NONLINEAR THEORY OF THE CONDUCTIVITY OF AN ELECTRON GAS IN A STRONG MAGNETIC FIELD

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Submitted July 1, 1970

Zh. Eksp. Teor. Fiz. 60, 175–181 (January, 1971)

For an electron gas in the presence of constant crossed E and H fields, an expression which is nonlinear in the electric field is obtained for the dissipative current in terms of the operator for scattering by randomly distributed impurities. In the particular case of an impurity having a short-range interaction, a formula is derived, describing the dependence of the transverse conductivity on E in the ultraquantum limit, in the region of electric field strengths which are not strong enough to cause heating of the electron system.

1. INTRODUCTION

In article [1] by the authors it was shown that in that case when scattering by impurities is the dominant scattering mechanism, the linear theory of galvanomagnetic phenomena is strictly speaking valid only for electric fields satisfying the condition $E \ll E_0 = \epsilon_0/ea$ ($a = (c/eH)^{1/2}$ is the magnetic length, e is the absolute value of the electron charge, and $\hbar = 1$). The energy parameter ϵ_0, which appears in the linear theory as the cut-off parameter in the logarithmically divergent expression for the transverse conductivity, may have a different physical origin. For a sufficiently large impurity concentration, ϵ_0 is determined by collision broadening of the one-electron energy levels; for a small impurity concentration the parameter ϵ_0 in order of magnitude determines the limiting value for the "longitudinal" energy of a particle, below this energy the Born approximation is not valid for the scattering. As a rule, ϵ_0 is the smallest energy measure in the theory, and in these cases the value E_0 associated with it is the smallest of the possible characteristic values of the electric field. In particular, E_0 is usually much smaller than the field strengths at which heating of the electron system appears ($E_0 \ll E_1 = v_s/ea^2$,[2] v_s is the velocity of sound).

In article [1] the nonlinear behavior of the transverse conductivity in constant crossed E and H fields was studied for the case when the electric field satisfies the condition $E_0 \ll E \ll E_1$. The lower limit on the electric field strength appeared as a result of the condition for validity of the Born approximation for the scattering and from the condition that collision broadening of the levels can be neglected. The transition to the linear theory ($E \to 0$) was achieved by the formal introduction of the cut-off parameter ϵ_0 in connection with integration over the energy. By the same token an interpolation formula was obtained, giving the correct behavior of the conductivity for $E \ll E_0$ and $E_0 \ll E \ll E_1$. However, the interpolation carried out in [1] was very arbitrary in nature, since in order to construct the function $\sigma_{xx}(E)$ only its limiting value at zero and its asymptotic behavior for $E \gg E_0$ were utilized. The basic object of the present work consists in the derivation of a formula which is also accurate in the intermediate region ($E \approx E_0$). In this connection the physical model assumed in article [3] will be considered: 1) the impurities have a short-range interaction ($r_0 \ll a$); 2) collision broadening of the levels is negligibly small.

2. DERIVATION OF THE FORMULA FOR THE DISSIPATIVE CURRENT

In order to derive the formula for the dissipative current in the steady-state case, it is convenient to start from the following expression which is cited in [4]:

$$j_x = \frac{ie}{m\omega V}\langle \mathrm{Sp}\{\rho[\mathscr{H}_{\mathrm{int}}, p_y]\}\rangle, \tag{1}$$

where $\mathscr{H}_{\mathrm{int}}$ is the Hamiltonian describing the interaction with the impurity centers, ω is the cyclotron frequency, p_y is the y-component of the electron's momentum operator, V is the volume of the system, the angular brackets denote averaging over the configurations of the impurities, and $\rho = \rho(\infty)$ is the steady-state density matrix.

In order to determine $\rho(\infty)$ we use the relation $\rho(\infty) = \bar{\rho}(+0)$, where

$$\bar{\rho}(s) = s\int_0^\infty \rho(t)e^{-st}\,dt$$

is the Laplace transform of the density matrix, satisfying the equation

$$[\mathscr{H}, \bar{\rho}(s)] - is\bar{\rho}(s) = -is\rho(0). \tag{2}$$

Here $\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_{\mathrm{int}}$, where $\mathscr{H}_0$ is the Hamiltonian of an electron in crossed E and H fields. The formal solution of Eq. (2) can be represented in terms of the resolvents $R^{\pm}(\epsilon) = (\epsilon - \mathscr{H} \pm is/2)^{-1}$:

$$\bar{\rho}(s) = \frac{s}{2\pi}\int_{-\infty}^{\infty} R^+(\varepsilon)\rho(0)R^-(\varepsilon)\,d\varepsilon. \tag{3}$$

Let us choose the initial density matrix $\rho(0)$ according to Adams and Holstein:[5]

$$\rho_{\alpha\beta}(0) = f(\varepsilon_\alpha^0)\delta_{\alpha\beta} \equiv f_\alpha\delta_{\alpha\beta},$$

where f is the Fermi function, ϵ_α^0 is the part of the electron's energy which does not depend on the electric field, and $\alpha = \{n, k_y, k_z\}$ is the set of quantum numbers which determine the electron's state in the presence of crossed fields. The choice of $\rho(0)$ which has been made assumes the absence of heating of the electron system. We note here that, to the lowest order in $1/\omega\tau$ (τ is the characteristic relaxation time), the entire investigation carried out below can be generalized to the case where heating is taken into consideration if as $\rho(0)$ one takes the asymptotic ($\omega\tau \to \infty$) density matrix.[4]

For the choice of $\rho(0)$ which has been made, substituting (3) into (1) and introducing the scattering operators $T^{\pm}(\epsilon)$:

$$R^{\pm}(\varepsilon) = R_0^{\pm}(\varepsilon) + R_0^{\pm}(\varepsilon)T^{\pm}(\varepsilon)R_0^{\pm}(\varepsilon), \quad R_0^{\pm}(\varepsilon) = (\varepsilon - \mathcal{H}_0 \pm is/2)^{-1},$$

we can reduce the expression for the current to the following form

$$j_x = \frac{ea^2}{2\pi iV}\lim_{s\to+0} s\sum_\alpha f_\alpha \int_{-\infty}^{\infty} d\varepsilon\langle\Lambda_{\alpha\alpha}(\varepsilon)\rangle. \qquad (4)$$

The operator $\Lambda(\epsilon)$ in Eq. (4) has the form

$$\Lambda(\varepsilon) = R_0^- p_y T^+ R_0^+ - R_0^- T^- p_y R_0^+ + R_0^- T^- R_0^- p_y T^+ R_0^+ - R_0^- T^- p_y R_0^+ T^+ R_0^+. \qquad (5)$$

In the limit $s \to +0$ from Eq. (5) we find

$$\lim_{s\to+0}\frac{s}{2\pi}\Lambda_{\alpha\alpha}(\varepsilon) = 2ip_{y\alpha}\,\mathrm{Im}\,T_{\alpha\alpha}^+(\varepsilon_\alpha)\delta(\varepsilon-\varepsilon_\alpha) + 2\pi i\sum_\beta \delta(\varepsilon-\varepsilon_\beta)|T_{\beta\alpha}^+(\varepsilon)|^2\delta(\varepsilon-\varepsilon_\alpha)p_{y\beta}. \qquad (6)$$

Having utilized the optical theorem

$$\mathrm{Im}\,T_{\alpha\alpha}^+(\varepsilon) = -\pi\sum_\beta \delta(\varepsilon-\varepsilon_\beta)|T^{\pm}(\varepsilon_\beta)|^2,$$

we arrive at the following expression for j_x:

$$j_x = \frac{e\pi}{V}\sum_{\alpha,\beta} X_{\alpha\beta}(f_\alpha - f_\beta)\delta(\omega_{\alpha\beta})\langle|T_{\beta\alpha}^+(\varepsilon_\beta)|^2\rangle. \qquad (7)$$

Here $\omega_{\alpha\beta} = \epsilon_\alpha - \epsilon_\beta$, $X_{\alpha\beta} = X_\alpha - X_\beta$, where $X_\alpha = -a^2k_{y\alpha} - v_d/\omega$, and $v_d = cE/H$. Confining our attention to the lowest-order approximation in the concentration of the impurity centers (as indicated in [6], in this case $T = \Sigma t(\mathbf{R}_i)$, where $t(\mathbf{R})$ denotes the operator for scattering by the impurity center located at the point $\mathbf{R}$), we obtain the final formula for the dissipative current:

$$t_x = e\pi n_d\sum_{\alpha,\beta} X_{\alpha\beta}(f_\alpha - f_\beta)\delta(\omega_{\alpha\beta})\langle|t_{\beta\alpha}^+(\varepsilon_\alpha, \mathbf{R})|^2\rangle, \qquad (8)$$

where n_d denotes the impurity concentration.

After linearization with respect to the electric field, formula (8) goes over into the expression cited in the review by Kubo et al.,[7] and its Born limit coincides with the expression for the current obtained in [4] and in [1]. However, if in (8) one simultaneously carries out linearization in E and passes to the Born limit, then we arrive at the well-known formula of Titeica.

In what follows we shall confine our attention to an examination of a short-range interaction potential ($r_0 \ll a$). In this case the scattering operator $t_{\beta\alpha}$ may be expressed in terms of the scattering length f (f denotes the amplitude for the scattering of a particle with zero energy in the absence of external fields). By the method used in article [3] we obtain

$$t_{\beta\alpha}^+(\varepsilon, \mathbf{R}) = \frac{2\pi f}{m}\frac{\varphi_\alpha(\mathbf{R})\varphi_\beta^*(\mathbf{R})}{1 + ifK(\varepsilon, \mathbf{R})}. \qquad (9)$$

Here φ_α is the wave function of an electron in crossed fields. The function $K(\epsilon, \mathbf{R})$ is determined by the relation

$$K(\varepsilon, \mathbf{R}) = -\frac{2\pi i}{m}\frac{\partial}{\partial|\mathbf{r}-\mathbf{R}|}[|\mathbf{r}-\mathbf{R}|G^r(\mathbf{r}, \mathbf{R}; \varepsilon)]\big|_{|\mathbf{r}-\mathbf{R}|=0}, \qquad (10)$$

where $G^r(\mathbf{r}, \mathbf{r}'; \epsilon)$ is the retarded Green's function of an electron in cross fields.

The expression for the scattering operator (9), corresponding in form with the analogous relation of article [3], differs from it by the fact that it includes the dependence on the electric field, which is contained both in the wave functions and in the function K. Furthermore, in our case the function K turns out to depend on the position of the scattering center.

Thus, from Eqs. (9) and (10) it is clear that the search for the scattering operator reduces to an investigation of the behavior of the Green's function $G^r(\mathbf{r}, \mathbf{r}'; \epsilon)$ for $|\mathbf{r} - \mathbf{r}'| \to 0$.

3. THE GREEN'S FUNCTION OF AN ELECTRON IN CROSSED FIELDS

The time-dependent Green's function

$$G^r(\mathbf{r}, \mathbf{r}'; t) = \begin{cases} \langle\mathbf{r}|e^{-i\mathcal{H}_0 t}|\mathbf{r}'\rangle, & t > 0 \\ 0, & t < 0 \end{cases}$$

for the problem under consideration may be expressed in closed form in terms of elementary functions with the aid of the methods developed by Schwinger[8] and Feynman.[9] Thereby we obtain a convenient integral representation of the required Green's function for the steady-state problem:

$$G^r(\mathbf{r}, \mathbf{r}'; \varepsilon) = \frac{1}{i}\int_0^\infty G^r(\mathbf{r}, \mathbf{r}'; t)e^{i(\varepsilon+i\eta)t}dt, \quad \eta \to +0. \qquad (11)$$

Without discussing the derivation, we cite the final expression for $G^r(\mathbf{r}, \mathbf{r}'; \epsilon)$:

$$G_r(\mathbf{r}, \mathbf{r}'; \varepsilon) = \frac{(2m)^{1/2}e^{3\pi i/4}}{8\pi^{3/2}a^2}\exp\left[-i\frac{(x+x')(y-y')}{2a^2} - i\frac{eE(y-y')}{\omega}\right] \times \int_0^\infty \frac{dt\, e^{i(\varepsilon+i\eta)t}}{\sqrt{t}\sin(\omega t/2)}\exp\left[\frac{i}{4a^2}|\rho-\rho'|^2\operatorname{ctg}\frac{\omega t}{2} + \frac{mi}{2t}(z-z')^2 + \frac{it}{2}eE(y-y')\operatorname{ctg}\frac{\omega t}{2} - \frac{it}{2}eE(x+x') - \frac{it}{2}mv_d^2 + \frac{it^2}{4a^2}v_d^2\operatorname{ctg}\frac{\omega t}{2}\right], \qquad (12)$$

where $|\rho - \rho'|^2 = (x - x')^2 + (y - y')^2$.

In the limit $E \to 0$ formula (12) goes over into the expression for the Green's function obtained by Schwinger's method in article [10]. (For this only certain transformations and the correction of the errors allowed in [10] are required.) For $H \to 0$ from Eq. (12) we obtain the Green's function in a homogeneous electric field and, finally, for $E \to 0$ and $H \to 0$ formula (12) goes over into the well-known expression for the Green's function of a free particle.

From expression (12) it follows that, as $\mathbf{r} \to \mathbf{R}$

$$G^r(\mathbf{r}, \mathbf{R}; \varepsilon) \approx -\frac{m}{2\pi}\left(\frac{1}{|\mathbf{r}-\mathbf{R}|} + iK(\varepsilon, X)\right), \qquad (13)$$

where $K(\epsilon, X)$ is defined by the integral

$$K(\varepsilon, X) = \frac{e^{i\pi/4}}{2a\sqrt{\pi}}\int_0^\infty \frac{dt}{\sqrt{t}}\left\{\frac{1}{t} - \frac{\exp[i(\varepsilon-\gamma+i\eta)t + i\gamma^2t^2\operatorname{ctg}t - 2i\gamma Xt]}{\sin t}\right\}. \qquad (14)$$

In formula (14) we arrived at dimensionless quantities: the energy is measured in units of $\omega/2$, the length in units of a. In addition the notation $\gamma = eEa/\omega$ has been introduced. One can regard the integral in (14) as the limit of an integration along a contour lying in the lower half-plane of t ($-\pi/4 < \arg t < 0$) upon the approach of this contour to the positive semi-axis, going around the points $t_n = n\pi$ from below. This makes it possible to replace the function $1/\sin t$ by a series representing it in the lower half-plane:

$$1/\sin t = 2i\sum_{n=0}^{\infty} e^{-(2n+1)it}.$$

Using this expression, let us write $K(\epsilon, X)$ in the form

$$K(\varepsilon, X) = \frac{ie^{i\pi/4}}{a}\sum_{n=0}^{\infty}(J_n^{(1)} - J_n^{(2)}(\varepsilon, X)), \tag{15}$$

where

$$J_n^{(1)} = \frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{\sin t}{t^{3/2}}e^{-(2n+1)it}\,dt = (\sqrt{2n+2} - \sqrt{2n})e^{-i\pi/4}, \tag{16}$$

$$J_n^{(2)} = \frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{dt}{\sqrt{t}}\exp[i\alpha_n t + i\gamma^2 t^2 \operatorname{ctg} t]; \quad \alpha_n = \varepsilon - \gamma^2 - 2\gamma X - (2n+1). \tag{17}$$

In the range of electric field strengths of interest to us, the quantity γ is small ($\gamma \ll 1$); therefore we may replace the integral $J_n^{(2)}$ by its asymptotic expression as $\gamma \to 0$. In order to obtain the asymptotic expression for $J_n^{(2)}$ let us rotate the contour of integration in the lower half-plane through a certain angle θ ($-\pi/4 \le \theta < 0$). We can verify that the integral along the infinitely distant arc vanishes, and to the lowest-order approximation in γ we obtain

$$J_n^{(2)} \approx \frac{e^{-i\theta/2}}{\sqrt{\pi}}\int_0^{\infty}\frac{dt}{\sqrt{t}}\exp[i\alpha_n t e^{-i\theta} - e^{-2i\theta}\gamma^2 t^2]$$

$$= \frac{1}{(2\gamma^2)^{1/4}}\exp\left(-\frac{\alpha_n^2}{8\gamma^2}\right)D_{-1/2}\left(-\frac{i\alpha_n}{\sqrt{2}\gamma}\right),$$

where $D_{-1/2}(x)$ is the parabolic cylinder function.

From formulas (16) and (18) it follows that for large values of n the terms of the series (15) behave like $n^{-3/2}$, i.e., the series converges. In the limit $\gamma = 0$ ($E = 0$) expression (15) takes the form

$$\lim_{\gamma\to 0} K(\varepsilon, X) = K_0(\varepsilon) = \frac{1}{a}\sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{\alpha_n^{(0)}}} + i\sqrt{2}(\sqrt{n+1} - \sqrt{n})\right], \tag{19}$$

where $\alpha_n^{(0)} = \epsilon - (2n+1)$. An approximate expression for $K_0(\epsilon)$ is given in article [11].

4. TRANSVERSE CONDUCTIVITY IN THE ULTRA-QUANTUM LIMIT

Let us perform a specific calculation of the conductivity under the assumptions which were adopted in article [1]: 1) the absence of heating ($eEa \ll v_s/a$), 2) Boltzmann statistics, and 3) the ultra-quantum limit ($\omega/T \gg 1$). Under these assumptions, after simple transformations we obtain the following results from formulas (8), (9), and (15)–(18):

$$\sigma_{xx} = \frac{2e^2 n n_d f^2}{(mT)^{3/2}}\mathscr{F}(\xi, \lambda), \tag{20}$$

where n denotes the density of the electrons, $\xi = eEa/T$, $\lambda = f^2/2a^2\gamma = \delta/\xi$ ($\delta = f^2/2ma^4T$ is the cut-off parameter introduced in [1]),

$$\mathscr{F}(\xi, \lambda) = 2\pi\int_0^{\infty} u^2 e^{-\xi u - u^2}(I_0(u^2) - I_1(u^2))\Phi(u, \lambda)\,du, \tag{21}$$

$$\Phi(u, \lambda) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{dx\, e^{-2(x-u)^2}}{W(x, \lambda)}. \tag{22}$$

$$W(x, \lambda) = 1 + \frac{\pi\lambda|x|e^{-x^2}}{2}\left[I_{1/4}^2\left(\frac{x^2}{2}\right) + I_{-1/4}^2\left(\frac{x^2}{2}\right)\right] + (\pi\lambda|x|)^{1/2}e^{-x^2/2}\left[I_{-1/4}\left(\frac{x^2}{2}\right) - I_{1/4}\left(\frac{x^2}{2}\right)\operatorname{sign} x\right]. \tag{23}$$

Here $I_\nu(x)$ are the Bessel functions of imaginary argument.

As $\lambda \to 0$ ($\xi \gg \delta$) the function $\Phi(u, \lambda) \to 1$ and Eq. (21) goes over into formula (7) of article [1]. In the opposite limiting case, when $\lambda \to \infty$ ($\xi \ll \delta$), the regions of large values of the integration variables $x \sim u \gtrsim \lambda$ give the major contribution to the integrals (21) and (22). Since for large values of x the function W changes little per unit interval, we obtain

$$\Phi(u, \lambda) \approx \frac{1}{W(u, \lambda)} = \frac{1}{1 + \lambda/u} + O(1/\lambda). \tag{24}$$

Substituting (24) into (21) and using the asymptotic behavior of the functions I_0 and I_1, we find

$$\lim_{\xi\to 0}\mathscr{F}(\xi, \lambda) = \sqrt{\frac{\pi}{2}}\int_0^{\infty}\frac{e^{-u}}{u+\delta}\,du, \tag{25}$$

which is equivalent to Skobov's result.[3]

The nature of the dependence of the change in the conductivity on the electric field is illustrated by the results cited below of a numerical calculation of the function $\mathscr{F}(\xi, \delta/\xi)$ for the value $\delta = 0.5 \times 10^{-3}$. (We obtain δ of the order of 10^{-3} assuming $f \approx 10^{-8}$ cm, $a \approx 10^{-6}$ cm, $H \approx 10^5$ oersted, $T \approx 10^{-15}$ erg, and $m \approx 10^{-28}$ g. For such values of the parameters, one does not need to take the heating of the electrons into consideration for $\xi < 10^{-1}$.) Values of the quantity $\mathscr{F}(\xi)$ are also given in the table; these values being determined by the interpolation formula derived in article [1] (formula (8)). From a comparison of both functions it is seen that interpolation leads to the appearance of a false maximum in the dependence of the conductivity on the electric field.

The nonlinearity in the behavior of the dissipative current which is being considered in the present work is of a weakly expressed character. Nevertheless, it appears to us that its experimental investigation may turn out to be rather interesting. Here it is important, in the first place, that an appreciable change of the conductivity already appears at very small values of the electric field, when the other possible nonlinear mechanisms have not yet come into play. In the second place, it is essential that the form of the dependence $\sigma_{xx}(E)$ is

ξ	10^{-5}	10^{-4}	$5\cdot10^{-4}$	$8\cdot10^{-4}$	10^{-3}	$2\cdot10^{-3}$	$5\cdot10^{-3}$	10^{-2}
$\mathfrak{F}(\xi)$	8.80	8.80	9.10	9.10	8.98	8.29	7.19	6.34
$\mathfrak{F}(\xi, \delta/\xi)$	8.80	8.69	8.30	8.08	7.95	7.47	6.68	5.98

only determined by the amplitude for the scattering by an individual impurity center, and it does not depend on the concentration of impurities. In connection with this, an experimental investigation of the behavior of $\sigma_{xx}(E)$ may be used for an independent determination of the scattering amplitude.

The authors sincerely thank É. G. Batyev, R. Z. Vitlinaya, I. A. Gilinskiĭ, A. V. Chaplik, and M. V. Éntin for a helpful discussion of the work, and they also express their gratitude to L. P. Adamchik for carrying out the numerical calculations.

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Translated by H. H. Nickle
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