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A STATISTICAL ANALYSIS OF GRAVITATIONAL INSTABILITY

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Density perturbations are considered as a random function of the coordinates. A solution of the Lifshitz equation is derived for linear perturbations in the range of wave numbers of the order of the Jeans value, for all isotropic cosmological models.

The perturbation spectra are used to obtain mass distributions of the objects formed. The teoretical formulas are compared with the observed statistics of galactic masses. It is pointed out that the Lemaitre model gives good agreement with the experimental data.

In the general theory of relativity the description of the observed universe takes its simplest form if we assume a uniform distribution of matter and an isotropic metric. That such cosmological models must be nonstationary is indicated by the red shift effect.

This effect, however, does not enable us to make a unique choice of one of the possible models. We cannot say with certainty whether the curvature of space is positive or negative, and we cannot assert that there is no cosmological constant in the Einstein equations.

On the other hand, models that assume uniformity have a serious shortcoming; the actual universe is uniform only when averaged on an extremely large scale. The greater part of the matter is grouped in galaxies (with masses 10^{42}–10^{45} g), which in turn occur in clusters of galaxies.

It is natural to assume that the nonuniformity of the universe has been formed as the result of instability. An investigation of gravitational instability in the linear approximation was made by E. Lifshitz[1] for open and closed models of the world. The extension to models with a cosmological constant has been made by the writer.[2]

A basic property of gravitational instability is that the only perturbations that grow are those with wavelength larger than a critical value, the Jeans length (named for James Jeans, who first discovered this fact). With expansion of the universe this boundary shifts, and owing to this the final spectrum of the perturbations falls off as we go to large wave numbers.

In the present paper gravitational instability is studied by the methods of the theory of random functions (Sec. 1). In Sec. 2 new solutions of the Lifshitz equation are found in the range of wave numbers of the order of the Jeans value, which are valid for all isotropic cosmological models. The next step in the study of the inhomogeneities is to try to connect the final spectrum of the perturbations with the mass distribution of the objects that are formed (Sec. 3). The equations obtained for the mass distributions can be used to compare observed and theoretical distributions (Sec. 4). It turns out that the observed distribution of galactic masses is close to the theoretical distribution for the Lematire model, but the difference between the estimate for the speed of sound and the accepted value does not allow us to draw any final conclusion about the cosmological model of the universe.

1. DENSITY PERTURBATIONS AS A RANDOM FUNCTION

Let us introduce a uniform function of the coordinates, ψ, as the fractional perturbation of the density ρ:

$$\psi(\mathbf{r}) = [\rho(\mathbf{r}) - \langle\rho\rangle] / \langle\rho\rangle. \tag{1.1}$$

Pointed brackets denote averages. Obviously $\langle \psi \rangle = 0$. We shall write out the relations that are known from the theory of random functions.[3]

We expand $\psi(\mathbf{r})$ in a Fourier series in a region of size much smaller than the radius of curvature a:

$$\psi(\mathbf{r}) = \sum_{\mathbf{n}} \psi_{\mathbf{n}} \exp\left\{ i\frac{\mathbf{nr}}{a} \right\}. \tag{1.2}$$

Here $\psi_{\mathbf{n}}$ is a random function of n, and $n = |\mathbf{n}|$ runs through integer values in the range $n \gg 1$.[1]

Owing to the uniformity of $\psi(\mathbf{r})$ the correlation function $K(R) = \langle \psi(\mathbf{r})\psi(\mathbf{r} + \mathbf{R})\rangle$ depends only on the absolute value R of the distance, so that we have the following condition on ψ_n:

$$\langle \psi_n \psi_{n'}^* \rangle = S_n \delta(\mathbf{n} - \mathbf{n}'). \tag{1.3}$$

The asterisk denotes the complex conjugate, and $\psi_n^* = \psi_{-n}$ because $\psi(\mathbf{r})$ is real.

The quantity S_n, which because of the uniformity does not depend on the angles, will be called the spectrum of the perturbations. The spectrum and the correlation are connected by a Fourier transformation (since $n \gg 1$ we have changed from summation to integration and carried out the integration over the angles):

$$K(R) = \frac{4\pi a}{R} \int_0^\infty S_n \sin \frac{nR}{a}\, n dn, \tag{1.4}$$

$$S_n = \frac{1}{2\pi^2 n a^2} \int_0^\infty K(R) \sin \frac{nR}{a}\, R dR. \tag{1.5}$$

Another important characteristic of a random function is its distribution law.

In the linear approximation $\psi \ll 1$, and the spectral amplitudes with different n are independent. By the central limit theorem the distribution law of ψ is normal (Gaussian),

$$W(\psi) = \frac{1}{\sqrt{2\pi K(0)}} \exp\left\{-\frac{\psi^2}{2K(0)}\right\}. \tag{1.6}$$

The condition for applicability of the linear approximation is

$$K(0) = \langle \psi^2(\mathbf{r}) \rangle \ll 1. \tag{1.7}$$

Subject to this the spectrum $S_n = \langle |\psi_n|^2 \rangle$ can be obtained from the equations of the linear approximation if the initial spectrum is known.

A random function with a normal distribution law is completely determined by its correlation function. This fact will be of use in Sec. 3.

2. THE FORM OF THE PERTURBATION SPECTRUM

The spectrum which is formed as the result of the growth of initial perturbations can be obtained from an analysis of equations given by E. Lifshitz, which describe gravitational instability in the linear approximation. From the results in[1] and[2] it is not hard to show that at the stage of the expansion of the universe when the pressure p is $p \ll \rho$ (with the speed of light $c = 1$), and consequently the speed of sound $u = (dp/d\rho)^{1/2} \ll 1$, the function ψ_n for short-wavelength perturbations ($n \gg 1$) satisfies the equation

$$\psi'' + \psi' \left(\frac{1}{a} + \frac{P'}{2P}\right) + \left[\frac{(nu)^2}{P} - \frac{1}{aP}\right]\psi = 0. \tag{2.1}$$

Here the prime denotes differentiation with respect to the radius of curvature a, and

$$P(a) = {}^2/_3 a \mp a^2 + {}^1/_3 \Lambda a^4. \tag{2.2}$$

The upper sign is for models with positive curvature, the lower for those with negative curvature. A quantity a_c with the dimensions of length has been set equal to unity. In models with positive curvature a_c is connected with the mass M of the universe: $a_c = \kappa M/4\pi^2$. For open and closed models the cosmological constant $\Lambda = 0$.

We note that for $u = 0$ (equation of state $p = 0$) one gets[2] as the exact solution of Eq. (2.1) the result

$$\psi = \text{const} \frac{\sqrt{P}}{a^2} \int \frac{a^3 da}{P^{3/2}}. \tag{2.3}$$

A more interesting region, however, is $nu \sim a^{-1/2}$. The speed of sound u is also a function of the time. It is well known that during the expansion of the world[4] the mean square speed of the particles varies as a^{-1}. We shall therefore suppose $u \sim a^{-1}$ (after recombination of the hydrogen). We use the notation $e = n^2 u_0^2$, where u_0 is the speed of sound for $a = 1$, and make the change of variables[1)]

$$\Phi = \int_a \frac{da}{a\sqrt{P}} \tag{2.4}$$

Then (2.1) can be written in a form analogous to that of the Schrödinger equation[2)]

$$\frac{d^2\psi}{d\Phi^2} + [e - a(\Phi)]\psi = 0, \tag{2.5}$$

where e plays the role of the energy, and $a(\Phi)$, the inversion of the integral (2.4), the role of the potential.

The function $a(\Phi)$ is monotonic in the interval $0 < \Phi < \infty$ and takes the particular forms

for the closed model a

for the open model a

for the Lemaitre model

$$a = \begin{cases} 6\Phi^{-2} & \text{for } \Phi \gg \ln \Delta^{-1}, \\ 1 - \sqrt{\Delta/3}\, \mathrm{sh}\,(\Phi - \Phi_0) & \text{for } |a - 1| \ll 1, \\ (4/3)^{1/4} \Phi^{-1/2} & \text{for } \Phi \ll 1. \end{cases} \tag{2.6}$$

The upper limit in (2.4) has been taken to be the maximum radius of curvature for each model:

$$\Phi_0 = \int_1^\infty \frac{da}{a\sqrt{P}} \sim \ln \Delta^{-1}.$$

Corresponding to the expansion of the universe there is a decrease of Φ from ∞ to 0. In models finite in time (closed models) Φ varies from 0 to $-\infty$ during the subsequent contraction.

Writing the equation for the density perturbations in the form (2.5) allows us to use the method of quasiclassical approximation in its solution. There is, however, the difference that in quantum mechanics one looks for the solution that falls off exponentially into the "classically inaccessible region" $a > e$, and for the study of the instability the solution of interest is a different one, which increases into this region.

Making appropriate changes in the connection formulas,[5] we get (going back to the variable a)

$$\psi(a < e) = \frac{C(e)}{(e-a)^{1/4}} \cos\left(\int_a^e (e-a)^{1/2} \frac{da}{a\sqrt{P}} + \frac{\pi}{4}\right), \tag{2.7}$$

$$\psi(a > e) = \frac{C(e)}{2(a-e)^{1/4}} \exp\left\{\int_e^a (a-e)^{1/2} \frac{da}{a\sqrt{P}}\right\}. \tag{2.8}$$

We note that in the most dangerous region (close to the singularity of the potential) $e \ll a \ll 1$ the solution (2.8) gives $\psi \sim a^{(3/2)1/2 - 1/4} \sim a^{0.975}$ instead of $\psi \sim a$, Eq. (2.3), and this characterizes the accuracy of the quasiclassical method.

Returning to the definition (1.3) of the spectrum,

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1)We point out that $\eta = \int da/\sqrt{P}$, $t = \int a da/\sqrt{P}$ and $\Phi(a)$ are the three fundamental elliptic integrals corresponding to the polynomial P(a).

2)It is interesting that this is possible only if u has the dependence a^{-1} on the radius of curvature.

we square (2.7) and (2.8) and average the rapidly oscillating square of the cosine. Expressing $C(e)$ in terms of the initial spectrum $S_i(e)$, we get for the final spectrum S_f in the region $a_i \ll e < a_f$ the expression

$$S_f(e, a_f) = \frac{1}{2} S_i(e) \sqrt{\frac{e}{a_f - e}} \exp\left\{2 \int_e^{a_f} (a-e)^{1/2} \frac{da}{a\sqrt{P}}\right\}. \quad (2.9)$$

The asymptotic behavior of the spectrum in the long-wavelength region $e \ll a_f$ does not depend on the form of the model,

$$S_f(e) \propto S_i(e) e^{-\sqrt{6}+1/2}. \quad (2.10)$$

Qualitatively this follows from the fact that for large-scale density perturbations the Jeans criterion has already begun to be satisfied, and their spectrum is determined by an earlier stage of the expansion of the universe, which is the same for all isotropic cosmological models.

Finally, in the region of still larger wavelengths the form of the spectrum cannot be established from Eq. (2.1), which applies only after the recombination of the hydrogen; thus there is a lower limit on the applicability of Eq. (2.9).

There is at present no reliable information as to the nature of the spectrum of the perturbations at the time of recombination. Therefore in the more detailed study of the spectra of different models (Sec. 4) we shall assume the initial spectrum to be uniform, $S_i(e) = \text{const}$.

3. THE CONNECTION BETWEEN THE SPECTRUM AND THE MASS DISTRIBUTION

The spectrum (2.9) has been derived from the equations of the linear approximation, which holds as long as the mean-square fluctuations remain small in comparison with the average density, Eq. (1.7). The formation of separate bodies occurs in the subsequent, nonlinear stage. As has been noted by Doroshkevich,[6] we can assume that the mass and angular-momentum distributions of the objects are determined already in the linear stage. It is of course not excluded that shock waves that arise in the nonlinear stage will lead to a redistribution of mass between the bodies. It will be assumed here that this effect does not change the distribution laws much.

On the other hand, the spectrum (2.9) itself is distorted in the nonlinear stage. It can, however, be shown, by analyzing the (nonrelativistic) second-approximation equations, that the correction to the spectrum is

$$S_2(e) \propto \left[\frac{25}{27} S_1(e) - \frac{29}{45} e \frac{dS_1}{de} + \frac{2}{5} e^2 \frac{d^2 S_1}{de^2}\right] \frac{\langle\rho^2\rangle - \langle\rho\rangle^2}{\langle\rho\rangle^2} \quad (3.1)$$

and consequently the spectrum is not much changed in form in its powerlaw and steeper parts. We shall show that the mass distribution does not depend on the actual height of the spectrum.

The question now is how to connect the spectrum and the density correlation function with the mass distribution of the objects which are formed. This problem is similar to the problem of the distribution of the zeroes (or passages through a given level) of a normal random function, which has not been solved for an arbitrary correlation function. We must make the transition from probability properties given on an infinite interval (volume) to the statistics of local formations. We therefore represent the random function—the density at a fixed time—in the form

$$\rho(\mathbf{r}) = \sum_\alpha b \exp\left\{-\frac{(\mathbf{r}-\mathbf{r}_\alpha)^2}{2\sigma_\alpha}\right\}, \quad (3.2)$$

where b is a constant, the points $\mathbf{r}_\alpha$ are distributed in space according to the Poisson law with mean density ν, and the variances (or cross sections) σ_α are distributed with an as yet unknown probability density $q(0 < \sigma < \infty)$.

For a random function given in this form we can, following Feynman and Hibbs,[7] construct the correlation function as follows:

$$\langle\rho(\mathbf{r})\rho(\mathbf{r}+\mathbf{R})\rangle = \nu \int_0^\infty q(\sigma)\, d\sigma \int \rho_\sigma(\mathbf{r}) \rho_\sigma(\mathbf{r}+\mathbf{R})\, d^3r,$$
$$\rho_\sigma(\mathbf{r}) = b \exp\{-r^2/2\sigma\}. \quad (3.3)$$

As is shown in[7], it is reasonable to approximate such a random function with a normal function. In our present case it is indeed normal in the linear region.

Since a random function with a normal distribution law is completely determined by its correlation function, by equating the correlation function (3.3) and the corresponding spectrum (2.9) we have made the random functions the same. Equating them gives an integral equation for $q(\sigma)$.

We now define the mass of an object with the parameter σ as

$$m = \int \rho_\sigma(r)\, d^3r = b(2\pi\sigma)^{3/2}, \quad (3.4)$$

which allows us to go over from the distribution $q(\sigma)$ to the mass distribution $p(m)$,

$$p(m)\, dm = q(\sigma)\, d\sigma. \quad (3.5)$$

Let us apply a Fourier transformation to Eq. (3.3). We multiply both sides by $\exp\{in r/a\}$ and integrate over all space. Then by Eq. (1.5) we get the spectrum on the left side, and consequently

$$\langle\rho\rangle^2 S_n = \frac{\nu b^2}{a^3} \int_0^\infty q(\sigma)\, d\sigma\, \sigma^3 \exp\left\{-\frac{n^2\sigma}{a}\right\}. \quad (3.6)$$

We change over to the variable $e = n^2 u_0^2$ and to the dimensionless $\tilde{\sigma} = \sigma a^{-2} u_0^{-2}$. Then

$$S(e) = \frac{\nu b^2 a^3 u_0^6}{\langle\rho\rangle^2} \int_0^\infty q(\tilde{\sigma}) \tilde{\sigma}^3 \exp\{-\tilde{\sigma} e\}\, d\tilde{\sigma}. \quad (3.7)$$

Here $S(e)$ is defined by Eq. (2.9). The coefficient of the integral is determined from the normalization of $q(\sigma)$. Multiplying by a power of e and integrating from 0 to ∞, we get the expression for an arbitrary moment of the distribution:

$$\langle\tilde{\sigma}^\beta\rangle = \frac{2}{\Gamma(3-\beta)} \int_0^\infty S(e) e^{2-\beta}\, de \Big/ \int_0^\infty S(e) e^2\, de. \quad (3.8)$$

This shows that the distribution $q(\sigma)$, and consequently the statistics of the masses, do not depend on the constant factor in the spectrum, i.e., on its height.

This result can also be expected from qualitative considerations. In fact, as an instability develops—the spectrum grows—the density increases where it was above the average value and decreases where it was below this value, and this does not lead to a redistribution of the masses given by (3.4).

When we use (3.5) we can directly write down an equation for $p(m)$, the probability density of the masses,

$$\frac{2S(e)}{\int\limits_0^\infty S(e)e^2de}=\int\limits_0^\infty p(m)\left(\frac{m}{m_0}\right)^2\exp\left\{-\left(\frac{m}{m_0}\right)^{2/3}e\right\}dm, \qquad (3.9)$$

$$m_0=(2\pi)^{3/2}b(au_0)^3.$$

Its solution is given by an inverse Laplace transformation and is therefore unique.

The asymptotic form of the distribution $p(m)$ in the region of large masses can be established by means of the asymptotic form of $S(e)$ for $e\to 0$, Eq. (2.10). Since $\langle m^\gamma\rangle$ diverges for $\gamma>(7-2\cdot 6^{1/2})/3$, we have to logarithmic accuracy for $m\to\infty$

$$p(m)\sim m^{-\gamma_0},\quad \gamma_0=(10-2\sqrt{6})/3\approx 1.7. \qquad (3.10)$$

This result, like Eq. (2.10), is not sensitive to the form of the model.

4. COMPARISON WITH THE OBSERVATIONAL DATA

The Jeans criterion obtained from the relativistic equation (2.5) for the density perturbations is exactly the same as the result of the nonrelativistic theory: perturbations with wave numbers less than the value given by

$$n_J^2=a/a_cu_0^2. \qquad (4.1)$$

will increase. Let us write this condition directly for the masses:

$$m_J\sim\rho(\pi a/n_J)^3\sim Mu_0^3(a_c/a)^{3/2}. \qquad (4.2)$$

When we apply this estimate (4.2) at the time of recombination, using the thermodynamic speed of sound $(T/m_H)^{1/2}\sim 8\cdot 10^5$ cm/sec, we find masses 10^5 to 10^6 $M_\odot$, corresponding to globular clusters.

It is not excluded, however, that the macroscopic speed of sound $u=(dp/d\rho)^{1/2}$ that appears in the equation for the perturbations may exceed the thermodynamic value, since the pressure p found from the energy-momentum tensor is an average over a scale so large that the corresponding relaxation times are larger than the age of the universe.

Since the Jeans mass value depends on the cube of the speed of sound, the characteristic mass of an instability may differ by a great deal (5 to 6 orders of magnitude) from the estimate based on the thermodynamic speed of sound. Therefore it is interesting to compare the results we have obtained with the observed mass distribution of galaxies ($m\sim 10^9$ to 10^{11} $M_\odot$). For this comparison we used the mass values for 78 galaxies (I. Genkin and L. Genkina[8]).

A correct comparison requires the construction of the distribution for all galaxies in a fixed volume. This is difficult, however, owing to the paucity of statistics. Therefore the histogram of the distribution of the 78 galaxies (Fig. 1) is evidently too high on the large-mass side.

We note that the maximum of the distribution

$$p(m)\sim\frac{dN}{dm}\sim\frac{1}{2,3m}\frac{dN}{d\lg m}$$

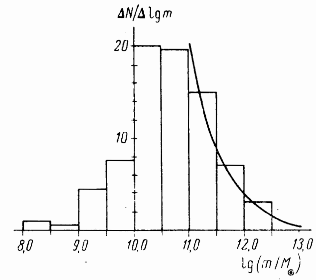

FIG. 1. Histogram of observed distribution $\Delta N/\Delta\log m = 2.3m\cdot\Delta N/\Delta m$; the curve is the asymptotic part of the theoretical distribution, $mp(m)\sim m^{-0.7}$.

is shifted toward smaller masses relative to the maximum of the histogram.

The numerical inversion of the Laplace transformation in (3.9) is inconvenient. Therefore for the comparison with the theory we have constructed the logarithm of the perturbation spectrum from the observed statistics:

$$\ln S_{\exp}(e)=\ln\left\langle\left(\frac{m}{m_0}\right)^2\exp\left\{-e\left(\frac{m}{m}\right)^{2/3}\right\}\right\rangle+\text{const.} \qquad (4.3)$$

Here the average is over the set of 78 masses, and the constant m_0 must be determined from the comparison with the theory. With this sort of processing of the observational data the random error is averaged, and the systematic error goes into the factor m_0.

For the numerical comparison our main attention was given to those models that are of interest practically or from the point of view of principle.

In the open model the present value of the radius of curvature is assumed large in comparison with a_c. Because the integral in (2.9) converges well its upper limit can be set equal to infinity. Then

$$\ln S_0(e)=\frac{2\sqrt{6}}{k}(\mathbf{K}-\mathbf{E})+\frac{1}{2}\ln e+\text{const}, \qquad (4.4)$$

where $\mathbf{K}$ and $\mathbf{E}$ are the complete elliptic integrals and $k^2=2/(2+3e)$.

In the closed model the radius of curvature is limited, and therefore the expression for the spectrum depends on the chosen value $a_f<2/3$:

$$\ln S_c(e)=2\sqrt{6}[F(k,\varphi)-E(k,\varphi)]+\frac{1}{2}\ln\frac{e}{a_f-e}+\text{const}, \qquad (4.5)$$

where $F(k,\varphi)$ and $E(k,\varphi)$ are the elliptic integrals and

$$k^2=\frac{2-3e}{2},\qquad \sin^2\varphi=\frac{1-e/a_f}{1-3e/2}.$$

In the Lemaitre model there is one further parameter $0<\Delta\ll 1$:

$$\ln S_L(e)=2\sqrt{3}\int\limits_e^{a_f}\frac{da}{a}\sqrt{\frac{a-e}{a(a-1)^2(a+2)+\Delta a^4}}+\frac{1}{2}\ln\frac{e}{a_f-e}+\text{const}. \qquad (4.6)$$

We calculate the integral approximately, breaking the range of integration up for this purpose into two parts $(e, 2/3)$ and $(2/3, a_f)$, assuming $a_f\geq 1$. We then neglect the fourth-order term in the first integral; since in the second integral the important region for the integration is the neighborhood of $a=1$, we replace a by unity where possible. We then get

$$\ln S_L(e)=2\sqrt{6}(\mathbf{K}-\mathbf{E})+L\sqrt{1-e}+\frac{1}{2}\ln\frac{e}{a_f-e}+\text{const}, \qquad (4.7)$$

where

$$L=2\ln\frac{2}{\Delta}\left(a_f-1+\sqrt{(a_f-1)^2+\frac{\Delta}{3}}\right)\qquad k^2=\frac{2-3e}{2}. \qquad (4.8)$$

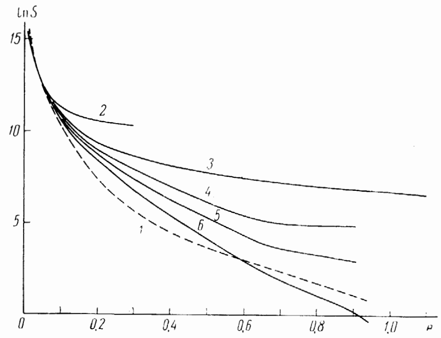

FIG. 2. Comparison of the spectrum calculated from the observed distribution (dashed curve 1) with the theoretical distributions (solid curves): 2, for the closed model, $a_f = 0.5$; 3, for the open model; 4, for the Lemaitre model, L = 3; 5, for the Lemaitre model, L = 6; 6, for the Lemaitre model, L = 10 (for curves 4–6, $a_f = 1.01$).

The first term in (4.7) is to be regarded as zero for $e > 2/3$. This estimate gives too-high values of the integral in the region $e < 2/3$, but is correct for small e and for e close to unity.

Curves of ln S(e) are shown in Fig. 2. The constants in Eqs. (4.3)–(4.7) were chosen by fitting the asymptotic forms for $e \ll 1$. The spectrum constructed from the experimental statistics is shown for the value $m_0 = 8 \cdot 10^7 M_\odot = 1.6 \cdot 10^{41}$ g. It is seen from the curves that the experimental spectrum is compatible with the curves for the Lemaitre model for $L \approx 6$ to 10 and does not agree with the curves for the other models.

Such a value of L agrees with the theoretical expression for the parameter Δ[9]

$$\Delta = 4\sigma\varkappa c^{-3} T_0^4 (z_c + 1)^4 a_c^2 \approx 1.3 \cdot 10^{-3}, \qquad (4.9)$$

derived on the assumption that the cosmological constant is exactly equal to the critical value and that the correction is due to black radiation (σ is the Stefan-Boltzmann constant, κ is the Einstein gravitational constant, $T_0 = 2.7°K$ is the present temperature of the residual radiation, and $z_c = 1.95$ is the red shift corresponding to the critical radius $a_c = 4.6 \cdot 10^{27}$ cm of the Lemaitre model).

The idea that our universe is described by the Lemaitre model was proposed by Petrosian, Salpeter, and Szekeres[10] and by Kardashev[11] in connection with studies of the red-shift distribution of quasars.

The results of this paper also favor the Lemaitre model, although they also do not exclude the usual open model with an appropriate initial spectrum.

In the Lemaitre model the minimal mass m_0 satisfies the Jeans condition for $a = a_f = a_c$, Eq. (4.2). Estimating the mass of the universe as $M = 4\pi^2 a_c / \kappa \sim 10^{56}$ g, we get

$$u_0 \sim (m_0 / M)^{1/3} \sim 10^{-5}. \qquad (4.10)$$

This value of the speed of sound ($3 \cdot 10^5$ cm/sec) for $a = a_c$ gives a value at the time of recombination which is two orders of magnitude larger than the thermodynamic speed of sound. This discrepancy, however, may be interpreted in the sense that the matter consists of separate formations of hydrogen, with mean-square speed corresponding to (4.10), and the temperature before recombination is determined by the temperature of the residual radiation.

After the formation of the galaxies the further development of the instability can be represented schematically in the following way. The mean-square speed of the galaxies is obviously larger than the speed of sound (4.10). This gives a new Jeans scale, associated with the characteristic size of clusters of galaxies.

The growth of the density perturbations ceases quickly after the completion of the Lemaitre plateau, and therefore the galactic clusters do not have time to be completely formed. In fact, one observes only correlations between the positions of individual galaxies, but it is difficult to distinguish one cluster clearly from another.

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