed fulfilled.

The binding energy of the localized surface state (9) is of the order $\hbar^2/2mL^2 \sim [10^{-1} - 10^{-3}]$ °K. Thus, to observe the states described in this section it is necessary to work in the temperature region $T \ll 0.1$ °K. In the opposite case $T > 0.1$ °K, the localization energy is found to be insignificant compared with the temperature, and the ground state of the surface electrons is, in practice, a surface wave of the form (1), with an effective mass not much different from the free electron mass.

MOBILITY OF SURFACE ELECTRONS IN THE REGION $T > 0.1$ °K

In the temperature region $T > 0.1$ °K the deformational interaction of an electron with free-surface vibrations can be assumed to be a small perturbation to the ground state (1). The magnitude of the mobility of an electron under these conditions can be determined by the known methods of the theory of the electron-phonon interaction (cf., e.g., the article by Bethe and Sommerfeld^[3]). The only question worth dwelling on first is connected with the quantization of the surface vibrations in liquid helium.

A. Quantization of the surface waves. This can be conveniently carried out following Pitaevskii's arguments,^[4] which he developed for the exchange case.

The Hamiltonian of the surface vibrations can be written with sufficient exactness in the following form:

$$\hat{H} = E_0 + \int \frac{\rho v^2}{2} dx dy dz + \int F(|\mathbf{r} - \mathbf{r}_1|) \delta(r) \delta(\mathbf{r}_1) d\mathbf{r}_1, \tag{10}$$

where ρ is the average helium density, \mathbf{v} is the local velocity, $\delta(r)$ is the deviation from the equilibrium position, \mathbf{r} is a two-dimensional vector in the plane of the free interface, $d\mathbf{r} = dx dy$, and $F(|\mathbf{r} - \mathbf{r}_1|)$ is the structure factor, which, by virtue of the homogeneity and isotropy of the liquid along the surface, depends only on the difference $|\mathbf{r} - \mathbf{r}_1|$. The velocity \mathbf{v} satisfies the equation $\text{div } \mathbf{v} = 0$. Moreover, the values of $\delta(r)$ are connected with the normal component of the velocity at the free surface by the relation

$$\frac{\partial \delta}{\partial t} = v_n|_{z=0}. \tag{11}$$

We introduce a scalar potential Φ by means of the equality $\mathbf{v} = \nabla \Phi$, so that $\Delta \Phi = 0$, and expand all the variables in Fourier series

$$\delta = \frac{2}{S^{1/2}} \sum_{\mathbf{k}} \delta_{\mathbf{k}} \cos kx \cos ky, \quad F = \frac{2}{S^{1/2}} \sum_{\mathbf{k}} F_{\mathbf{k}} \cos kx \cos ky,$$

$$\Phi(xyz) = \frac{2}{S^{1/2}} \sum_{\mathbf{k}} \Phi_{\mathbf{k}} \cos kx \cos ky \exp(\sqrt{2}kz),$$

$$z \leq 0.$$

Putting these series into (10) and taking the condition (11) into account, we bring the Hamiltonian (10) to the form

$$\hat{H} = E_0 + \frac{3}{8\gamma^2} \rho \sum_k \frac{1}{k} \left(\frac{d\delta_k}{dt} \right)^2 + \frac{S^{1/2}}{4} \sum_k F_k \delta_k^2.$$

The total excitation energy in this expression breaks down into a sum of energies of independent oscillators with characteristic frequencies

$$\omega_k^2 = S^{1/2} \frac{2\gamma^2}{3} F_k \frac{k}{\rho}. \quad (12)$$

Furthermore, by analogy with [4], we can indicate one more link between $\hbar\omega_k$ and the average potential energy of the k -th oscillator

$$1/4 \hbar\omega_k = 1/4 S^{1/2} F_k |\overline{\delta_k}|^2. \quad (13)$$

The equalities (12) and (13) enable us to write the dispersion law in terms of $|\overline{\delta_k}|^2$:

$$\hbar\omega_k = \frac{2\gamma^2}{3} \frac{\hbar^2 k}{\rho |\overline{\delta_k}|^2} \quad (14)$$

The quantity $|\overline{\delta_k}|^2$ cannot be calculated, but it can, in principle, be determined experimentally for all k . In the volume case, the corresponding program has already been carried through successfully. As regards the surface case, the problem of $|\overline{\delta_k}|^2$ has not been investigated practically. Concrete ideas on the properties of $|\overline{\delta_k}|^2$ can be expressed only for the limiting case of $k \rightarrow 0$. In this limit, the dispersion law of the surface waves has the form [5]

$$\omega_k^2 = gk + \frac{\sigma}{\rho} k^3,$$

where g is the acceleration due to gravity and σ is the surface tension. Consequently, as $k \rightarrow 0$ we have $\omega \propto k^{1/2}$, and, for this, $|\overline{\delta_k}|^2$ from (14) must behave like $|\overline{\delta_k}|^2_{k \rightarrow 0} \propto k^{1/2}$. The latter assertion is, however, somewhat formal. The point is that, for example, for the temperature dependence of the surface tension in liquid helium, values of $k \lesssim 10^5 \text{ cm}^{-1}$ must be considered essentially as the long-wave limit, since these k correspond to a temperature $T \lesssim 10^{-2} - 10^{-3} \text{ }^\circ\text{K}$. But for these k , the surface tension term in the dispersion law is the principal term in comparison with the gravitational one, and only when $k \lesssim 10 \text{ cm}^{-1}$ are the two terms in the dispersion law comparable in magnitude. Consequently, in all thermodynamic applications the dispersion law of the surface waves in the long-wave limit has essentially the form $\omega^2 = \sigma k^3 / \rho$. And this means that $|\overline{\delta_k}|^2_{k \rightarrow 0} \propto k^{-1/2}$, i.e., $|\overline{\delta_k}|^2$ has an essential singularity as $k \rightarrow 0$. And only in the region of very small $k \lesssim 10 \text{ cm}^{-1}$ is this radical increase replaced by the dependence $|\overline{\delta_k}|^2 \propto k^{1/2}$.

To conclude this section, we cite an expression for the displacement $\delta(\mathbf{r})$ in terms of creation and destruction operators for surface phonons

$$\delta(\mathbf{r}) = \frac{1}{S^{1/2}} \sum_{\mathbf{k}} \left(\frac{\hbar k}{2\rho\omega_k} \right)^{1/2} (a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k}}^+ e^{-i\mathbf{k}\cdot\mathbf{r}}), \quad (15)$$

where $\omega_{\mathbf{k}}$ is the surface-wave dispersion law.

B. Calculation of the mobility. The starting equation for the calculation of the matrix elements determining

the magnitude of the mobility is, as in the previous section, Eq. (2a). The unperturbed wavefunctions of this equation have the form $f_n(\mathbf{z}) e^{i\mathbf{k}\cdot\mathbf{r}}$, where the number n takes the discrete series of values $n = 0, 1, 2, 3, \dots$, and \mathbf{k} is a continuous two-dimensional vector. In the general case, the perturbing terms of Eq. (2a) can induce transitions of electrons from one state to another, both with changes of wave vector \mathbf{k} and with change of n . However, in the temperature region $0.1 \text{ }^\circ\text{K} \lesssim T < 1 \text{ }^\circ\text{K}$, the probability that the electron is in a state with $n > 0$ is exponentially small.⁴⁾ Under these conditions, the interaction of electrons with surface waves can be accompanied only by change of the electron wave vector \mathbf{k} with unchanged $n = \text{const} = 0$. Consequently, to describe electron states in the temperature region $0.1 \text{ }^\circ\text{K} \lesssim T < 1 \text{ }^\circ\text{K}$ (we confine ourselves to treating this) we can use (1) with a fixed function $f(\mathbf{z})$.

Having determined the electron states, it is easy to see that there are non-zero matrix elements in these states only for the perturbation $eE_{\perp} \delta$. As regards the perturbing terms

$$\frac{\hbar^2}{2m} \Delta \delta - \frac{\partial}{\partial \xi}, \quad \frac{\hbar^2}{2m} \delta' - \frac{\partial^2}{\partial r \partial \xi},$$

which, from estimates, are much greater than $eE_{\perp} \delta$, these have no effect, for the reasons indicated in footnote 3, in the approximation linear in δ , if the initial and final electron states have the same functions $f_0(\mathbf{z})$.

Taking the above remarks into account, calculating the matrix elements for the perturbation $eE_{\perp} \delta$ (in which the quantity δ is expressed in terms of the operators $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$ using (15)) between the electron states from (1) with fixed $f_0(\mathbf{z})$, and then continuing as in [3], we write down the kinetic equation for the electron distribution function $f(\mathbf{k})$:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{field}} + \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = 0, \quad (16)$$

$$f(\mathbf{k}) = f_0(k) [1 + f_1(\mathbf{k})], \quad f_1 \ll 1, \quad (16a)$$

$$\left(\frac{\partial f}{\partial t} \right)_{\text{field}} = -eE_{\perp} \frac{\partial f_0}{\partial \epsilon_{\mathbf{k}}} \frac{\hbar k}{m} \cos \theta, \quad f_0 = \text{const} \cdot \exp \left(-\frac{\epsilon_{\mathbf{k}}}{k_0 T} \right) \quad (16b)$$

Here $f_0(k)$ is the equilibrium electron distribution function, f_1 is the increment arising in an external field \mathbf{E}_{\parallel} directed along the boundary surface, $\epsilon_{\mathbf{k}}$ is the energy of an electron with wave vector \mathbf{k} , $\epsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m$, θ is the angle between the directions of \mathbf{k} and \mathbf{E}_{\parallel} , k_0 is Boltzmann's constant,

$$\begin{aligned} \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} &= \Lambda f_0(k) \sum_{\mathbf{q}} \frac{q}{\omega_{\mathbf{q}}} N_{\mathbf{q}} \left\{ \delta(\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} - \hbar\omega_{\mathbf{q}}) \right. \\ &\times [f_1(\mathbf{k} + \mathbf{q}, \epsilon_{\mathbf{k}} + \hbar\omega_{\mathbf{q}}) - f_1(\mathbf{k}, \epsilon_{\mathbf{k}})] + \exp \left(\frac{\hbar\omega_{\mathbf{q}}}{k_0 T} \right) \delta(\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} + \hbar\omega_{\mathbf{q}}) \\ &\left. \times [f_1(\mathbf{k} + \mathbf{q}, \epsilon_{\mathbf{k}} - \hbar\omega_{\mathbf{q}}) - f_1(\mathbf{k}, \epsilon_{\mathbf{k}})] \right\}, \quad (16c) \end{aligned}$$

$$\Lambda = \frac{\pi e^2 E_{\perp}^2}{S \rho}, \quad N_{\mathbf{q}} = \left[\exp \left(\frac{\hbar\omega_{\mathbf{q}}}{k_0 T} \right) - 1 \right]^{-1},$$

⁴⁾The energy levels of the unperturbed Eq. (2a) are defined by the expression [2]

$$\lambda_n - \hbar^2 k^2 / 2m = -m\alpha^2 / 2\hbar^2 (n+1)^2,$$

where α is as in (1). Hence $\lambda_0 \approx -30^\circ\text{K}$, $\lambda_1 \approx -10^\circ\text{K}$ and $\Delta\lambda_{0,1} \approx 20^\circ\text{K}$. The electron transition probability from the ground level $n = 0$ to levels with $n > 0$ is small, if $\Delta\lambda_{0,1} \gg T$. For $T < 1^\circ\text{K}$ this inequality is in fact fulfilled.

q and ω_q are the wave vector and dispersion law of the surface waves, ρ is the density of liquid helium, and S is the free-surface area of the helium. Equation (16c) is the two-dimensional analog of Eq. (45.5) in [3].

The energy conservation law appearing in (16c) through the δ -functions can be written in explicit form as:

$$\frac{\hbar^2(\mathbf{k} + \mathbf{q})^2}{2m} - \frac{\hbar^2\mathbf{k}^2}{2m} = \pm \hbar \left(\frac{\sigma}{\rho} q^2 \right)^{1/2}, \quad (17)$$

or

$$\hbar \frac{k}{m} \cos \varphi + \frac{\hbar q}{2m} = \pm \left(\frac{\sigma}{\rho} q \right)^{1/2},$$

where φ is the angle between the vectors \mathbf{k} and \mathbf{q} . Of the two terms containing q in the latter equation, the term $\hbar q/2m$ is the principal one for all characteristic q in the temperature region $0.1^\circ\text{K} \lesssim T < 1^\circ\text{K}$ of interest to us. It therefore follows with good accuracy from (17) that

$$q + 2k \cos \varphi = 0, \quad q_{\min} = 0, \quad q_{\max} = 2k. \quad (17a)$$

Below we shall need a further estimate of the characteristic energies $(\hbar\omega_q)_{\max}$ in comparison with k_0T . If we take into account that $q_{\max} = 2k$ and $\epsilon_k \sim k_0T$, this estimate appears as follows:

$$(\hbar\omega_q)_{\max} \approx 4 \left(\frac{\sigma}{\rho\hbar} \right)^{1/2} (mk_0T)^{1/2} \ll k_0T. \quad (17b)$$

Thus, in the process of scattering of free electrons by surface vibrations, only the long-wavelength phonons, with energies much smaller than thermal energies, take part. For such phonons we can use the expression $\omega_q^2 = \sigma q^3/\rho$ as the dispersion law. (In the dispersion law as written, the gravitational term gq is absent. The simplification is valid, by a large margin, for the whole of the temperature range of interest to us.)

The solution of Eq. (16) can be sought in the three-dimensional case, in the form

$$f_1(\mathbf{k}, \epsilon_k) = k_{\parallel} c(\epsilon_k), \quad k_{\parallel} = k \cos \theta. \quad (18)$$

Substituting f_1 from (18) into (16), we find an equation for $c(\epsilon_k)$ ⁵⁾

$$\Lambda f_0(k) \frac{m}{\hbar^2} c(\epsilon_k) \sum_q \frac{2qN_q}{\omega_q} \delta \left(\cos \varphi + \frac{q}{2k} \right) \cos \varphi = -eE_{\perp} \frac{\partial f_0}{\partial \epsilon_k} \frac{\hbar k}{m}. \quad (19)$$

In Eq. (19) the argument of the δ -function is written in the form (17a) and, in addition, the simplifications associated with the inequality (17b) have been used. In particular, $\exp(\hbar\omega_q/k_0T) \approx 1$, and $N_q \approx kT/\hbar\omega_q$.

Going over in (19) from the summation over q to the corresponding integration over $|q|$ and φ , we find, after simple calculations, an expression for $c(\epsilon_k)$ or $c(k)$:

$$c(k) = 2E_{\parallel} \hbar^4 \sigma k^2 / eE_{\perp}^2 m^2 (k_0T)^2. \quad (20)$$

Thus, the required distribution function $f(\mathbf{k}, \theta)$ is

$$f(\mathbf{k}, \theta) = f_0(k) [1 + kc(k) \cos \theta],$$

where $c(k)$ is given by (20).

The corresponding mobility μ_{σ} of the surface elec-

trons due to scattering by surface waves is equal to

$$\mu_{\sigma} = 4\hbar\sigma / eE_{\perp}^2 m. \quad (21)$$

Apart from the surface waves, gas atoms can be a cause of scattering of surface electrons. The gas mobility μ_{gas} of the surface electrons is in principle different from the volume gas electron mobility. However, assuming in this paper that the reflection of electrons from a free helium surface is purely mirror reflection, it is possible to suppose that the surface gas mobility is of the same order of magnitude as the volume gas mobility, which is given by the following relation:^[6]

$$\mu_{\text{gas}} = \frac{1}{3} e / n\sigma_0 (2\pi m k_0 T)^{1/2},$$

where n is the density of gas atoms, σ_0 is the total cross section for scattering of an electron by an individual atom, and m is the free electron mass. The value of σ_0 for scattering of slow electrons by an individual helium atom is approximately $\approx 10\pi a_0^2$,^[7] where a_0 is the Bohr radius. The density n of the saturated vapors depends very strongly on the temperature. Thus, at $T \approx 1^\circ\text{K}$, we have $n \approx 10^{18} \text{ cm}^{-3}$, while for $T \approx 0.5^\circ\text{K}$, $n \approx 3 \times 10^{12} \text{ cm}^{-3}$ and continues to fall exponentially on further lowering of the temperature. Consequently, unlike the mobility μ_{σ} , which does not depend on temperature at all, the mobility μ_{gas} increases exponentially as T is lowered. Numerical estimates which for μ_{gas} are in good agreement with the experimental data,^[6] show that for $T \sim 1^\circ\text{K}$ we have $\mu_{\sigma}/\mu_{\text{gas}} \approx 10$, while for $T \approx 0.5^\circ\text{K}$, $\mu_{\sigma}/\mu_{\text{gas}} \sim 10^{-3}$. In these estimates it is assumed that $E_{\perp} \approx 300 \text{ V/cm}$. Thus, there exists a completely defined temperature region $0.1^\circ\text{K} \lesssim T < 1^\circ\text{K}$ in which scattering by surface vibrations makes the main contribution to the mobility of the surface electrons. On raising the temperature into the region $T \gtrsim 1^\circ\text{K}$, collisions between the electrons and gas atoms begin to play a significant role in the slowing down of the electrons. However, one should remember that in this temperature region additional channels of scattering by surface vibrations, associated with the perturbations of the form $\Delta\delta\partial/\partial\xi$, also come into play. Therefore, in the temperature region $T \gtrsim 1^\circ\text{K}$, the mobility μ_{σ} is no longer given by the expression (21), and the actual relationship between μ_{σ} and μ_{gas} requires a special treatment.

CONCLUSION

The above treatment shows that surface electrons interact appreciably with the surface of liquid helium. In the region of very low temperatures, this interaction leads to the formation of a self-consistent complex of an electron and a static deformation of the surface beneath the electron. This complex moves adiabatically over the helium surface as a united whole, with a mobility which apparently differs greatly from the mobility of free electrons in a gas of saturated vapors (an explicit calculation of this mobility will be performed in a separate paper). In the opposite limiting case of high temperatures $T \gtrsim 0.1^\circ\text{K}$, the thermal fluctuations destroy the self-localized states and the motion of electrons over the helium surface is practically free; collisions with the thermal surface excitations can be taken account of by simple perturbation theory. The mobility of the surface electrons under these conditions turns

⁵⁾In fact, on substitution of f_1 from (18) into (16), not all the terms of the resulting equation are proportional to $\cos \theta$. Thus, from $f_1(\mathbf{k} + \mathbf{q}, \epsilon_k) = (k_{\parallel} + q_{\parallel})c(\epsilon_k)$, where $q_{\parallel} = q \cos(\theta - \varphi)$, a term appears with $\sin \theta$. However, this term goes to zero as a result of the subsequent integration over the angle φ .

out to be independent of temperature and comparable in magnitude with the gas mobility of electrons in a gas of saturated vapors at $T \approx 1^\circ\text{K}$. In the temperature region $0.1^\circ \leq T < 1^\circ\text{K}$, the principal mobility is μ_σ .

An important role in the formation of the interaction of electrons with a free helium surface is played by the electric field which exerts an additional pull on the electron towards the free surface. However, this means of increasing the strength of the deformational interaction is not the only one; the increase in the interaction of electrons with a free liquid helium surface should be very marked when the liquid helium is in the form of a thin film lying on a dielectric backing. In this case, an additional image force, associated with the dielectric properties of the support, acts on the free electron situated in the gas phase. The corresponding deformational interaction energy W_d is given by the expression

$$W_d \approx \frac{\delta}{(d+l)^2} \frac{e^2}{2\epsilon_1} \left(\frac{2\epsilon_1}{\epsilon_1 + \epsilon_2} \right)^2 \frac{\epsilon_2 - \epsilon_3}{\epsilon_2 + \epsilon_3}.$$

Here ϵ_1 , ϵ_2 and ϵ_3 are the dielectric constants of the gas, the liquid and the dielectric support, d is the thickness of the helium film, and l is the characteristic distance of the electron from the free liquid helium surface. Comparing W_d with $W_{E_\perp} = eE_\perp \delta$ for the values $\epsilon_1 \approx \epsilon_2 \approx 1$, $\epsilon_3 \sim 2-5$, $d \sim 10^{-5}$ cm and $l \sim 10^{-6}$ cm, we can conclude that the effect of the support on a surface electron is equivalent to a constricting field of intensity E_\perp

≈ 300 V/cm. On decrease of the film thickness to values $d \sim 10^{-6}$ cm, this influence of the support on the interaction of electrons with the surface waves is further increased by an order of magnitude.

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¹M. W. Cole and M. H. Cohen, Phys. Rev. Lett. 23, 1238 (1969).

²V. B. Shikin, Zh. Eksp. Teor. Fiz. 58, 1748 (1970) [Sov. Phys.-JETP 31, 936 (1970)].

³A. Sommerfeld and H. Bethe, Electron Theory of Metals, in Handbuch der Physik, Vol. XXIV/2, Springer-Verlag, 1934. (Russ. Transl., Gostekhizdat, 1938, p. 236).

⁴L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. 31, 536 (1956) [Sov. Phys.-JETP 4, 439 (1957)].

⁵L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1953, p. 291 [Addison-Wesley, 1959].

⁶J. L. Levine and T. M. Sanders, Jr., Phys. Rev. 154, 138 (1967).

⁷N. F. Mott and H. S. W. Massey, The Theory of Atomic Collisions (p. 572), Oxford, 1965 (Russ. Transl. Mir, 1969, p. 515).

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78