LONGITUDINAL OSCILLATIONS OF RELATIVISTIC ELECTRON-ION PLASMA AT ZERO TEMPERATURE

V. V. KOSACHEV and B. A. TRUBNIKOV

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An explicit expression is obtained for the longitudinal dielectric constant $\epsilon(\mathbf{k}, \omega)$ of a fully degenerate relativistic electron gas, and the low-frequency ion-acoustic and high-frequency electronic plasma oscillations are considered in detail. It is shown that allowance for the relativism influences only the determination of the main parameters of the plasma, without changing essentially the character of the oscillation spectra.

1. We have obtained an explicit expression for the longitudinal dielectric constant $\epsilon(\mathbf{k}, \omega)$ of an electron gas with zero temperature, with allowance taken for the quantum and relativistic effects, and have considered oscillations of a relativistic degenerate electron-ion plasma, which are determined, as is well known, by the equation

$$\varepsilon(\mathbf{k}, \omega) = 0.$$

We recall that in the classical case ϵ is determined by an integral (see, e.g.,[1])

$$\varepsilon(\mathbf{k}, \omega) = 1 - \frac{4\pi e^2}{k^2} \int \frac{d\mathbf{p}}{\mathbf{kv} - \omega} \left(\mathbf{k} \frac{\partial f(\mathbf{p})}{\partial \mathbf{p}} \right) \tag{1}$$

and if we substitute here the Fermi distribution for fully degenerate electrons ($f(p) = 2/(2\pi h)^3$ for $p < p_0$ $= \hbar(3\pi^2 n)^{1/3}$—the Fermi momentum), then we obtain a quasiclassical approximation suitable for $k \ll k_0 = p_0/\hbar$:

$$\varepsilon(k, \omega) = 1 + \frac{e^2}{\pi\hbar v_0} \left(\frac{2k_0}{k} \right)^2 \left(1 - \frac{1}{2N\beta_0} \ln \frac{1 + N\beta_0}{1 - N\beta_0} \right), \tag{2}$$

where $N = kc/\omega$ is the refractive index and $\beta_0 = v_0/c$, so that $N\beta_0 = kv_0/\omega = v_0/v_{\text{phase}}$. It will be shown later that expression (2) is valid for $k \ll k_0$ also in the relativistic case, when the velocity on the Fermi boundary is equal to $v_0 = p_0/m\gamma_0$, where $\gamma_0 = [1 + (p_0/mc)^2]^{1/2}$ $= (1 - \beta_0^2)^{-1/2}$. In the quantum nonrelativistic case we have (see[1,2])

$$\varepsilon(\mathbf{k}, \omega) = 1 - \frac{4\pi e^2}{\hbar k^2} \int \frac{d\mathbf{p}}{\mathbf{kv} - \omega} \left[f\left(\mathbf{p} + \frac{\hbar \mathbf{k}}{2}\right) - f\left(\mathbf{p} - \frac{\hbar \mathbf{k}}{2}\right) \right]$$
$$= 1 - \frac{4\pi e^2}{m} \int \frac{f\, d\mathbf{p}}{(\mathbf{kv} - \omega)^2 - (\hbar k^2/2m)^2}, \tag{3}$$

and for $T = 0$ we can obtain from this

$$\varepsilon(k, \omega) = 1 + \frac{e^2}{2\pi\hbar v_0} \left(\frac{2k_0}{k} \right)^2 \left\{ 1 + \frac{k_0}{2k} \left[(1 - \sigma_+^2) \ln \frac{\sigma_+ + 1}{\sigma_+ - 1} + (1 - \sigma_-^2) \ln \frac{\sigma_- + 1}{\sigma_- - 1} \right] \right\}, \tag{4}$$

where $\sigma_\pm = (k/2k_0) \pm (\omega/kv_0)$.

The most general expression for $\epsilon(\mathbf{k}, \omega)$, with account taken of all the quantum and relativistic corrections, including the effects of recoil, the possibility of production of electron-positron pairs, and the contribution of the positrons, was obtained by Tsytovich[3] and presented by him in the form (in the formula (5) we assume for brevity the units $\hbar = 1$ and $c = 1$)

$$\varepsilon(\mathbf{k}, \omega) = 1 - \frac{4\pi e^2}{\omega^2} \int f\, d\mathbf{p} \left[\frac{\varepsilon_\mathbf{p} - \varepsilon_{\mathbf{p}-\mathbf{k}}}{(\varepsilon_\mathbf{p} - \varepsilon_{\mathbf{p}-\mathbf{k}})^2 - \omega^2} \Lambda_- + \frac{\varepsilon_\mathbf{p} + \varepsilon_{\mathbf{p}-\mathbf{k}}}{(\varepsilon_\mathbf{p} + \varepsilon_{\mathbf{p}-\mathbf{k}})^2 - \omega^2} \Lambda_+ \right] + \delta\varepsilon, \tag{5}$$

where $f = (n_\mathbf{p}^- + n_\mathbf{p}^+)/4\pi^3$ ($n_\mp$—Fermi distributions of the electrons and positrons),

$$\text{for } \varepsilon_\mathbf{p} = \sqrt{m^2 + \mathbf{p}^2}, \quad \Lambda_\pm = 1 \pm \frac{\varepsilon_\mathbf{p}^2 + \mathbf{kp} - 2(\mathbf{kp})^2 k^{-2}}{\varepsilon_\mathbf{p} \varepsilon_{\mathbf{p}-\mathbf{k}}},$$

and the quantity $\delta\epsilon$ takes into account the polarization of the vacuum.

In the present paper, the integral in (5) is calculated for the case $T = 0$, when there are no free positrons ($n_\mathbf{p}^+ = 0$), and the electrons are fully degenerate ($n_\mathbf{p}^- = 1$). Individual stages of the calculations are briefly indicated in Appendices I and II. The result can be represented in the form (for region I of Fig. 1)

$$\varepsilon^\text{I} = 1 + \frac{e^2/\hbar c}{4\pi} \left\{ -\frac{4}{3} \ln \frac{1 + \beta_0}{1 - \beta_0} + \frac{8}{3} \frac{\gamma_0 \rho_0}{x^2} + \frac{\gamma_0 a}{x^3} \ln \frac{(\rho_0 - \xi_+)(\rho_0 - \xi_-)}{(\rho_0 + \xi_+)(\rho_0 + \xi_-)} + \frac{yb}{x^3} \ln \frac{(\mathscr{B}_+ + \beta_0)(\mathscr{B}_- - \beta_0)}{(\mathscr{B}_+ - \beta_0)(\mathscr{B}_- + \beta_0)} + \frac{c\sqrt{\varkappa}}{y^3} \ln \frac{(\mathscr{B}_+ + \beta_0)(\mathscr{B}_- + \beta_0)}{(\mathscr{B}_+ - \beta_0)(\mathscr{B}_- - \beta_0)} \right\}. \tag{6}$$

We have introduced here the notation

$$x = \hbar k / 2mc, \qquad y = \hbar\omega / 2mc^2,$$

$$\varkappa = y^2 + \frac{y^2}{x^2 - y^2} \qquad \xi_\pm = x \pm \sqrt{\varkappa}, \quad \mathscr{B}_\pm = \frac{\xi_\pm}{\sqrt{1 + \xi_\pm^2}}$$

and the coefficients

$$a = x^2 - y^2 - {}^1\!/_3\gamma_0^2, \quad b = x^2 - \gamma_0^2 - {}^1\!/_3 y^2, \quad c = y^2 - {}^1\!/_3\varkappa \tag{7}$$

From formula (6) in the limit as $\hbar \to 0$ we can obtain the quasiclassical case (2) indicated above, and in the limit as $c \to \infty$ we can obtain the nonrelativistic case (4).

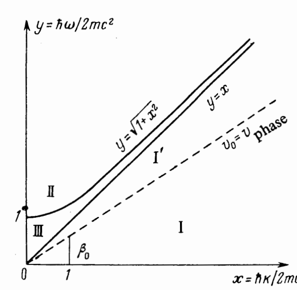

FIG. 1. The x, y plane: region I) $\kappa > 0$, $N > 1$, $v_{\text{phase}} < v_0$; region I′) $\kappa > 0$, $N > 1$, $v_{\text{phase}} > v_0$; region II) $\kappa > 0$, $N < 1$, $v_{\text{phase}} > c$; region III) $\kappa < 0$, $N < 1$.

2. Let us consider longitudinal oscillations in the described system. Since it should contain ions that compensate for the charge of the electrons, the equation for the oscillations is of the form

$$\varepsilon(k,\omega)+\Delta\varepsilon_i=0, \tag{8}$$

where $\Delta\epsilon_i = -(\omega_{0i}/\omega)^2$ is the ion increment (the ions are assumed to be non-degenerate, and $\omega_{0i}^2 = 4\pi n_i e_i^2/M_i$ is their plasma frequency). Equation (8) has, as is well known, two roots, one of which corresponds to low-frequency ion-acoustic oscillations and the second to high-frequency electronic oscillations.

For low frequencies, using formula (6) and expanding $\epsilon(\mathbf{k}, \omega)$ in a series in small y with $x < \rho_0$, we obtain accurate to terms of order y^2

$$\varepsilon(k,\omega)=\varepsilon(k,0)+y\varphi_1(x)+y^2\varphi_2(x)+\dots \tag{9}$$

The first term, corresponding to the penetrability for zero frequency, is equal here to

$$\varepsilon(k,\omega)|_{\omega=0}=1+\frac{e^2/\hbar c}{4\pi}\Big\{-\frac{4}{3}\ln\frac{1+\beta_0}{1-\beta_0}+\frac{8}{3}\frac{\gamma_0\rho_0}{x^2}+\frac{\gamma_0}{x^3}\Big(x^2-\frac{\gamma_0^2}{3}\Big)\ln\Big(\frac{\rho_0-x}{\rho_0+x}\Big)^2+\frac{\sqrt{1+x^2}}{3x^3}(2x^2-1)\ln\Big(\frac{\beta_0\sqrt{1+x^2}+x}{\beta_0\sqrt{1+x^2}-x}\Big)^2\Big\}. \tag{10}$$

At small $x \ll \rho_0$ this expression, as expected, takes the form

$$\varepsilon(k,0)|_{x\ll\rho_0}=1+\frac{e^2}{\pi\hbar c}\frac{\gamma_0\rho_0}{x^2}=1+\frac{1}{k^2d^2}, \tag{11}$$

where $d = (\frac{1}{3}p_0v_0/4\pi ne^2)^{1/2}$ is the radius of the Debye screening of the charges in the plasma.

The term linear in y in formula (9) corresponds to the imaginary part

$$y\varphi_1(x)=i\frac{e^2}{\hbar c}y\frac{(\gamma_0^2-x^2)}{2x^3} \tag{12}$$

and describes the damping of the waves, while the term quadratic in y, which represents a correction to the real part of ϵ, is equal to

$$y^2\varphi_2(x)=\frac{e^2}{4\pi\hbar c}y^2\frac{\gamma_0}{x^4}\Big\{4\rho_0\frac{2x^2-\gamma_0^2}{\rho_0^2-x^2}+x\ln\Big(\frac{\rho_0+x}{\rho_0-x}\Big)^2-\frac{1}{\gamma_0x\sqrt{1+x^2}}\ln\frac{(\rho_0\sqrt{1+x^2}+x\sqrt{1+\rho_0^2})^2}{(\rho_0^2-x^2)}\Big\}. \tag{13}$$

When x approaches the point $x^* = \rho_0$, this correction becomes anomalously large, so that the expansion (9) is not valid near this point. We note also that when x approaches the point $x^* = \rho_0$, the imaginary part decreases, and when $x > \rho_0$ it is entirely absent (see Fig. 2).

Assuming y to be small and neglecting in the expansion (9) all the terms except the zeroth $\epsilon(k, 0)$, we obtain from (8) an approximate law for the dispersion of the ion-acoustic waves $\omega^2 = \omega_{0i}^2/\epsilon(k, 0)$. At small k ($x \ll \rho_0$) it takes the form $\omega = c_s k$, where $c_s = \omega_{0i} d$ is the speed of sound for long waves.

Let us examine in somewhat greater detail expression (10) for $\epsilon(k, 0)$, writing it in the form (see (11))

$$\varepsilon(k,0)=1+\alpha(k)\,/\,k^2d^2. \tag{14}$$

The difference between the coefficient $\alpha(k)$ and unity characterizes the distortion of the Debye screening and,

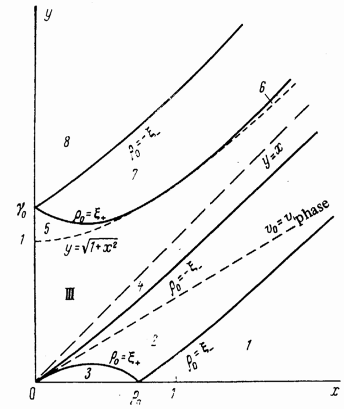

FIG. 2. Family of hyperbolas $\rho_0 = \pm\xi_\pm$. For regions 3 and 5 in the integrals (I.11)–(I.13) we have a pole $\rho_\parallel = \xi_\pm < \rho_0$, and for regions 2, 3, 5, and 7 we have a pole $|\rho_\parallel{}'| = |\xi_-| < \rho_0$ and in these regions Im $\epsilon \neq 0$.

by definition, we obtain for $\alpha(k)$ from formulas (10) and (11)

$$\alpha(k)=\frac{2}{3}+\frac{\frac{1}{3}\gamma_0^2-x^2}{4\rho_0x}\ln\Big(\frac{\rho_0+x}{\rho_0-x}\Big)^2+\frac{\sqrt{1+x^2}}{12\gamma_0\rho_0x}(2x^2-1)\ln\Big(\frac{\beta_0\sqrt{1+x^2}+x}{\beta_0\sqrt{1+x^2}-x}\Big)^2-\frac{x^2}{3\gamma_0\rho_0}\ln\frac{1+\beta_0}{1-\beta_0}. \tag{15}$$

In the nonrelativistic approximation ($c \to \infty$) this expression takes the form ($\sigma = 1/s = x/\rho_0 = k/2k_0$)

$$\alpha_{n.r}=\frac{1}{2}+\frac{(1-\sigma^2)}{4\sigma}\ln\frac{1+\sigma}{1-\sigma}=\frac{1}{2}+\frac{s-1}{4s}(s+1)\ln(s+1)-\frac{(s+1)}{4s}(s-1)\ln(s-1) \tag{16}$$

and can be obtained also from formula (4). In the ultrarelativistic limit ($\rho_0 \to \infty$), on the other hand, we obtain from the general formula (15)

$$\alpha_{n.y.}=\frac{2}{3}\Big[1+\frac{1+2\sigma}{4\sigma}(1-\sigma)^2\ln\frac{1+\sigma}{1-\sigma}+\sigma^2\ln\frac{\sigma}{1+\sigma}\Big]=\frac{2}{3}\Big[1+\frac{s-2}{4s^2}(s+1)^2\ln(s+1)-\frac{s+2}{4s^2}(s-1)^2\ln(s-1)\Big], \tag{17}$$

where, as before, $\sigma = 1/s = x/\rho_0 = k/2k_0$. Expressions (16) and (17) have no singularities at the point $x = \rho_0$ and differ little from each other (see Fig. 3), so that practically for all k the dispersion of the sound is given by the law

$$\omega_{ac}=\omega_{0i}\frac{kd}{\sqrt{1+k^2d^2}}\approx\omega_{0i}\quad(\text{for } kd\gg1). \tag{18}$$

The damping decrement defined by the imaginary part (12) is equal to

$$\gamma(k)=\operatorname{Im}\omega=\frac{\pi}{12}\frac{k\rho_0}{M_i/Z_i}\frac{1-(x/\gamma_0)^2}{(1+k^2d^2)^2}, \tag{19}$$

where M_i is the mass and $Z_i = e_i/e$ is the charge of the ions. When $x > \rho_0$, there is no damping.

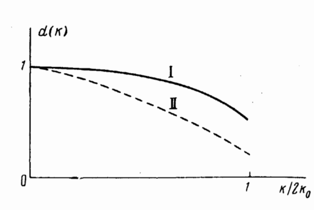

FIG. 3. The function $\alpha(k)$ (see formula (14)) which characterizes the quantum distortion of the Debye screening for nonrelativistic (I) and ultrarelativistic (II) cases.

3. For the high-frequency electronic oscillations, the ion increment to Eq. (8) is negligible, and the root of the equation $\epsilon(k, \omega) = 0$ fall in region I if $x > y$ and in region III if $x < y$. In the latter region, the expression for ϵ takes the form

$$\varepsilon = 1 + \frac{e^2/\hbar c}{4\pi}\left\{-\frac{4}{3}\ln\frac{1+\beta_0}{1-\beta_0} + \frac{8}{3}\frac{\gamma_0\rho_0}{x^2} + \frac{\gamma_0 a}{x^3}\lambda_1 + \frac{yb}{x^3}\lambda_3 + \frac{c\sqrt{\varkappa}}{y^3}\lambda_2\right\}, \tag{20}$$

where $\lambda_{1,2,3}$ are determined by formulas (I, 23) (see the Appendix I). Expanding in terms of small k, we get

$$\varepsilon(k, \omega) = \varepsilon(0, \omega) + x^2\psi_2(y) + \ldots. \tag{21}$$

Here

$$\varepsilon(0, \omega) = 1 + \frac{e^2/\hbar c}{4\pi}\left\{-\frac{4}{3}\ln\frac{1+\beta_0}{1-\beta_0} - \frac{4}{3}\frac{\gamma_0\rho_0}{y^2} - \frac{2}{3}\frac{\sqrt{1-y^2}}{y^3}(2y^2+1)\,\mathrm{arctg}\,\frac{2\beta_0 y\sqrt{1-y^2}}{y^2(1+\beta_0^2)-1}\right\}. \tag{22}$$

For small $y \ll 1$ we have

$$\varepsilon(0, \omega) = 1 - \frac{e^2/\hbar c}{3\pi}\frac{\rho_0^2\beta_0}{y^2} = 1 - \frac{\omega_0^2}{\omega^2}, \tag{23}$$

where $\omega_0^2 = 4\pi n e^2/m\gamma_0$ is the square of the plasma frequency with allowance for the relativism. In the nonrelativistic case $\gamma_0 = 1$. The correction term in formula (21) is equal to

$$x^2\psi_2(y) = x^2\frac{e^2/\hbar c}{4\pi}\left\{\frac{4\rho_0\gamma_0}{(\gamma_0^2-y^2)^2}\left[\frac{\gamma_0^2+{}^1/_2}{3y^2} - \frac{\gamma_0^2+2\gamma_0^4+2}{10y^4}\right] - \frac{1}{y^5\sqrt{1-y^2}}\,\mathrm{arctg}\,\frac{2\beta_0 y\sqrt{1-y^2}}{y^2(1+\beta_0^2)-1}\right\}. \tag{24}$$

When $y \ll 1$, this term is equal to

$$x^2\psi_2(y)\,|_{y\ll 1} = -x^2\frac{e^2/\hbar c}{5\pi}\frac{\rho_0^2\beta_0^3}{y^4} = -\frac{3}{5}\frac{\omega_0^2}{\omega^2}\left(\frac{kv_0}{\omega}\right)^2. \tag{25}$$

Using formulas (23)–(25) and solving the equation $\epsilon(k, \omega) = 0$, we obtain the dispersion law for the plasma waves in the case $y \ll 1$ with allowance for relativism

$$\omega^2 \cong \omega_0^2(1 + {}^9/_5 k^2 d^2 + \ldots), \tag{26}$$

where $d^2 = (1/3)p_0 v_0/4\pi n e^2$ is the square of the relativistic Debye radius (see (11)). Formula (26) coincides with the corresponding nonrelativistic formula[1], so that allowance for the relativism influences only the determination of the quantities ω_0 and d. We note also that there is no imaginary part of ϵ in region III, and therefore the plasma oscillations are not damped.

Formula (26) is valid when $y \approx \hbar\omega_0/2mc^2 \ll 1$ or in other words $\rho_0^3/\gamma_0 \ll 3\pi\hbar c/e^2$, which is satisfied approximately up to densities $n \lesssim 10^{34}\ \mathrm{cm}^{-3}$, including also the ultrarelativistic case ($p = mc$ at $n = 5.9 \times 10^{29}\ \mathrm{cm}^{-3}$). At y close to unity it is necessary to use the exact formula (22). From it we find, in particular, that $y = 1$, i.e., $\omega = 2mc^2/\hbar$ satisfies the equation for plasma oscillations $\epsilon(\omega, 0) = 0$ at a density $n_e = 2.7 \times 10^{34}\ \mathrm{cm}^{-3}$. At still higher densities, the branch of the longitudinal plasma oscillations falls in region II.

4. In region II expression (20) for ϵ remains in force, and λ_3 can be determined from formula (I.23), whereas λ_2 can be represented for region II in the form

$$\lambda_2 = 2\,\mathrm{Arth}\left[\frac{2\beta_0 y(1-N^2)\sqrt{\varkappa}}{y^2(\beta_0^2-N^2)+\varkappa(1-N^2\beta_0^2)}\right]. \tag{27}$$

At small $x \ll 1$, expanding $\epsilon(k, \omega)$ in powers of x, we obtain

$$\varepsilon^{\mathrm{II}}(k, \omega) = \varepsilon(0, \omega) + x^2\psi_2^{\mathrm{II}}(y) + \ldots + i\,\mathrm{Im}\,\varepsilon. \tag{28}$$

Here

$$\varepsilon(0, \omega) = 1 + \frac{e^2/\hbar c}{4\pi}\left\{-\frac{4}{3}\ln\frac{1+\beta_0}{1-\beta_0} - \frac{4}{3}\frac{\gamma_0\rho_0}{y^2} + \frac{2}{3}\frac{\sqrt{y^2-1}}{y^3}(2y^2+1)\,\mathrm{Arth}\,\frac{2\beta_0 y\sqrt{y^2-1}}{y^2(1+\beta_0^2)-1}\right\}, \tag{29}$$

which is an analytic continuation of formula (22) into the region $y > 1$.

Assuming here that $y \sim 1$ and $\gamma_0 \sim \rho_0 \gg 1$, we can discard all the terms in the curly brackets except the second, and then we have for the plasma waves

$$\varepsilon(0, \omega) \cong 1 - \frac{e^2/\hbar c}{3\pi}\frac{\gamma_0\rho_0}{y^2} = 1 - \frac{\omega_0^2}{\omega^2}, \tag{30}$$

which practically coincides with (23). The correction factor $x^2\psi_2^{\mathrm{II}}(y)$ in (28) is obtained from (24) by making the substitution $\tan^{-1} \to \tanh^{-1}$. The imaginary part of ϵ in (28) is described by formula (II.9) and is negative, which should lead to a buildup of the oscillations. This unnatural result is obtained because in our calculations we did not take into account the vacuum polarization. Tsytovich[3] calculated the vacuum correction $\delta\epsilon_{\mathrm{vac}}$ for the imaginary part (we note incidentally that in formula (27) of[3] the factor 2 has been left out in error) and has shown that it canceled out completely the imaginary part of (II.9), thereby eliminating the indicated paradox. Thus, in region 5 the plasma oscillations of a fully degenerate electron gas are undamped.

5. In conclusion, it can be stated that the allowance for relativism made in this article influences only the determination of the main parameters of the plasma—the Debye radius d and the plasma frequency ω_0, and does not change significantly the character of the spectra of both the ion-acoustic and plasma electron oscillations (see, in particular, Fig. 3 and formula (26)).

APPENDIX I

Calculation of the Integral (5) for $\epsilon(k, \omega)$

Since the calculations are quite cumbersome it is apparently useful to indicate briefly the individual stages. If we introduce the momenta $p_\parallel$ and $p_\perp$, which are longitudinal and transverse relative to the wave vector **k**, and introduce the dimensionless notation

$$x = \frac{\hbar k}{2mc}, \quad y = \frac{\hbar\omega}{2mc^2}, \quad N = \frac{kc}{\omega} = \frac{x}{y}, \quad \rho_0 = \frac{p_0}{mc} \quad \gamma_0 = \sqrt{1+\rho_0^2}, \tag{I.1}$$

then, introducing also the variables

$$\boldsymbol{\rho} = \frac{\mathbf{p}}{mc}, \quad \rho_{\parallel,\perp} = \frac{p_{\parallel,\perp}}{mc}, \quad \gamma = \sqrt{1+\rho_\parallel^2+\rho_\perp^2}, \quad \gamma_\parallel = \sqrt{1+\rho_\parallel^2}, \tag{I.2}$$

we can, after reducing the expression in the square brackets to a common denominator, represent the integral (5) in the form

$$\varepsilon = 1 - \frac{e^2/\hbar c}{4\pi y^2}\left\{\frac{1}{\pi}\int_0^{\rho_0}\frac{d\rho}{\gamma} + \frac{1}{y}\int_{-\rho_0}^{+\rho_0} d\rho_\parallel[(\rho_\parallel - x)^2 - y^2][\lambda_+(\rho_\parallel) - \lambda_-(\rho_\parallel)]\right\}. \tag{I.3}$$

Here $\lambda_\pm(\rho_\parallel)$ are logarithms defined by the integrals with respect to the variable momenta:

$$\lambda_\pm(\rho_\parallel) = \int_{\rho_\perp=0}^{\sqrt{\rho_0^2-\rho_\parallel^2}} \frac{\rho_\perp d\rho_\perp}{\gamma(\gamma \pm S)} = \int_{t_1=\gamma_\parallel}^{t_2=\gamma_0}\frac{dt}{t \pm S} = \ln\frac{S\pm\gamma_0}{S\pm\gamma_\parallel}, \tag{I.4}$$

where $S = N\rho_\parallel + y(1 - N^2)$. Thus, we first integrated with respect to $p_\perp$. We note that the logarithms $\lambda(\rho_\parallel)$ vanish at $\rho_\parallel = \pm\rho_0$, and therefore it is necessary to satisfy for them the identities

$$\int_{-\rho_0}^{+\rho_0}\left(\frac{\partial}{\partial\rho_\parallel}\lambda_\pm\right)d\rho_\parallel = 0.$$

It is convenient to represent the derivatives in the form

$$\frac{\partial}{\partial\rho_\parallel}\lambda_\pm = \frac{N}{S\pm\gamma_0} - A \pm (yB + N\varkappa C), \tag{I.5}$$

where $\kappa = y^2 + 1/(N^2 - 1)$, and the quantities A, B, and C, which depend on the integration variable $\rho_\parallel$, are equal to

$$A = \frac{(\rho_\parallel - x)\gamma_\parallel}{\gamma_\parallel[(\rho_\parallel - x)^2 - \varkappa]}, \quad B = \frac{(\rho_\parallel - x)}{\gamma_\parallel[(\rho_\parallel - x)^2 - \varkappa]},$$
$$C = \frac{1}{\gamma_\parallel[(\rho_\parallel - x)^2 - \varkappa]}, \tag{I.6}$$

and thus the indicated identities can be written in the form

$$\int_{-\rho_0}^{+\rho_0}\frac{d\rho_\parallel}{S\pm\gamma_0} = \frac{1}{N}\int_{-\rho_0}^{+\rho_0} d\rho_\parallel[A \mp (yB + N\varkappa C)]. \tag{I.7}$$

Calculating now the last integral in (I.3) by parts, we get

$$\varepsilon = 1 + \frac{e^2/\hbar c}{4\pi y^2}\left\{-\frac{1}{\pi}\int_0^{\rho_0}\frac{d\rho}{\gamma} + \frac{1}{y}\int_{-\rho_0}^{+\rho_0} d\rho_\parallel \mu \frac{\partial}{\partial\rho_\parallel}(\lambda_+ - \lambda_-)\right\}, \tag{I.8}$$

where $\mu(\rho_\parallel) = (1/3)(\rho_\parallel - x)^3 - y^2(\rho_\parallel - x)$, and taking into account the identities (I.7), it is easy to verify that the integrand here can be replaced by the equivalent form (we leave out terms that are certainly odd in $\rho_\parallel$):

$$\mu\frac{\partial}{\partial\rho_\parallel}(\lambda_+ - \lambda_-) \to \frac{2\gamma_0}{3N}\left(\frac{2y^2}{x} + x\right) + \frac{2y}{3}\gamma_\parallel - \frac{4y}{3\gamma_\parallel}(1 + y^2) \tag{I.9}$$
$$+ \frac{2\gamma_0}{N^3}aA - \frac{2}{N^3}(y^2 b + \varkappa N^4 c)B - 2y\varkappa\left(\frac{b}{N^2} + c\right)C.$$

For brevity we have introduced here the following coefficients that do not depend on $\rho_\parallel$:

$$a = x^2 - y^2 - \frac{\gamma_0^2}{3}, \quad b = x^2 - \gamma_0^2 - \frac{y^2}{3}, \quad c = y^2 - \frac{\varkappa}{3}. \tag{I.10}$$

In addition to trivial terms, the answer contains thus three integrals of the expressions A, B, and C.

The appearance of the imaginary part of the dielectric constant is connected with the vanishing of the denominator $[(\rho_\parallel - x)^2 - \kappa]$ in these integrals. When $\kappa < 0$, there is certainly no imaginary part, and when $\kappa > 0$ the imaginary part can occur following half a circuit around the indicated pole.

Let us calculate the integrals with A, B, and C, assuming for concreteness that $\kappa > 0$. For the integral with A we have (see (I.6))

$$\int_{-\rho_0}^{+\rho_0} A\, d\rho_\parallel = \int_{-\rho_0}^{+\rho_0}\frac{(\rho_\parallel - x)d\rho_\parallel}{(\rho_\parallel - \xi_+)(\rho_\parallel - \xi_-)} = \frac{1}{2}\ln\frac{(\rho_0 - x)^2 - \varkappa}{(\rho_0 + x)^2 - \varkappa}$$
$$= \frac{1}{2}\ln\frac{(\rho_0 - \xi_+)(\rho_0 - \xi_-)}{(\rho_0 + \xi_+)(\rho_0 + \xi_-)}. \tag{I.11}$$

We have introduced here the convenient notation $\xi_\pm = x \pm \sqrt{\kappa}$. For the two other integrals we obtain (see (I.6))

$$\int_{-\rho_0}^{+\rho_0} B\, d\rho_\parallel = \frac{1}{2}\int_{-\rho_0}^{+\rho_0}\frac{d\rho_\parallel}{\sqrt{1+\rho_\parallel^2}}\left(\frac{1}{\rho_\parallel - \xi_+} + \frac{1}{\rho_\parallel - \xi_-}\right) = \frac{1}{2}[I(\xi_+) - I(\xi_-)], \tag{I.12}$$

$$\int_{-\rho_0}^{+\rho_0} C\, d\rho_\parallel = \frac{1}{2\sqrt{\varkappa}}\int_{-\rho_0}^{+\rho_0}\frac{d\rho_\parallel}{\sqrt{1+\rho_\parallel^2}}\left(\frac{1}{\rho_\parallel - \xi_+} - \frac{1}{\rho_\parallel - \xi_-}\right) = \frac{1}{2\sqrt{\varkappa}}[I(\xi_+) - I(\xi_-)]. \tag{I.13}$$

Here the function $I(\xi)$ is equal to (we make the substitution $\rho_\parallel = (1/2)(z - 1/z)$)

$$I(\xi) = \int_{-\rho_0}^{+\rho_0}\frac{d\rho_\parallel}{\sqrt{1+\rho_\parallel^2}}\frac{1}{(\rho_\parallel - \xi)} = 2\int_{\gamma_0-\rho_0}^{\gamma_0+\rho_0}\frac{dz}{(z-\xi)^2 - (1+\xi^2)}$$
$$= -\frac{1}{\sqrt{1+\xi^2}}\ln\frac{\mathscr{B}(\xi) + \beta_0}{\mathscr{B}(\xi) - \beta_0}, \tag{I.14}$$

where we put $\mathscr{B}(\xi) = \xi/\sqrt{1+\xi^2}$. Substituting (I.9) into the integral (I.8) with allowance for the formulas (I.11)–(I.13), we obtain for ϵ

$$\varepsilon = 1 + \frac{e^2/\hbar c}{4\pi}\left\{-\frac{4}{3}\ln\frac{1+\beta_0}{1-\beta_0} + \frac{8\gamma_0\rho_0}{3x^2} + \frac{\gamma_0}{x^3}a\ln\frac{(\rho_0 - x)^2 - \varkappa}{(\rho_0 + x)^2 - \varkappa}\right.$$
$$\left. - \left(b + cN^3\frac{\sqrt{\varkappa}}{y}\right)\frac{(y^2 + x\sqrt{\varkappa})}{x^3}I(\xi_+) - \left(b - cN^3\frac{\sqrt{\varkappa}}{y}\right)\frac{(y^2 - x\sqrt{\varkappa})}{x^3}I(\xi_-)\right\}. \tag{I.15}$$

Further simplifications and an analysis of these formulas is best carried out separately in different regions of the (x, y) plane (see Fig. 1). If, given κ and x, we solve the equation

$$\varkappa = y^2 + y^2/(x^2 - y^2) \tag{I.16}$$

relative to y, then we obtain respectively for regions I, II, and III

$$y_{\rm I} = {}^1\!/_2\left(\sqrt{1+\xi_+^2} - \sqrt{1+\xi_-^2}\right), \quad y_{\rm II,\,III} = {}^1\!/_2\left(\sqrt{1+\xi_+^2} + \sqrt{1+\xi_-^2}\right), \tag{I.17}$$

and therefore we get for the coefficients preceding the functions $I(\xi_\pm)$ in (I.15)

$$(y^2 \pm x\sqrt{\varkappa})_{\rm I} = \pm y\sqrt{1+\xi_\pm^2}, \quad (y^2 \pm x\sqrt{\varkappa})_{\rm II,\,III} = y\sqrt{1+\xi_\pm^2}. \tag{I.18}$$

Taking these relations into account, we can write $\epsilon(k, \omega)$ in region I in the form (here $\mathscr{B}_\pm = \mathscr{B}(\xi_\pm)$)

$$\varepsilon^{(\rm I)} = 1 + \frac{e^2/\hbar c}{4\pi}\left\{-\frac{4}{3}\ln\frac{1+\beta_0}{1-\beta_0} + \frac{8}{3}\frac{\gamma_0\rho_0}{x^2}\right.$$
$$+ \frac{\gamma_0 a}{x^3}\ln\frac{(\rho_0 - \xi_+)(\rho_0 - \xi_-)}{(\rho_0 + \xi_+)(\rho_0 + \xi_-)} + \frac{yb}{x^3}\ln\frac{(\mathscr{B}_+ + \beta_0)(\mathscr{B}_- - \beta_0)}{(\mathscr{B}_+ - \beta_0)(\mathscr{B}_- + \beta_0)}$$
$$\left. + \frac{c\sqrt{\varkappa}}{y^3}\ln\frac{(\mathscr{B}_+ + \beta_0)(\mathscr{B}_- + \beta_0)}{(\mathscr{B}_+ - \beta_0)(\mathscr{B}_- - \beta_0)}\right\}. \tag{I.19}$$

For regions II and III it is necessary to interchange the coefficients in front of the last two logarithms. It is very convenient to use the parametric representation of $\epsilon(k, \omega)$ with the aid of hyperbolic functions. It is necessary to use here a different notation for the arguments in different regions of the xy plane.

In region I we have $N = x/y > 1$ and therefore it is convenient to put ($s \geq 0$, $t \geq 0$, $f \geq 0$)

$$x = \mathrm{ch}\, t\, \mathrm{sh}\, s,\quad y = \mathrm{sh}\, t\, \mathrm{sh}\, s,\quad N = x/y = \mathrm{cth}\, t > 1,\quad \rho_0 = \mathrm{sh}\, f. \tag{I.20}$$

We then get

$$\sqrt{\varkappa} = \mathrm{sh}\, t\, \mathrm{ch}\, s,\quad \xi_\pm = x \pm \sqrt{\varkappa} = \mathrm{sh}\,(s \pm t),\quad \beta_0 = \mathrm{th}\, f,\quad \mathscr{B}_\pm = \mathrm{th}\,(s \pm t), \tag{I.21}$$

so that formula (I.19) for ϵ can be written in this region in the form

$$\varepsilon = 1 + \frac{e^2/\hbar c}{4\pi}\left\{-\frac{8}{3}f + \frac{4}{3}\frac{\mathrm{sh}\, 2f}{x^2} + \frac{\gamma_0 a}{x^3}\lambda_1 + \frac{yb}{x^3}\lambda_2 + \frac{c\sqrt{\varkappa}}{y^3}\lambda_3\right\}, \tag{I.22}$$

where $\lambda_{1,2,3}$ are logarithms with respect to the values

$$\lambda_1 = \ln\left[\mathrm{th}\frac{f-s-t}{2}\,\mathrm{th}\frac{f-s+t}{2} \Big/ \mathrm{th}\frac{f+s+t}{2}\,\mathrm{th}\frac{f+s-t}{2}\right],$$
$$\lambda_2 = \ln\frac{\mathrm{sh}(f-s+t)\,\mathrm{sh}(f+s+t)}{\mathrm{sh}(f+s-t)\,\mathrm{sh}(f-s-t)},$$
$$\lambda_3 = \ln\frac{\mathrm{sh}(f+s+t)\,\mathrm{sh}(f+s-t)}{\mathrm{sh}(f-s-t)\,\mathrm{sh}(f-s+t)}. \tag{I.23}$$

We see here quite clearly that the four roots $f = \pm s \pm t$ are the same for all three logarithms.

For regions II and III formula (I.22) retains its form, but it is necessary in it to interchange the coefficients yb/x^3 and $c\sqrt{\kappa}/y^3$ in front of the last two logarithms. In addition, the connection between x, y and the parameters s and t changes. Namely, in region II, where $y > 1$, it is necessary to assume

$$x = \mathrm{sh}\, s\, \mathrm{ch}\, t,\quad y = \mathrm{ch}\, s\, \mathrm{ch}\, t, \tag{I.24}$$

and for region III it is necessary to make the substitution $t \to i\tau$ and to put

$$x = \mathrm{sh}\, s \cos\tau,\quad y = \mathrm{ch}\, s \cos\tau. \tag{I.25}$$

APPENDIX II

Imaginary Part of $\epsilon(k, \omega)$

Let us determine the imaginary part of $\epsilon(k, \omega)$, which arises when the poles $\rho_{\parallel} = \xi_\pm$ of the integrals (I.11)–(I.13) fall in the integration interval $-\rho_0 < \rho_{\parallel} < +\rho_0$. Confining ourselves to the main quadrant $x > 0$ and $y > 0$ of the (x, y) plane, we find the regions of values of x and y for which $0 < \xi_+ < \rho_0$ and $-\rho_0 < \xi_- < \rho_0$. When $\kappa < 0$, i.e., in the region III, there is no imaginary part, and for regions I or II, where $\kappa > 0$, we have the following situation: In the region I, where $y = \frac{1}{2}(\sqrt{1+\xi_+^2} - \sqrt{1+\xi_-^2})$ (see (I.17)), the three curves $\rho_0 = \xi_+$ and $\rho_0 = \pm\xi_-$ correspond to the three hyperbolas

$$y^{(I)}_{\rho_0=\xi_+} = \tfrac{1}{2}\left(\gamma_0 - \sqrt{1+(2x-\rho_0)^2}\right),\qquad y^{(I)}_{\rho_0=\xi_-} = \tfrac{1}{2}\left(\sqrt{1+(2x-\rho_0)^2} - \gamma_0\right),$$
$$y^{(I)}_{\rho_0=-\xi_-} = \tfrac{1}{2}\left(\sqrt{1+(2x+\rho_0)^2} - \gamma_0\right), \tag{II.1}$$

shown in Fig. 2. In region II, where $y = \frac{1}{2}(\sqrt{1+\xi_+^2} + \sqrt{1+\xi_-^2})$, we obtain correspondingly the hyperbolas

$$y^{(II)}_{\rho_0=\xi_+} = \tfrac{1}{2}\left(\gamma_0 + \sqrt{1+(2x-\rho_0)^2}\right),\qquad y^{(II)}_{\rho_0=-\xi_-} = \tfrac{1}{2}\left(\sqrt{1+(2x+\rho_0)^2} + \gamma_0\right);$$
$$y^{(II)}_{\rho_0=\xi_-} = \tfrac{1}{2}\left(\sqrt{1+(2x-\rho_0)^2} + \gamma_0\right). \tag{II.2}$$

We note here that the same branch of the hyperbola $y = \frac{1}{2}(\gamma + \sqrt{1+(2x-\rho)^2})$ is first the curve $\rho_0 = \xi_+$ from $x = 0$ to $x = \rho_0$, but for $x > \rho_0$ it continues already as the curve $\rho_0 = \xi_-$. All these hyperbolas are described by the canonical equation

$$\frac{(y \pm \gamma_0/2)^2}{1/4} - \frac{(x \pm \rho_0/2)^2}{1/4} = 1. \tag{II.3}$$

Figure 2 shows all these hyperbolas, which together with the boundaries of the region III divide the plane of the main quadrant into eight regions, which are also shown in Fig. 2. The imaginary part of $\epsilon(k, \omega)$ is represented by four different methods. In the Cerenkov case for $N > 1$ we have (the index denotes the region in Fig. 2)

$$\mathrm{Im}\,\varepsilon_2 = \frac{e^2}{4\hbar c}\left(-\frac{\gamma_0 a}{x^3} - \frac{yb}{x^3} + \frac{c\sqrt{\varkappa}}{y^3}\right) = \frac{e^2}{4\hbar c}\left(Q - P_+ - \frac{Q^3 - P_+^3}{3}\right), \tag{II.4}$$

where for brevity we put $Q = \sqrt{\kappa}/y$ and $P_+ = (y + \gamma_0)/x$.

In the nonrelativistic limit ($c \to \infty$) we obtain from th this the known expression

$$\mathrm{Im}\,\varepsilon_2 = \frac{e^2}{\hbar v_0}\frac{(1-\sigma_-^2)}{(k/k_0)^3}, \tag{II.5}$$

where $k_0 = mv_0/\hbar$, $\sigma_- = (k/2k_0) - (\omega/kv_0)$.

In region 3 there are two poles $\rho_{\parallel} = \xi_\pm$, and we obtain there

$$\mathrm{Im}\,\varepsilon_3 = \frac{e^2}{4\hbar c}\left(-2\frac{yb}{x^3}\right) = \frac{e^2}{\hbar c}\,y\,\frac{\gamma_0^2 - x^2 + y^2/3}{2x^3}. \tag{II.6}$$

In the nonrelativistic limit we have

$$\mathrm{Im}\,\varepsilon_3 = \frac{e^2}{\hbar v_0}\frac{2(\omega/kv_0)}{(k/k_0)^2}. \tag{II.7}$$

We consider further region II, where $N < 1$. There we have for the region 7 the pole $\rho_{\parallel} = \xi_-$, and we get

$$\mathrm{Im}\,\varepsilon_7 = \frac{e^2}{4\hbar c}\left(-\frac{\gamma_0 a}{x^3} + \frac{yb}{x^3} - \frac{c\sqrt{\varkappa}}{y^3}\right) = -\frac{e^2}{4\hbar c}\left(Q - P_- - \frac{Q^3 - P_-^3}{3}\right), \tag{II.8}$$

where $Q = (\sqrt{\kappa}/y) < 1$, $P_- = (y - \gamma_0)/x$. In region 5 there are both poles $\rho_{\parallel} = \xi_\pm$, and there we have

$$\mathrm{Im}\,\varepsilon_5 = \frac{e^2}{4\hbar c}\left(-2\frac{c\sqrt{\varkappa}}{y^3}\right) = -\frac{e^2}{\hbar c}\frac{Q}{2}\left(1 - \frac{Q^2}{3}\right). \tag{II.9}$$

Inasmuch as $y > 1$ in region II, the transition to the nonrelativistic limit in formulas (II.8) and (II.9) is impossible. The imaginary parts in this region are due to the fact that the wave can generate electron-positron pairs (see[3]).

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