STABILIZATION OF A UNIVERSAL INSTABILITY

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Submitted April 13, 1970; resubmitted June 30, 1970

Zh. Eksp. Teor. Fiz. 59, 2252–2255 (December, 1970)

Normal modes of a "universal" instability of plasma in a magnetic field with a large rotation of the lines of force of the magnetic field (shear) ($\Theta > a_i/R_p$) are considered. A criterion is obtained for the stabilization of these modes which differs from that found recently in the paper by Berk and Pearlstein.[3]

DRIFT instabilities present a great danger from the point of view of confinement of plasma in magnetic traps. For a long time it was thought that rotation of lines of force of the magnetic field (shear), which is defined by the ratio of the characteristic length for variation in the plasma density to the rotation length ($\Theta = R_p/L_s$), easily stabilizes the so-called "universal" mode of the drift instability of a plasma. In [1,2] it was shown that the corresponding problem of the stability of a plasma with respect to drift waves reduces to the problem of eigenvalues of an equation of the type of a Schrödinger equation with a complex potential. Investigation of this equation was usually restricted to the case $\Theta \le a_i/R_p$ (a_i is the Larmor radius for the ions). In the opposite case the real part of the potential, generally speaking, does not have the shape of a well. From this the conclusion was drawn that for $\Theta > a_i/R_p$ the normal modes of a drift instability are not present.

Recently Berk and Pearlstein[3] have noted that this conclusion is not justified and they have attempted to find the normal modes of the drift instability for a comparatively large value of $\Theta > a_i/R_p$. For a normal solution in [3] a solution was adopted in the form of waves propagating from the potential hill in both directions. It was assumed that these oscillations would be damped out at infinity due to interaction with the ions.

In order to obtain the stabilization criterion a comparison was made in [3] of the flux of energy away from the region in which the oscillations are generated with the energy flow into the perturbations due to the interaction with resonance electrons. As a result a criterion was obtained for the stabilization of normal modes by the rotation of the lines of force of the magnetic field:

$$\Theta > (m/M)^{1/3}(k_y a_i)^{2/3}. \quad (1)$$

The aim of the present communication is to show that although the initial assumptions of Berk and Pearlstein are sensible, the subsequent solution and the criterion (1) are not valid. The basic error in [3] consists of the fact that the energy balance was evaluated at a single point where the interaction with the resonance electrons is at a maximum. Moreover, in determining the energy balance the Landau damping by the ions was not taken into account. In the present paper a finite solution has been constructed which is obtained upon taking into account the interaction of the perturbations with the resonance ions. A knowledge of the complete profile $\Phi(x)$ of the solution enabled us to calculate the energy balance over the whole region within which the perturbations are localized. We have obtained a criterion of stabilization which is less rigid than (1) and which in order of magnitude can be written in the following form (cf., with the criterion (6)):

$$\Theta > (m/M)^{1/2} k_y a_i. \quad (1a)$$

We note that an analogous criterion was obtained by Jukes[4] (cf., also the review by Davidson and Kammach[5]). The difference between Jukes' analysis and ours consists of the fact that he considered a solution the finiteness of which is guaranteed by the interaction of the oscillations with electrons, while the interaction with the ions was completely neglected. However the authors of [4,5], apparently, did not note that the region of localization (and correspondingly $k_\parallel$) turns out to be so small that the condition for the applicability of the equations utilized in [4] ($\omega/k_\parallel \ll v_{Te}$) is violated. Therefore the result of [4] cannot be considered as proven.

1. The well-known second order equation for the perturbation of the electric field $\Phi(x)\exp(ik_y y - i\omega t)$ has the form,[3] under the usual assumption ($k_x^2 \ll k_y^2$):

$$\bar{a}_i^2 \frac{d^2\Phi(x)}{dx^2} - [Q_R(x) + iQ_I(x)]\Phi(x) = 0. \quad (2)$$

Here we have

$$Q_R(x) = -1 + \frac{1+\eta}{I_0 e^{-b}} \frac{\omega}{\omega-\omega_i^*}\left[1 - \left(\frac{k_y v_{Ti}}{\omega L_s} x\right)^2\right],$$

$$Q_I(x) = \sqrt{\frac{\pi}{2}} \frac{1+\eta}{I_0 e^{-b}} \frac{\omega}{\omega-\omega_i^*}\left[\frac{\eta}{1+\eta}\frac{\omega-\omega_e^*}{|k_\parallel v_{Te}|} + \frac{\omega}{|k_\parallel v_{Ti}|}\exp\left(-\frac{\omega^2}{2k_\parallel^2 v_{Ti}^2}\right)\right],$$

$$\eta = \frac{T_i}{T_e}, \quad \omega_j^* = k_y \frac{cT_j}{e_j H_0}\frac{1}{n_0}\frac{dn_0}{dx},$$

$$\frac{\omega}{v_{Ti}} \gg k_\parallel \equiv k_y \frac{x}{L_s} \gg \frac{\omega}{v_{Te}},$$

$$\bar{a}_i^2 = -a_i^2 \frac{d}{db}\ln I_0(b)e^{-b}, \quad b = k_y^2 a_i^2,$$

a_j and v_{Tj} are the Larmor radius and the thermal velocity of the particles, $I_0(b)$ is the modified Bessel function.

In contrast to [3], in addition to the interaction of the perturbation with the resonance electrons we have also taken into account the Landau damping due to the ions.

The solution of (2) is well known if one neglects the imaginary part of the potential ($Q_I(x) \equiv 0$):

$$\Phi_R(x) = H_n((i\sigma)^{1/2}x)e^{-i\sigma x^2/2}, \quad n = 0, 1, \ldots, \tag{3}$$

$H_n(\xi)$ are the Hermite polynomials,

$$\sigma = \sigma_0 + i\delta\sigma = \frac{1}{\bar{a}_i}\left[\frac{1+\eta}{I_0 e^{-b}}\frac{\omega}{\omega - \omega_i^*}\right]^{1/2}\frac{k_y v_{Ti}}{\omega L_s}$$

$$\omega = \omega_0 + i\,\mathrm{Im}^{(0)}\omega = \frac{-\omega_i^* I_0 e^{-b}}{1+\eta - I_0 e^{-b}} - i\frac{\bar{a}_i}{L_s}k_y v_{Ti}\frac{1+\eta}{1+\eta - I_0 e^{-b}}(2n+1).$$

We see that the obtained solution decays in time due to the outflow of energy and is infinite in space. In order to calculate the work done by the particles in the field of the wave it is more convenient to utilize the finite solution which is obtained by taking into account the Landau damping due to the ions.

We seek a finite solution of Eq. (2) in the form

$$\Phi(x) = \Phi_R(x)\Phi_I(x), \tag{4}$$

where $\Phi_I(x)$ is a slowly varying function compared to $\Phi_R(x)$. Substituting (4) into (2), we obtain the following equation for $\Phi_I(x)$:

$$2\sigma x\frac{d\Phi_I(x)}{dx} + \frac{1}{\bar{a}_i^2}Q_I^{(i)}(x)\Phi_I(x) = 0.$$

Here we have neglected the term containing the second derivative of the slowly varying function and have restricted ourselves to a consideration of the most unstable mode n = 0.

After integrating this equation we obtain (setting $\omega = \omega_0$ in $Q_I(x)$):

$$\Phi_I(x) = \exp\left(-\frac{1}{2\Theta_{eff}}J_i\left(\frac{x}{\alpha_i}\right)\right),$$

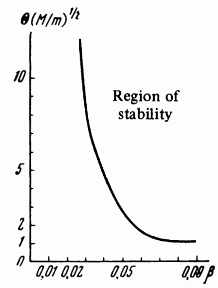
where

$$\Theta_{eff} = \sqrt{\frac{2}{\pi}}\sigma_0\bar{a}_i^2, \qquad J_i(\xi) = \int_{1/|\xi|}^{\infty}\exp(-t^2/2)\,dt, \qquad \alpha_i = \frac{\omega_0 L_s}{k_y v_{Ti}}$$

2. In order to determine the stabilization criterion we make use of the energy balance equation for the perturbations. The latter is obtained by multiplying Eq. (2) by $\Phi^*(x)$ and by subsequent integration from 0 to ∞. Integrating further the term involving the second derivative by parts we represent the energy balance in the form

$$\frac{\bar{a}_i^2}{2i}\left(\Phi^*\frac{d\Phi}{dx} - \Phi\frac{d\Phi^*}{dx}\right)\Big|_0^\infty - \int_0^\infty\left[Q_I^{(i)}(x) + Q_I^{(e)}(x)\right]_{\omega=\omega_0}|\Phi|^2dx$$
$$- \int_0^\infty \mathrm{Im}\,\omega\left[\frac{\partial Q_R(x)}{\partial\omega}\right]_{\omega=\omega_0}|\Phi|^2dx = 0. \tag{5}$$

The first term in this equation represents the outflow of energy from the region in which the oscillations are generated, while the second determines the magnitude of the energy exchange between the resonance particles and the perturbation. The contribution of the electrons to the energy balance diverges logarithmically for small x. However, for $x < x_A = \omega_0 L_S/k_y v_A$ (v_A is the Alfven velocity), the oscillations become nonpotential



ones, since for $x < x_A$, $\omega/k_\parallel > v_A$. It is well known that in this case the drift waves are damped. Therefore it appears to be sensible to cut off the integral in (5) at $x = x_A$. As a result the stabilization criterion may be written in the form

$$\Theta\Lambda(\Theta_{eff}) \geq \left(\frac{m\eta}{M}\right)^{1/2}\frac{a_i}{\bar{a}_i}\frac{1 - I_0 e^{-b}}{1+\eta - I_0 e^{-b}}\sqrt{\frac{\pi}{2}}\int_{\sqrt{\beta\eta}}^{\infty}\frac{dt}{t}\exp\left(-\frac{1}{\Theta_{eff}}J_i(t)\right), \tag{6}$$

where

$$\Lambda(\Theta_{eff}) = \int_0^\infty \exp\left(-\frac{1}{\Theta_{eff}}J_i(t)\right)dt, \qquad \beta = \frac{8\pi n_0 T_e}{H_0^2}.$$

On the other hand, it follows from [6] that for finite β the universal instability is stabilized even for $\Theta = 0$.

In [6] a function $\beta = \beta(k_y a_i)$ is obtained which separates the regions of stability and instability in the (β, $k_y a_i$) plane. We assume that the corresponding region of stability in any case remains stable also for $\Theta \neq 0$. Then, eliminating $k_y a_i$ from $\beta = \beta(k_y a_i)$ of [6] and substituting into (6) we obtain the desired stability condition $\Theta = \Theta(\beta)$. In explicit form this condition would have a very awkward form. The diagram shows the dependence $\Theta = \Theta(\beta)$ obtained by numerical integration for a deuterium plasma $(M/m)^{1/2} = 60$ and $\eta = 1$.

The author expresses his gratitude to A. A. Galeev and R. Z. Sagdeev for posing the problem and for their continued interest in this work.

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Translated by G. Volkoff
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