NONLINEAR EFFECTS IN THE ELECTRODYNAMICS OF SUPERCONDUCTORS IN AN ALTERNATING FIELD

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We investigate nonlinear effects in superconductors placed in a superposition of a constant electromagnetic field and an alternating one (of frequency $\omega \ll \Delta$). We show that even when $l \gg \xi_0$ it is necessary to take the impurities into account in nonlinear non-equilibrium problems. The behavior of the gap and of the nonlinear surface imepdance turns out to depend on the parameter $\lambda = (\tau\sqrt{2\Delta\omega})^{-1}$. We calculate the coordinate dependence of the gap harmonic of frequency ω and the nonlinear correction to the surface impedance. We show that the oscillations of the gap lead to a reversal of the sign of the nonlinear correction to the impedance on going through the frequency $\omega \sim \Delta^2/T_c$.

MUCH attention has recently been paid to phenomena occurring in superconductors under the influence of strong alternating electromagnetic fields. From the theoretical point of view, this question was considered in greatest detail by Gor'kov and Éliashberg[1], who proposed a general approach to the analysis of nonstationary phenomena in superconductors. In the present paper we employ the calculation procedure developed in[1] to investigate nonlinear effects in the electrodynamics of pure superconductors, i.e., superconductors satisfying the condition $l \gg \xi_0$, where l is the electron mean free path and ξ_0 is the correlation parameter. We investigate the coordinate dependence of the superconductivity parameter $\Delta(\mathrm{rt})$ and the influence of the constant magnetic field on the surface impedance of the superconductor. We show that even if $l \gg \xi_0$, allowance for the impurities is essential in principle in non-equilibrium nonlinear problems. Both the superconductivity parameter $\Delta(\mathrm{rt})$ and the surface impedance depend significantly on the parameter $\lambda = (\tau\sqrt{2\Delta\omega})^{-1}$, where τ is the free path time, ω the frequency of the alternating field, and Δ the equilibrium value of the gap.

The problem is solved for an infinite superconductive half-space, and the reflection of the electrons from the boundary is assumed to be specular. The field is considered in accordance with perturbation theory. We use a gauge $\operatorname{div}\mathbf{A} = 0$, which makes it possible to regard $\Delta(\mathrm{rt})$ as a reql quantity. The temperature is limited by the condition $1 - T/T_c \gg \kappa^2$.

1. ROLE OF IMPURITIES AND DEPENDENCE OF THE SUPERCONDUCTIVITY PARAMETER ON THE FIELD

Being interested in the correction $\Delta_1(\mathrm{rt}) = \Delta(\mathrm{rt}) - \Delta$ to its equilibrium value, we write down the equation for Δ, using the thermodynamic approach, up to second order in the field, and then carry out an analytic continuation to the real-frequency axis by the method of[1]; we then obtain, in the Fourier representation, the following equation[1)]:

$$L(k)\Delta_1(k) = \iint\limits_{-\infty}^{+\infty} \frac{dk_1 dk_2}{(2\pi)^2} \delta(k - k_1 - k_2) L_2(kk_1k_2)\mathbf{A}(k_1)\mathbf{A}(k_2), \quad (1.1)$$

where k stands for $(\mathbf{q}\omega)$, and the vector $\mathbf{q}$ is directed along the Z axis, which is perpendicular to the surface of the superconductor. $\mathbf{A}(k)$ in formula (1.1) is the solution of the linear equation for the field:

$$\mathbf{A}(k) = \frac{2H_{\mathrm{sur}}(\omega)}{q^2 + K(q\omega)} = 2H_{\mathrm{sur}}(\omega)Y(k), \quad (1.2)$$

$H_{\mathrm{sur}}(\omega)$ is the Fourier component of the field on the surface, and K(k) is the kernel connecting the field with the current in the linear approximation; this kernel was calculated in[3]:

$$\mathbf{j}(k) = -(c/4\pi)K(k)\mathbf{A}(k). \quad (1.3)$$

Analogously, the kernel L(k) is a "loop" of two Green's functions of the superconductor, in which an analytic continuation has been carried out to the real-frequency axis. The analytic expressions for the kernels L(k) and K(k) will be given later.

The kernel $L_2(kk_1k_2)$ is the product of three Green's functions of the superconductor. It is of greatest interest to us, since its properties determine the frequency region in which it is possible to go over in (1.1) to the static case. We shall see presently that when the limit $\omega_{1,2} \to 0$ is taken, $L_2(kk_1k_2)$ goes over into its static expression only if account is taken of the dependence on the electron mean free path l. This is connected with the fact that after averaging over the impurities, the dependence of the free-path time τ is contained in L_2 in such a way that the limiting transition $\omega \to 0$, $\tau \to \infty$ will depend on the sequence in which the limits are taken. This is easiest to see by considering in (1.1) large values of the momentum $qv \gg 1/\tau$ (v is the Fermi velocity), i.e., by investigating the behavior of the gap at distances $z \ll l$. It can be shown (see Appendix 1) that to average (1.1) over the impurities in this case it is not necessary to take into account the "ladder of dashed lines," and it suffices merely to replace the Green's function of the pure superconductor by the Green's functions of the alloy.

We write

$$L_2(kk_1k_2) \equiv L_2\begin{pmatrix} qq_1q_2 \\ \omega\omega_1\omega_2 \end{pmatrix} \quad (k = k_1 + k_2)$$

[1)]This equation was investigated for $\omega = 0$ and $l \gg \xi_0$ by Rusinov and Shapoval [2], whose notation as a rule coincides with ours.

in the form

$$L_2\begin{pmatrix} qq_1q_2 \\ \omega\omega_1\omega_2\end{pmatrix} = L_2^{(0)}\begin{pmatrix} qq_1q_2 \\ 000\end{pmatrix} + L_2^{(1)}\begin{pmatrix} qq_1q_2 \\ \omega\omega_1\omega_2\end{pmatrix}, \qquad (1.4)$$

where $L_2^{(0)}(qq_1q_2)$ is the static result. After rather cumbersome integrations we obtain for the quantity $L_2^{(1)}\begin{matrix} q & q_1 & q_2 \\ \omega & \omega & 0\end{matrix}$, which we shall need later on, under the condition that $qv \gg 1/\tau$ and $\omega_{1,2} \ll \Delta$, T, the following result (see Appendix 2):

$$L_2^{(1)}\begin{pmatrix} q & q_1 & q_2 \\ \omega & \omega & 0\end{pmatrix} = -\frac{i\pi\Delta^3}{8v}\left(\frac{ev}{c}\right)^2\frac{\omega}{2T}\operatorname{ch}^{-2}\frac{\Delta}{2T}\int_c\frac{d\varepsilon}{\tilde{\gamma}\gamma_1^2}$$

$$\times\left[\frac{1}{2}\frac{1-|qq_1|/qq_1}{|q_2|(\gamma_1+\tilde{\gamma}+i/\tau)} + \frac{1-|qq_2|/qq_2}{\gamma_1|q+q_1|+\tilde{\gamma}|q_2|+i|q_1|/\tau}\right]$$

$$+\begin{pmatrix} q \leftrightarrow q_1 \\ q_2 \leftrightarrow -q_2\end{pmatrix}, \quad \gamma_1(\varepsilon) = \gamma(\varepsilon-\omega). \qquad (1.5)$$

In formula (1.5), γ and $\tilde{\gamma}$ are the roots of $\sqrt{\epsilon^2-\Delta^2}$, determined with the cut $(-\infty, -\Delta)$ $(\Delta, +\infty)$ and taken respectively with the complex ϵ plane approached from below or from above. The unambiguous choice of the branches is determined by the condition $\gamma(0) = i\Delta$, from which it follows that all the roots have a positive imaginary part in the entire complex ϵ plane. The integration is along a contour that encompasses the cut $(-\infty, -\Delta)$ $(\Delta, +\infty)$ (Fig. 1).

We note now that at finite τ, none of the denominators in the square brackets of (1.5) has zeroes in the entire complex ϵ plane, since $\operatorname{Im}(\gamma_1, \tilde{\gamma}) \geq 0$. Bearing this in mind and using the condition $\omega \ll \Delta$ and the fact that the integrals in (1.5) converge within distances $\sim\omega$ near the points $|\epsilon| = \Delta$, we can transform the expression for $L_2^{(1)}$ into

$$L_2^{(1)}\begin{pmatrix} qq_1q_2 \\ \omega\omega 0\end{pmatrix} = -\frac{i\pi}{16v}\left(\frac{ev}{c}\right)^2\frac{\Delta}{2T}\operatorname{ch}^{-2}\frac{\Delta}{2T}$$

$$\times\left[\frac{1}{2q_2}\left(\frac{|q_1|}{q_1}-\frac{|q|}{q}\right)I_0(\lambda) + \frac{1}{q_1}\left(\frac{|q_2|}{q_2}-\frac{|q|}{q}\right)I_\beta(\lambda)\right]$$

$$+(q \leftrightarrow q_1, \; q_2 \leftrightarrow -q_2),$$

$$I_\beta(\lambda) = -\frac{i\pi}{1-\beta^2+i\lambda} \qquad (1.6)$$

$$+2\int_0^\infty\frac{dx}{(x^2+1)(x+\beta^2/x+i\lambda)} + 2\int_0^\infty\frac{dx}{(x^2-1)(x-\beta^2/x+i\lambda)},$$

$$I_0(\lambda) = I_\beta(\lambda)|_{\beta=0}, \quad \beta^2 = q/q_1, \quad \lambda = 1/\tau\sqrt{2\Delta\omega}. \qquad (1.7)$$

The integrals in (1.7) can be calculated in elementary fashion, but are rather complicated functions of the parameters β and λ. We do not present their expressions here, since the coordinate dependence of $\Delta_1(\mathbf{r}t)$ and the surface impedance can be calculated only in the limiting cases $\lambda \gtrless 1$. The limiting values of functions $I_\beta(\lambda)$ and $I_0(\lambda)$ are:

$$I_\beta(\lambda) \to \frac{i\pi\lambda}{\beta(1-\beta^2)}\cdot\left[\frac{\beta-1}{\beta+1}+i\frac{\beta^2+1}{\beta^2-1}\right] \text{ as } I_0(\lambda) \to -i\pi, \qquad (1.8)$$

$$I_0(\lambda) \to I_\beta(\lambda) \to -\frac{\pi}{\lambda}(1+i) \to 0 \text{ as } \lambda \to 0. \qquad (1.9)$$

It is important, however, that when $\lambda \to 0$ (i.e., $1/\tau \to 0$ at finite ω), we have $I_\beta(\lambda) \to 0$, whereas $I_0(\lambda) \to -i\pi$. This means that had we not introduced from the very outset the finite path time τ, and assumed $1/\tau = 0$, then the expression for the kernel $L_2(kk_1k_2)$ would not go over into its static expression as $\omega_{1,2} \to 0$. On the other hand, if $\lambda \to \infty$ ($\omega \to 0$ at finite τ), then $I_0(\lambda)$ and $I_\beta(\lambda)$ tend to zero, as they should.

A similar situation arises in all the higher-order kernels in the expansion of $\Delta(\mathbf{r}t)$ or of the current in powers of the field. Such a frequency dispersion at low frequencies and at a large electron mean free path is undoubtedly due to the presence of bound electron states near the surface of the metal[2)][4]. Since we take only terms $\sim H_0H_1$ into account in (1.1), this corresponds to a calculation of a "loop" of Green's functions (Fig. 2), expanded in a series in the field H_0 to first order only; in this loop, $\hat{G}(H_0)$ is the Green's functions of the superconductor in the presence of only a constant magnetic field. In the case of an impurity-free superconductor, these functions have singularities in H_0 due to the presence of surface levels, so that our series expansion of $\Delta(\mathbf{r}t)$ is, generally speaking, not valid. However, the presence of impurities in the superconductor, as we shall see, eliminates these singularities and makes it possible to solve the problem by perturbation theory. We note now that in the linear "responses" L(k) and K(k), the limiting transition $\omega \to 0$, $\tau \to \infty$ does not depend on the sequence in which the limits are taken, and they must therefore be calculated at $1/\tau = 0$.

Proceeding now to the solution of (1.1), we see that the frequency dispersion in the kernel L_2 becomes appreciable starting with a frequency $\Omega \sim \Delta/(\Delta\tau)^2$. As to the kernels L(k) and K(k), it will be shown below that in the temperature region $(-\infty, -\Delta)$ $(\Delta, +\infty)$, the frequency dispersion in these kernels will assume a role starting with the frequency $\Omega_0 \sim \Delta^2/T_c$. At sufficiently large τ (which we assume) we have $\Omega \ll \Omega_0$ and it is necessary to take into account only the dispersion of the kernel L_2. We shall calculate the coordinate dependence of the superconductivity parameter $\Delta(\mathbf{r}t)$ for this case, assuming that the field on the surface of the superconductor is given by $\mathbf{H}(t) = \mathbf{H}_0 + \mathbf{H}_1 \sin\omega t$ ($H_1 \ll H_0$). We confine ourselves to the case when the vectors $\mathbf{H}_0$ and $\mathbf{H}_1$ are parallel.

From (1.1) we have for the harmonic of Δ with frequency ω:

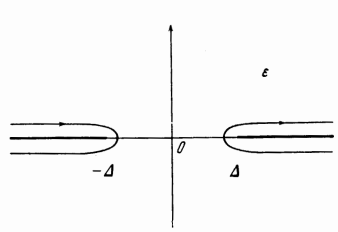

FIG. 1

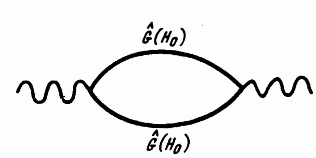

FIG. 2

[2)]This circumstance was pointed out to the author by L. P. Gor'kov.

$$\Delta_\omega(q)=\frac{8\pi i H_0 H_1}{L(q0)}\iint_{-\infty}^{\infty} dq_1\,dq_2\,\delta(q-q_1-q_2)L_2\begin{pmatrix} qq_1q_2 \\ \omega\omega 0\end{pmatrix}Y(q_10)Y(q_20). \tag{1.10}$$

For L(q0) and K(q0) we readily obtain

$$L(q0)=\pi T\sum_{\omega_n}\frac{1}{\tilde\omega_n}\left[1-2\frac{\omega_n^2}{\tilde\omega_n qv}\operatorname{arctg}\frac{qv}{2\tilde\omega_n}\right]; \tag{1.11}$$

here $\tilde\omega_n=\sqrt{\omega_n^2+\Delta^2}$, $\omega_n=\pi T(2n+1)$.

$$K(q0)=\frac{q_0^3}{|q|},\quad q_0^3=\frac{\pi p_0^2e^2\Delta}{c^2}\operatorname{th}\frac{\Delta}{2T}. \tag{1.12}$$

The quantity $L_2^{(0)}(qq_1q_2)$ was calculated in[2]:

$$L_2^{(0)}=\frac{\pi^2}{16v}\left(\frac{ev}{c}\right)^2\left\{\left[\frac{1}{q_2}\left(\frac{|q_1|}{q_1}-\frac{|q|}{q}\right)+\frac{1}{q_1}\left(\frac{|q_2|}{q_2}-\frac{|q|}{q}\right)\right]S_1\left(\frac{\Delta}{T}\right)+\frac{1}{q}\left(\frac{|q_1|}{q_1}+\frac{|q_2|}{q_2}\right)S_2\left(\frac{\Delta}{T}\right)\right\}, \tag{1.13}$$

$$S_{1,2}=\operatorname{th}\frac{\Delta}{2T}\pm\frac{\Delta}{2T}\operatorname{ch}^{-2}\frac{\Delta}{2T}. \tag{1.14}$$

Using (1.13) and the limiting expressions for $L_2^{(1)}\begin{pmatrix} qq_1q_2 \\ \omega\omega 0\end{pmatrix}$, we reduce (1.10), after simple transformations, to the form (H_{c_0} is the critical field at T = 0):

a) $\omega\ll\Omega\sim\Delta/(\Delta\tau)^2$

$$\frac{\Delta_\omega(q)}{\Delta}=-\frac{4\pi i}{q_0}\left(\frac{H_0H_1}{H_{c0}}\right)^2\left(\frac{\Delta_0}{\Delta}\right)^2\frac{1}{\operatorname{th}(\Delta/2T)L(q0)}\cdot \times\left\{F_1(q/q_0)\left[\operatorname{th}\frac{\Delta}{2T}+\left(1-\frac{1-i}{\lambda}\right)\frac{\Delta}{2T}\operatorname{ch}^{-2}\frac{\Delta}{2T}\right]+F_2(q/q_0)\left[\operatorname{th}\frac{\Delta}{2T}-\left(1-\frac{1-i}{\lambda}\right)\frac{\Delta}{2T}\operatorname{ch}^{-2}\frac{\Delta}{2T}\right]\right\}; \tag{1.15}$$

b) $\Omega\ll\omega\ll\Delta$

$$\frac{\Delta_\omega(q)}{\Delta}=-\frac{4\pi i}{q_0}\left(\frac{H_0H_1}{H_{c0}}\right)^2\left(\frac{\Delta_0}{\Delta}\right)^2\frac{1}{\operatorname{th}(\Delta/2T)L(q0)}\cdot \times\left[F_1(q/q_0)\left(S_1\left(\frac{\Delta}{T}\right)-\frac{\Delta}{4T}\operatorname{ch}^{-2}\frac{\Delta}{2T}\right)+F_2(q/q_0)S_2\left(\frac{\Delta}{T}\right)\right]. \tag{1.16}$$

Here

$$F_1(y)=\int_0^\infty\frac{x\,dx}{(x^3+1)((x+y)^3+1)}=\frac{2\pi}{3\sqrt3}\begin{cases}{}^1/_3(1-y), & 0<y\ll1,\\ y^{-3}, & y\gg1,\end{cases} \tag{1.17}$$

$$F_2(y)=\frac{1}{2y}\int_0^y\frac{x(y-x)\,dx}{(x^3+1)((y-x)^3+1)}=\begin{cases}{}^1/_{12}y^2, & 0<y\ll1,\\ 2\pi/3\sqrt{3}y^3, & y\gg1.\end{cases}$$

It is seen from (1.15) and (1.16) that on going to the frequency $\omega\sim\Omega$ the function $\Delta_\omega(q)$ changes by one order of magnitude. The coordinate dependence of $\Delta_\omega(z)$ is determined by the inverse Fourier transformation of formulas (1.15) and (1.16). Just as in the static case[2], there exist two temperature regions, in each of which the expression for L(q0) simplifies:

$$\begin{gathered}1-T/T_c\ll\varkappa^{4/5}\quad L(q)\approx\frac{1-T/T_c}{\varkappa^2}(2\varkappa^2+(q\delta_L)^2),\\ 1-T/T_c\gg\varkappa^{4/5},\quad L(q)\approx\ln(q\xi_0)\approx\ln(q_0\xi_0)\end{gathered} \tag{1.18}$$

δ_L is the London depth of penetration: $\delta_L=\kappa v/\Delta\sqrt6$.

In the first region ($\kappa^2\ll1-T/T_c\ll\kappa^{4/5}$) the major contribution to the integrals with respect to q is made by $q\sim\kappa/\delta_L\ll q_0$. Replacing $F_1(y)$ and $F_2(y)$ by their values at y = 0, we obtain

a) $\omega\ll\Omega$

$$\frac{\Delta_\omega(z)}{\Delta}=-\frac{4\pi^2 i}{9\sqrt6}\left(\frac{H_0H_1}{H_{c0}}\right)^2\frac{\varkappa}{1-T/T_c}\frac{1}{q_0\delta_L}\left(\frac{\Delta_0}{\Delta}\right)^2\left(1-\frac{1-i}{\lambda}\right)e^{-\sqrt{2}\varkappa z/\delta_L} \tag{1.19}$$

b) $\omega\gg\Omega$

$$\frac{\Delta_\omega(z)}{\Delta}=-\frac{4\pi^2 i}{9\sqrt6}\left(\frac{H_0H_1}{H_{c0}}\right)^2\frac{\varkappa}{1-T/T_c}\frac{1}{q_0\delta_L}\left(\frac{\Delta_0}{\Delta}\right)^2 e^{-\sqrt{2}\varkappa z/\delta_L} \tag{1.20}$$

Thus, in this temperature region the first harmonic of the gap decreases in general exponentially.

In the temperature region $1-T/T_c\gg\kappa^{4/5}$ we have

a) $\omega\ll\Omega$

$$\frac{\Delta_\omega(z)}{\Delta}=-4\pi i\left(\frac{H_0H_1}{H_{c0}}\right)^2\left(\frac{\Delta_0}{\Delta}\right)^2\frac{1}{\operatorname{th}(\Delta/2T)\ln(q_0\xi_0)}\cdot\left\{\Phi_1(zq_0)\left[\operatorname{th}\frac{\Delta}{2T}+\left(1-\frac{1-i}{\lambda}\right)\frac{\Delta}{2T}\operatorname{ch}^{-2}\frac{\Delta}{2T}\right]+\Phi_2(zq_0)\left[\operatorname{th}\frac{\Delta}{2T}-\left(1-\frac{1-i}{\lambda}\right)\frac{\Delta}{2T}\operatorname{ch}^{-2}\frac{\Delta}{2T}\right]\right\}; \tag{1.21}$$

b) $\omega\gg\Omega$

$$\frac{\Delta_\omega(z)}{\Delta}=-4\pi i\left(\frac{H_0H_1}{H_{c0}}\right)^2\left(\frac{\Delta_0}{\Delta}\right)^2\frac{1}{\operatorname{th}(\Delta/2T)\ln(q_0\xi_0)}\times\left\{\Phi_1(zq_0)\left(S_1\left(\frac{\Delta}{T}\right)-\frac{\Delta}{4T}\operatorname{ch}^{-2}\frac{\Delta}{2T}\right)+\Phi_2(zq_0)S_2\left(\frac{\Delta}{T}\right)\right\} \tag{1.22}$$

$$\Phi_{1,2}(z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F_{1,2}(y)e^{iyz}\,dy, \tag{1.23}$$

$$\Phi_1(z)=\begin{cases}4\pi/27-(1/9\pi)\psi'(1/3)\approx0.108, & z=0,\\ 2\sqrt3/27z^2, & z\gg1;\end{cases}$$

$$\Phi_2(z)=\begin{cases}5\pi/216\approx0.073, & z=0,\\ 4/\pi z^6, & z\gg1.\end{cases}$$

We see that at distances $zq_0\gg1$, i.e., $q_0^{-1}\ll z\ll\xi_0$, the amplitude of the first harmonic of the gap decreases like $1/z^2$. On the other hand, at distances exceeding ξ_0, the decrease is mainly exponential.

2. DEPENDENCE OF THE SURFACE IMPEDANCE OF A SUPERCONDUCTOR ON AN EXTERNAL CONSTANT MAGNETIC FIELD

As usual[5], we define the surface impedance by the relation

$$\mathbf{E}(\omega)=\zeta(\omega)[\mathbf{H}(\omega)\mathbf{n}]. \tag{2.1}$$ *

Here $\mathbf{E}(\omega)$ and $\mathbf{H}(\omega)$ are the values of the amplitudes of the alternating electric and magnetic fields on the surface of the superconductor, and $\mathbf{n}$ is the inward normal to the metal. As in the preceding section, we assume that the magnetic field on the surface of the superconductor is given by

$$\mathbf{H}_{\rm sur}(t)=\mathbf{H}_0+\mathbf{H}_1\sin\omega t.$$

We shall solve Maxwell's equations by perturbation theory. In the zeroth approximation $\mathbf{H}_0=0$ and we obtain for the impedance the well-known relation

$$\zeta_0(\omega)=-2i\frac{\omega}{c}\int_{-\infty}^{+\infty}\frac{dq}{2\pi}Y(q\omega). \tag{2.2}$$

If $\mathbf{H}_0\neq0$, we include in the current the next higher nonlinear term of the expansion in $\mathbf{A}$:

$$\mathbf{j}(q\omega)=\mathbf{j}_1(q\omega)+\mathbf{j}_3(q\omega). \tag{2.3}$$

Substituting (2.3) in Maxwell's equations, we obtain for the correction to the electric field on the surface

$$\mathbf{E}_1(\omega)=i\frac{\omega}{c}\int_{-\infty}^{+\infty}\frac{dq}{2\pi}Y(q\omega)\frac{4\pi}{c}\mathbf{j}_3(q\omega). \tag{2.4}$$

We shall calculate the difference between the surface impedance of the superconductor in the presence of a constant magnetic field and the value of the impedance at $\mathbf{H}_0=0$, referred to the surface resistance of the normal metal. This is precisely the quantity measured in the experiments. Thus

*$[\mathbf{H}(\omega)\mathbf{n}]\equiv\mathbf{H}(\omega)\times\mathbf{n}$.

$$\eta = (\zeta(H_0) - \zeta(0)) / R_N, \tag{2.5}$$

$$R_N = \frac{2}{3\sqrt{3}}\frac{\omega}{c}\delta_s = \frac{2}{3\sqrt{3}}\frac{\omega}{c}\left(\frac{3\pi^2 Ne^2}{mc^2}\frac{\omega}{v}\right)^{-1/3}. \tag{2.6}$$

Taking (2.4) into account, we obtain

$$\eta = -\frac{\omega}{c}\frac{2}{H_1 R_N}\int_{-\infty}^{+\infty}\frac{dq}{2\pi}Y(q\omega)\frac{4\pi}{c}j_3(q\omega). \tag{2.7}$$

The expression for the current j_3 will be written in the form:

$$\frac{4\pi}{c}j_3(k) = \iiint_{-\infty}^{+\infty}\frac{dk_1\,dk_2\,dk_3}{(2\pi)^4}\delta(k-k_1-k_2-k_3)K_3(kk_1k_2k_3)A(k_1)$$

$$\times A(k_2)A(k_3) + \iint_{-\infty}^{+\infty}\frac{dk_1\,dk_2}{(2\pi)^2}\delta(k-k_1-k_2)K_2(kk_1k_2)A(k_1)\Delta_1(k_2),$$

$$A(k) = |\mathbf{A}|. \tag{2.8}$$

The meaning of the kernels K_2 and K_3 will be made clear subsequently. Substituting (2.8) in (2.7) and using (1.1), we obtain the following general expression for the correction η:

$$\eta = -3i\frac{\omega}{c}\frac{1}{R_N}\delta_1(\omega), \tag{2.9}$$

$$\delta_1(\omega) = \frac{H_0^2}{3\pi^3}\Big[2\iint\iint_{-\infty}^{+\infty} dq\,dq_1\,dq_2\,dq_3\delta(q-q_1-q_2-q_3)K_3\begin{pmatrix} q & q_1 & q_2 & q_3 \\ \omega & \omega & 0 & 0\end{pmatrix}$$

$$\times Y(q\omega)Y(q_1\omega)Y(q_20)Y(q_30) + \iint\iint_{-\infty}^{+\infty} dq\,dq_1\,dq_2\,dq_3\delta(q-q_1-q_2-q_3)$$

$$\times K_3\begin{pmatrix} q & q_1 & q_2 & q_3 \\ \omega & 0 & \omega & 0\end{pmatrix}Y(q\omega)Y(q_10)Y(q_2\omega)Y(q_30) + 2\int_{-\infty}^{+\infty}\frac{dq_2}{L(q_20)}$$

$$\times\iint_{-\infty}^{+\infty} dq\,dq_1\delta(q-q_1-q_2)K_2\begin{pmatrix} q & q_2 & q_1 \\ \omega & \omega & 0\end{pmatrix}Y(q\omega)Y(q_10)$$

$$\times\iint_{-\infty} dq_3\,dq_4\delta(q_2-q_3-q_4)L_2\begin{pmatrix} q_2 & q_3 & q_4 \\ \omega & \omega & 0\end{pmatrix}Y(q_3\omega)Y(q_40)$$

$$+\int_{-\infty}^{+\infty}\frac{dq_2}{L(q_20)}\iint_{-\infty}^{+\infty} dq\,dq_1\delta(q-q_1-q_2)K_2\begin{pmatrix} q & q_1 & q_2 \\ \omega & \omega & 0\end{pmatrix}Y(q\omega)Y(q_1\omega)$$

$$\times\iint_{-\infty}^{+\infty} dq_3\,dq_4\delta(q_2-q_3-q_4)L_2^{(0)}(q_2q_3q_4)Y(q_30)Y(q_40)\Big]. \tag{2.10}$$

In deriving (2.10) we used the symmetry properties of the kernels L_2, K_2, and K_3:

$$L_2(kk_1k_2) = L_2(kk_2k_1),$$
$$K_2(kk_1k_2) = K_2(kk_2k_1), \quad K_3(kk_1k_2k_3) = K_3(kk_3k_2k_1).$$

If $\omega = 0$, then $\delta_1(\omega)$ is pure real, and (2.10) coincides with the corresponding expression obtained by Rusinov and Shapoval[2] for the correction to the depth of penetration of a constant static magnetic field. If $\omega \neq 0$, then $\delta_1(\omega)$ is a complex quantity whose imaginary part determines the nonlinear correction to the absorption of the electromagnetic waves. It is our purpose to calculate the first term of the expansion of $\delta_1(\omega)$ in powers of ω. The result depends significantly on the relations between the different parameters that characterize the superconductor. Actually, we wish to take into account the frequency dispersion in the kernels L, K, L_2, K_2, and K_3. There are several characteristic frequencies, starting with which it is necessary to take into account the frequency dispersion in each of these kernels.

First among these is the dispersion connected with the impurities, which was discussed in the preceding section. It plays a role in the kernels L_2, K_2, and K_3, starting with the frequency $\Omega \sim \Delta/(\Delta\tau)^2$, which, as we shall see, depends strongly on the temperature and the number of impurities in the superconductor.

Second, there is the frequency dispersion in the kernel K(k) which enters in Y(k); this dispersion is connected with the fact that the penetration of the electromagnetic field into the superconductor changes its character with increasing alternating-field frequency, in which case it is governed not by the equation for the Meissner effect but by the equations for the skin effect. Mathematically this means that the first and second terms in the expression for the kernel K(k) (see[3])

$$K(q\omega) = \frac{q_0^3}{|q|} - i\frac{\omega}{c}\frac{p_0^2e^2}{c^2}\Big[1 - \operatorname{th}\frac{\Delta}{2T}$$

$$+\frac{\Delta}{T}\operatorname{ch}^{-2}\left(\frac{\Delta}{2T}\right)\ln 2\sqrt{\frac{2\Delta}{\omega}} - 2\frac{\Delta}{T}P\left(\frac{\Delta}{T}\right)\Big],$$

$$P(x) = \int_1^{+\infty}\frac{d\varepsilon}{\varepsilon^2-1}\frac{\operatorname{ch}x\varepsilon - \operatorname{ch}x}{(\operatorname{ch}x\varepsilon+1)(\operatorname{ch}x+1)} \tag{2.11}$$

which describe the Meissner effect and the anomalous skin effect, respectively, become equalized at the frequency $\Omega_1 \sim \Delta^2/T_c$.

Finally, at the frequency Ω_0, which is also of the order of Δ^2/T_c in a "pure" superconductor at temperatures $1 - T/T_c \gg \kappa^2$, the frequency dispersion comes into play in the kernel $L(q\omega)$, i.e., starting with this frequency, an important influence on the dynamics of the superconductor is exerted by oscillations of the superconductivity parameter. Indeed, the expression for $L(q\omega)$, like (1.4), can be written in the form

$$L(q\omega) = L^{(0)}(q0) + L^{(1)}(q\omega), \tag{2.12}$$

$L^{(0)}(q0)$ is the static result of (1.11).

Calculating $L^{(1)}(q\omega)$ in analogy with the calculation of $L_2^{(1)}$ in Appendix 2, we can obtain

$$L^{(1)}(q\omega) = \frac{\pi^2\omega\Delta}{8|q|vT}\operatorname{ch}^{-2}\frac{\Delta}{2T} + \frac{i\pi\omega}{2|q|v}\Big[\operatorname{th}\frac{\Delta}{2T} - \operatorname{th}\frac{\sqrt{\Delta^2+(qv/2)^2}}{2T}$$

$$-\frac{\Delta}{T}\operatorname{ch}^{-2}\frac{\Delta}{2T}\ln 2\sqrt{\frac{2\Delta}{\omega}} + 2\frac{\Delta}{T}P\left(\frac{\Delta}{T}\right) + \frac{\Delta}{2T}M\left(\frac{\Delta}{2T},\frac{qv}{2\Delta}\right)\Big]. \tag{2.13}$$

This result is valid if

$$\omega \ll \Delta,\ T, \quad qv \gg \sqrt{2\Delta\omega} \gg 1/\tau.$$

Here

$$M(xy) = \int_{\sqrt{1+y^2}}^{\infty}\frac{d\varepsilon}{\varepsilon^2-1}\frac{1}{\operatorname{ch}^2 x\varepsilon},$$

$$M(0y) = \ln\frac{\sqrt{y^2+1}+1}{y}, \quad x \ll 1,$$

$$M(xy) = -(1 - \operatorname{th}x\sqrt{1+y^2})/x, \quad x \gg 1. \tag{2.14}$$

If we assume that $\Delta \ll T$, then (2.13) coincides with the formula obtained by Kemoklidze[6]. A comparison of (2.13) and (1.11) shows that $L^{(1)}(q\omega)$ becomes of the order of $L^{(0)}(q\omega)$ at a frequency $\Omega_0 \sim \Delta^2/T_c$.

In calculating the correction to the impedance, we first take into account the contribution made to η by the dispersion on the impurities. Putting $\omega = 0$ in $Y(q\omega)$ and $K(q\omega)$ and using the relations

$$K_2^{(1)}\begin{pmatrix} q & q_1 & q_2 \\ \omega & \omega & 0\end{pmatrix} = 16\pi\frac{mp_0}{2\pi^2}L_2^{(1)}\begin{pmatrix} q & q_1 & q_2 \\ \omega & \omega & 0\end{pmatrix},$$

$$K_2^{(0)}(qq_1q_2) = 16\pi \frac{mp_0}{2\pi^2} L_2^{(0)}(q_2 - q_1q),$$

$$L_2^{(1)}\begin{pmatrix} q & q_1 & q_2 \\ \omega & \omega & 0 \end{pmatrix} = L_2^{(1)}\begin{pmatrix} q_1 & q & -q_2 \\ \omega & \omega & 0 \end{pmatrix},$$

we reduce the expression for $\delta_1(\omega)$ to the form

$$\delta_1(\omega) - \delta_1(0)$$

$$= \frac{H_0^2}{3\pi^3} \iiiint\limits_{-\infty}^{+\infty} dq\, dq_1\, dq_2\, dq_3 \delta(q - q_1 - q_2 - q_3) Y(q) Y(q_1) Y(q_2) Y(q_3)$$

$$\times \left[K_3^{(1)}\begin{pmatrix} q & q_1 & q_2 & q_3 \\ \omega & 0 & \omega & 0 \end{pmatrix} + 2K_3^{(1)}\begin{pmatrix} q & q_1 & q_2 & q_3 \\ \omega & \omega & 0 & 0 \end{pmatrix} \right]$$

$$+ \frac{H_0^2}{3\pi^3} 16\pi \frac{mp_0}{2\pi^2} \int\limits_{-\infty}^{+\infty} \frac{dq}{L(q0)} \iint\limits_{-\infty}^{+\infty} dq_1\, dq_2 \delta(q - q_1 - q_2) L_2^{(0)}(qq_1q_2) Y(q_1) Y(q_2) \cdot$$

$$\times \iint\limits_{-\infty}^{+\infty} dq_3\, dq_4 \delta(q - q_3 - q_4) Y(q_3) Y(q_4)$$

$$\times \left[4L_2^{(1)}\begin{pmatrix} q & q_3 & q_4 \\ \omega & \omega & 0 \end{pmatrix} + L_2^{(1)}\begin{pmatrix} q_3 & -q_4 & q \\ \omega & \omega & 0 \end{pmatrix} \right]. \qquad (2.15)$$

The calculation of the kernels $K_3^{(1)}$ at $\lambda \lesssim 1$ is similar to that used in Appendix 2, but is much more cumbersome. As a result we can obtain from (2.15) the following limiting formulas:

a) $\bar{\omega} \ll \Omega$

$$\delta_1(\omega) - \delta_1(0) = \frac{1-i}{2\pi^3\lambda}\left(\frac{\Delta_0}{\Delta}\right)^2 \frac{\Delta}{T} \frac{1 - \mathrm{th}^2(\Delta/2T)}{\mathrm{th}^2(\Delta/2T)} \frac{1}{q_0}\left(\frac{H_0}{H_{c0}}\right)^2$$

$$\times \left[\frac{q_0 v}{(2\pi)^2} \frac{1}{\omega\lambda^2} C_2 + \frac{5}{3} \int\limits_{-\infty}^{+\infty} \frac{dx}{L(q_0x)} A_1(x)(F_1(x) - F_2(x)) \right]; \quad (2.16)$$

b) $\omega \gg \Omega$

$$\delta_1(\omega) - \delta_1(0) = \frac{1}{\pi^3}\left(\frac{\Delta_0}{\Delta}\right)^2 \frac{\Delta}{T} \frac{1 - \mathrm{th}^2(\Delta/2T)}{\mathrm{th}^2(\Delta/2T)} \frac{1}{q_0}\left(\frac{H_0}{H_{c0}}\right)^2$$

$$\times \left[-\frac{2i}{\pi^3} \frac{q_0 v}{\omega} C_1 + \frac{1}{3} \int\limits_{-\infty}^{+\infty} \frac{dx}{L(q_0x)} A_1(x)(F_1(x) - F_2(x)) \right]. \quad (2.17)$$

In these formulas, $A_1(x) = F_1(x)S_1(\Delta/T) + F_2(x)S_2(\Delta/T)$, $C_2 = 0.69$ and $C_1 = 0.056$ are the values of certain of the definite integrals, which were determined numerically. In calculating the integrals in (2.16) and (2.17), as in the preceding section, two temperature regions appear. If $1 - T/T_c \ll \kappa^{4/5}$, we assume the arguments of $F_1(x)$ and $F_2(x)$ to be equal to zero and take into account the pole in $L(q_0x)$. As a result we obtain

a) $\omega \ll \Omega$, $\varkappa^2 \ll 1 - T/T_c \ll \varkappa^{4/5}$

$$\delta_1(\omega) - \delta_1(0)$$

$$= 10^{-2}\delta_0 \left(\frac{H_0}{H_{c0}}\right)^2 \frac{1-i}{\lambda(1 - T/T_c)^{3/2}} \left[\frac{0{,}17}{\lambda^2} \frac{T_c}{\omega\varkappa} + \frac{0{,}65\varkappa^{1/3}}{(1 - T/T_c)^{2/3}} \right]; \qquad (2.18)$$

b) $\omega \gg \Omega$, $\varkappa^2 \ll 1 - T/T_c \ll \varkappa^{4/5}$

$$\delta_1(\omega) - \delta_1(0)$$

$$= 10^{-2}\delta_0 \left(\frac{H_0}{H_{c0}}\right)^2 \frac{1}{(1 - T/T_c)^{3/2}} \left[-i0{,}91 \frac{T_c}{\omega\varkappa} + \frac{2{,}1\varkappa^{4/3}}{(1 - T/T_c)^{2/3}} \right]; \qquad (2.19)$$

δ_0 is the coefficient in the formula for the depth of penetration near T_c: $\delta = \delta_0(1 - T/T_c)^{-1/2}$.

In the temperature region $1 - T/T_c \gg \kappa^{4/5}$, replacing $L(q_0x)$ by $\ln(q_0\xi_0)$ and using the results of the numerical calculations of[2]

$$\int_0^\infty F_1^2(x)\,dx = 0{,}066, \quad \int_0^\infty F_2^2(x)\,dx = 0{,}011, \quad \int_0^\infty F_1(x)F_2(x)\,dx = 0{,}010,$$

we obtain

a) $\omega \ll \Omega$, $1 - T/T_c \gg \varkappa^{4/5}$

$$\delta_1(\omega) - \delta_1(0) = 10^{-3} \frac{1-i}{\lambda q_0}\left(\frac{\Delta_0}{\Delta}\right)^2 \frac{\Delta}{T} \frac{1 - \mathrm{th}^2(\Delta/2T)}{\mathrm{th}^2(\Delta/2T)} \left(\frac{H_0}{H_{c0}}\right)^2$$

$$\times \left[0{,}27 \frac{q_0 v}{\omega\lambda^2} + \frac{3S_1(\Delta/T) - 0{,}53S_2(\Delta/T)}{\ln(q_0\xi_0)} \right]; \qquad (2.20)$$

b) $\omega \gg \Omega$, $1 - T/T_c \gg \varkappa^{4/5}$

$$\delta_1(\omega) - \delta_1(0) = 10^{-3} \frac{1}{q_0}\left(\frac{\Delta_0}{\Delta}\right)^2 \frac{\Delta}{T} \frac{1 - \mathrm{th}^2(\Delta/2T)}{\mathrm{th}^2(\Delta/2T)} \left(\frac{H_0}{H_{c0}}\right)^2$$

$$\times \left[-0{,}11i \frac{q_0 v}{\omega} + \frac{0{,}64S_1(\Delta/T) + 0{,}052S_2(\Delta/T)}{\ln(q_0\xi_0)} \right]. \qquad (2.21)$$

We now take into account the contribution made to η by the dispersion of the kernels $K(q\omega)$ and $L(q\omega)$. The dispersion of these kernels becomes significant if the condition $\Omega_0 \sim \Delta^2/T_c \ll \Omega_1 \sim \Delta/(\Delta\tau)^2$ is satisfied, as is possible near T_c. After certain transformations of (2.10), we obtain

$$\eta = -i\left(\frac{H_0}{H_{c0}}\right)^2 \frac{C^{-1/3}(\Delta/T)}{\mathrm{th}^2(\Delta/2T)} \frac{9\sqrt{3}}{8\pi^3} \frac{q_0^6}{\delta_s} \tilde{\eta}, \qquad (2.22)$$

$$\tilde{\eta} = -\frac{1}{(2\pi)^2} \frac{v}{\Delta_0} \iiiint\limits_{-\infty}^{+\infty} dq_1\, dq_2\, dq_3\, dq_4 \delta(q_1 + q_2 + q_3 + q_4) \tilde{K}_3^{(0)}(q_1q_2q_3q_4)$$

$$\times Y(q_1\omega)\,[Y(q_20)Y(q_30)Y(q_4\omega) + Y(q_20)Y(q_3\omega)Y(q_40)$$

$$+ Y(q_2\omega)Y(q_30)Y(q_40)] + \frac{2}{3}\int\limits_{-\infty}^{+\infty} \frac{dq}{L(q\omega)} A_1^2(q\omega)$$

$$+ \frac{1}{3}\int\limits_{-\infty}^{+\infty} \frac{dq}{L(q0)} A_0(q\omega) A_2(q\omega). \qquad (2.23)$$

$C(\Delta/T)$ denotes here the expression in the square brackets in (2.11),

$$A_0(q0) = \iint\limits_{-\infty}^{+\infty} dq_1\, dq_2 \delta(q + q_1 + q_2) \tilde{L}_2^{(0)}(qq_1q_2) Y(q_10) Y(q_20),$$

$$A_1(q\omega) = \iint\limits_{-\infty} dq_1\, dq_2 \delta(q + q_1 + q_2) \tilde{L}_2^{(0)}(qq_1q_2) Y(q_10) Y(q_2\omega),$$

$$A_2(q\omega) = \iint\limits_{-\infty}^{+\infty} dq_1\, dq_2 \delta(q + q_1 + q_2) \tilde{L}_2^{(0)}(qq_1q_2) Y(q_1\omega) Y(q_2\omega),$$

$$\tilde{L}_2^{(0)}(qq_1q_2) = \left(S_2 - \frac{1}{2}\left(\frac{\Delta}{\Delta_0}\right)^2 S_4\right) \frac{|q_1| + |q_2| - |q|}{q_1q_2} - + \frac{1}{2}\left(\frac{\Delta}{\Delta_0}\right)^2 S_4 \frac{1}{q}\left(\frac{|q_1|}{q_1} + \frac{|q_2|}{q_2}\right),$$

$$\tilde{K}_3^{(0)} = \left(S_4 - \frac{3}{4}\left(\frac{\Delta}{\Delta_0}\right)^2 S_6\right)\left(\frac{|q_1|}{q_2(q_1 + q_4)} + \frac{|q_4|}{q_3(q_1 + q_4)} + \frac{|q_1 + q_2|}{q_2q_3}\right)$$
$$+ (1234) + (13)(24) + (4321) + (123) + (432) + (12) + (34)$$
$$- \frac{3}{4}\left(\frac{\Delta}{\Delta_0}\right)^2 S_6 \frac{|q_1| + |q_3| - |q_2| - |q_4|}{(q_1 + q_2)(q_2 + q_3)}.$$

In these formulas, S_n are the dimensionless temperature functions introduced in[2] (which must not be confused with (1.14)); (abc) denotes a permutation of the arguments, with the aid of which the corresponding term can be obtained from the first. For example, (123) denotes the substitution $q_1 \to q_2 \to q_3$, $q_3 \to q_1$.

Concrete calculations of η using formulas (2.23) are exceedingly cumbersome. Depending on the values of the parameters, we used for the kernels K and L one of the limiting expressions, and then we evaluated the integrals of (2.23) by expansion in powers of a suitable small parameter. For example, when $\omega \ll \Omega_0$ the second term of (2.11) is small compared with the first, i.e., $q_0 \gg \delta_s^{-1}$, corresponding to the fact that the penetration of the field into the superconductor is determined by the Meissner effect, and consequently we have in the integrals (2.23) the small parameter $(q_0\delta_S)^{-1} \ll 1$, in terms of which the expansion was carried out. In the same frequency region, for the kernel $L(q\omega)$, the second term in (2.13) was also small compared with $L^{(0)}(q0)$.

In the other limiting case $\omega \gg \Omega_0$, the kernel $K(q\omega)$ contains the parameter $q_0\delta_S \ll 1$. In the kernel $L(q\omega)$, in this frequency region, $L^{(1)}(q\omega)$ becomes the principal term. Since an important role is played here by the values $qv \gg \Delta$, we can use the expression obtained by Abrahams and Tsuneto[7]:

$$L^{(1)}(q\omega) = -i\pi\omega/8T_c.$$

This formula is obtained from (2.13) under the condition $\omega \ll \Delta \ll qv \ll T_c$.

We present now final expressions for the correction to the impedance at different values of the parameters (we used in the calculations the condition $\kappa \ll 1$). The temperature region is $\kappa^2 \ll 1 - T/T_c \ll \kappa^{4/5}$.

1) $\omega \ll \Omega_0 \sim \Delta^2/T_c$

$$\eta = \left(\frac{H_0}{H_{c0}}\right)^2 \Big[-i5{,}2\cdot 10^{-2}\left(\frac{\omega}{\varkappa^2 T_c}\right)^{1/3}(1-T/T_c)^{-1} - i\cdot 10^{-1}\left(\frac{\omega}{T_c}\right)^{1/3}\varkappa^{2/3}(1-T/T_c)^{-13/6} + 10^{-3}\varkappa^{-2/3}\left(\frac{\omega}{T_c}\right)^{4/3}(1-T/T_c)^{-2} + 2{,}2\cdot 10^{-3}\varkappa^{2/3}\left(\frac{\omega}{T_c}\right)^{4/3}(1-T/T_c)^{-19/6} + 2{,}9\cdot 10\varkappa^{2/3}\left(\frac{\omega}{T_c}\right)^{4/3}(1-T/T_c)^{-19/6}\ln^2(2\Delta/\omega)\Big], \quad (2.24)$$

2) $\omega \gg \Omega_0 \sim \Delta^2/T_c$

$$\eta = \left(\frac{H_0}{H_{c0}}\right)^2 \Big[(\sqrt{3}+i)\cdot 2{,}9\cdot 10^{-2}\varkappa^{-2/3}\frac{T_c}{\omega}(1-T/T_c)^{1/3} + (\sqrt{3}+i)\cdot 3{,}2\cdot 10^{-1}\varkappa^{2/3}\frac{T_c}{\omega}(1-T/T_c)^{-5/6} + (i-1)\cdot 2{,}3 \times 10\varkappa^{2/3}\left(\frac{T_c}{\omega}\right)^{13/6}(1-T/T_c)^{1/2}\Big]. \quad (2.25)$$

The different terms in these formulas are connected with different terms of the initial formula (2.23). It is important to note that the last term in (2.25) describes the gap oscillations and corresponds to the second term in (2.23). We see that it makes the main contribution to η, and its comparison with the terms of (2.24) shows that the real and imaginary parts of η reverse sign with increasing frequency on going through $\omega \sim \Omega_0$. This change of sign is connected with allowance for the dependence of the kernel $L(q\omega)$ on ω in the second term of (2.23).

Let us finally write the expression for η in the region $\kappa^{4/5} \ll 1 - T/T_c \ll 1$, $\omega \ll \Omega_0$:

$$\eta = \left(\frac{H_0}{H_{c0}}\right)^2 \Big[-i2{,}3\cdot 10^{-2}\left(\frac{\omega}{\varkappa^2 T_c}\right)^{1/3}(1-T/T_c)^{-1} - i8{,}5\cdot 10^{-3}\frac{1}{\ln(1/\varkappa)}\left(\frac{\omega}{T_c}\right)^{1/3}(1-T/T_c)^{-4/3} + 10^{-3}\varkappa^{-2/3}\left(\frac{\omega}{T_c}\right)^{4/3} \times (1-T/T_c)^{-2} + 2{,}4\cdot 10^{-2}\left(\frac{\omega}{T_c}\right)^{4/3}(1-T/T_c)^{-7/3}\frac{1}{\ln(1/\varkappa)} + 10\varkappa^{2/3}\left(\frac{\omega}{T_c}\right)^{4/3}(1-T/T_c)^{-19/6}\ln^2(2\Delta/\omega)\Big]. \quad (2.26)$$

Formulas (2.24)—(2.26) can be illustrated by a qualitative plot (see Fig. 3).

In the temperature region $1 - T/T_c \sim 1$ the formulas for η differ from (2.6) only in that they depend little on

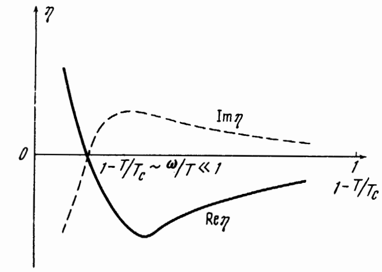

FIG. 3

the temperature and are much more complicated in form.

In the temperature region $1 - T/T_c \ll \kappa^2$, the question of the influence of the constant magnetic field on the surface impedance of a superconductor was investigated by Kemoklidze[6], who took into account only the second term of formula (2.23). As regards the other theoretical papers, for example[8,9], only the influence of a constant magnetic field on the spectrum of the quasiparticles was taken into account there and the oscillations of the ordering parameter and the contribution of these oscillations to the surface impedance of the superconductor were not investigated. Our results (Fig. 3) show that the gap oscillations lead to interesting features in the behavior of the surface impedance.

There are presently reports of experimental investigations of nonlinear effects in superconductors, particularly the influence of a constant magnetic field on the surface impedance of a superconductor[3]. However, a quantitative comparison of our formulas with the corresponding experiments is impossible, for several reasons. First, as we have already seen, all the nonlinear effects in superconductors are strongly influenced by even negligible amounts of ordinary impurities[4]; second, the existing experiments were performed with superconductors of the intermediate type, $\kappa \sim 1$, whereas our formulas are applicable to distinct Pippard superconductors, $\kappa \ll 1$. Nonetheless, the obtained formulas are in qualitative agreement with experiment. In particular, the reversal of the sign of $\mathrm{Re}\,\eta$ was observed in several experiments and is a reliably established fact. The external-field frequency at which $\mathrm{Re}\,\eta$ goes through zero satisfies furthermore the relation $\omega \sim \Delta^2/T_c$, in agreement with our formulas.

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APPENDIX 1

We shall show that if $qv \gg 1/\tau$ it is possible to replace the mean value of a product of several Green's functions by the product of the mean values. For simplicity we consider the product of two Green's functions. When averaging over the impurities, it is convenient to employ the method proposed in[12], that of reducing, in

[3]For details, see [10], which contains a bibliography on this problem.

[4]The influence of impurities on absorption in a Pippard superconductor (Al) in the presence of a constant magnetic field was studied in [11], but at frequencies $\omega \sim \Delta$.

a definite manner, the averaging of the superconducting Green's functions of the impurities to an averaging over the Green's functions of the normal metal. For example, for averaging of the product of two Green's functions the matter reduces to averaging the combination

$$\langle (G_{\varepsilon_1}{}^R(\mathbf{xx}_1) - G_{\varepsilon_1}{}^A(\mathbf{xx}_1))\,(G_{\varepsilon_2}{}^R(\mathbf{x}_1\mathbf{x}) - G_{\varepsilon_2}{}^A(\mathbf{x}_1\mathbf{x}))\rangle,$$

where $G^{R(A)}$ are the retarded (advanced) Green's functions of the normal metal, and ϵ_1 and ϵ_2 are frequency arguments corresponding to the energy variables ξ and ξ_1 in the initial superconducting Green's functions. In this mean value, only the two terms (RA) and (AR) differ from zero, and for one of them the summation of the "ladder of dashed lines" leads to the following equation for the vertex (isotropic scattering):

$$\Pi^{AR}(\mathbf{q}) = 1 - \Pi^{AR}(\mathbf{q})\frac{n|u|^2}{(2\pi)^3}\int d\mathbf{p}\,G_{\varepsilon_1}{}^A(\mathbf{p})\,G_{\varepsilon_2}{}^R(\mathbf{p}-\mathbf{q}).$$

Here n is the electron density and u the Fourier component of the impurity potential. After integration we obtain

$$\Pi^{AR}(\mathbf{q}) = \left[1 - \frac{i}{2\tau}\int_{-1}^{+1}\frac{d\mu}{\varepsilon_2 - \varepsilon_1 - \mathbf{qv} + i/\tau}\right]^{-1}.$$

If now qv $\gg 1/\tau$, then small angles μ play a role in the integral, and we can extend the integration to infinity and confine ourselves to a contribution of the pole. We immediately find that the second term in the denominator is small compared with the first by a factor (qv $(qv\tau)^{-1} \ll 1$. Neglect of this term is equivalent to neglecting the "ladder of dashed lines."

APPENDIX 2

In the temperature technique we obtain for $L_2(kk_1k_2)$ after averaging over the impurities:

$$L_2(kk_1k_2) = -\frac{\Delta}{4}\left(\frac{ev}{c}\right)^2 \iiint_{-\infty}^{+\infty}\frac{d\varepsilon_1\,d\varepsilon_2\,d\varepsilon_3}{(2\pi i)^3}\int_{-1}^{+1}(1-\mu^2)\,d\mu$$
$$\times T\sum_{\varepsilon}\frac{\Delta^2 + \varepsilon(\varepsilon-\omega_1) + (\varepsilon-\omega_1)(\varepsilon-\omega) + \varepsilon(\varepsilon-\omega) + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_3}{(\varepsilon^2 - E_1{}^2)((\varepsilon-\omega_1)^2 - E_2{}^2)((\varepsilon-\omega)^2 - E_3{}^2)}$$
$$\times\int_{-\infty}^{+\infty} d\xi\,(G_{\varepsilon_1}{}^A(\mathbf{p}) - G_{\varepsilon_1}{}^R(\mathbf{p}))\,(G_{\varepsilon_2}{}^A(\mathbf{p}-\mathbf{q}_1) - G_{\varepsilon_2}{}^R(\mathbf{p}-\mathbf{q}_1))$$
$$\times(G_{\varepsilon_3}{}^A(\mathbf{p}-\mathbf{q}) - G_{\varepsilon_3}{}^R(\mathbf{p}-\mathbf{q})),$$
$$\varepsilon = i\pi T(2n+1),\quad E_i = \sqrt{\varepsilon_i{}^2 + \Delta^2},\quad G_\varepsilon{}^{R(A)}(\mathbf{p}) = (\varepsilon - \xi \pm i/2\tau)^{-1}. \tag{A2.1}$$

By analytic continuation of the product of the thermodynamic Green's functions we get

$$T\sum_{\varepsilon}(\ldots) \to -\frac{1}{4\pi i}\int_{-\infty}^{+\infty}d\varepsilon\left[\operatorname{th}\frac{\varepsilon}{2T}(\ldots)^{AAA} - \operatorname{th}\frac{\varepsilon-\omega}{2T}(\ldots)^{RRR}\right.$$
$$\left.-\left(\operatorname{th}\frac{\varepsilon}{2T} - \operatorname{th}\frac{\varepsilon-\omega_1}{2T}\right)(\ldots)^{RRA} - \left(\operatorname{th}\frac{\varepsilon-\omega_1}{2T} - \operatorname{th}\frac{\varepsilon-\omega}{2T}\right)(\ldots)^{RAA}\right].$$

Here (...) denotes the product of three Green's functions, and the indices R and A denote which of them are taken to be retarded and which advanced after the continuation.

After integrating with respect to ξ and making the transformation $(\epsilon_1\epsilon_2\epsilon_3) \to (\xi\xi_1\xi_2)$, we obtain from (A2.1)

$$L_2(kk_1k_2) = \frac{\Delta}{2}\left(\frac{ev}{c}\right)^2\int_{-1}^{+1}(1-\mu^2)\,d\mu\,T\sum_{\varepsilon}\cdot$$
$$\times\iiint_{-\infty}^{+\infty}\frac{d\xi\,d\xi_1\,d\xi_2}{(2\pi i)^3}\;\frac{\Delta^2 + \varepsilon\varepsilon_1 + \varepsilon\varepsilon_0 + \varepsilon_1\varepsilon_0 + \xi\xi_1 + \xi_1\xi_2 - \xi\xi_2}{(\xi^2 - \gamma^2)(\xi_1{}^2 - \gamma_1{}^2)(\xi_2{}^2 - \gamma_0{}^2)}$$
$$\times\left[\frac{1}{(\xi_2 - \xi + x + i/\tau)(\xi_2 - \xi_1 + x_2 + i/\tau)} + \begin{pmatrix}\xi_1 \leftrightarrow \xi_2\\ x \to x_1\\ x_2 \to -x_2\end{pmatrix} + \begin{pmatrix}\xi_2 \to \xi \to \xi_1 \to \xi_2\\ x \to -x_1\\ x_2 \to -x\end{pmatrix}\right],$$
$$x = qv\mu,\quad x_1 = q_1v\mu,\quad x_2 = q_2v\mu,$$
$$\gamma = \sqrt{\varepsilon^2 + \Delta^2},\quad \gamma_1 = \sqrt{(\varepsilon-\omega_1)^2 - \Delta^2},\quad \gamma_0 = \sqrt{(\varepsilon-\omega)^2 - \Delta^2}.$$

Let us integrate with respect to ξ, ξ_1 and ξ_2:

$$L_2(kk_1k_2) = \frac{i\pi\Delta}{8}\left(\frac{ev}{c}\right)^2\int_{-1}^{+1}(1-\mu^2)\,d\mu\,T\sum_{\varepsilon}\frac{1}{\gamma\gamma_1\gamma_0}$$
$$\times\left[\frac{\alpha + \gamma_0\gamma_1 - \gamma\gamma_1 + \gamma\gamma_0}{(\gamma + \gamma_1 - x_1 + i/\tau)(\gamma + \gamma_0 - x + i/\tau)} + \begin{pmatrix}\gamma \leftrightarrow \gamma_0\\ x_1 \leftrightarrow x_2\end{pmatrix}\right.$$
$$\left.+\frac{\alpha - \gamma_0\gamma_1 - \gamma\gamma_1 - \gamma\gamma_0}{(\gamma_1 + \gamma - x_1 + i/\tau)(\gamma_1 + \gamma_0 + x_2 + i/\tau)}\right],\qquad \alpha = \Delta^2 + \varepsilon\varepsilon_1 + \varepsilon\varepsilon_0 + \varepsilon_1\varepsilon_0$$

After analytic continuation we obtain

$$L_2(kk_1k_2) = -\frac{\Delta}{32}\left(\frac{ev}{c}\right)^2\int_{-1}^{+1}(1-\mu^2)\,d\mu\int_{-\infty}^{+\infty}d\varepsilon\cdot$$
$$\times\left\{\frac{\operatorname{th}(\varepsilon/2T)}{\gamma\gamma_1\gamma_0}\left[\frac{\alpha + \gamma_0\gamma_1 - \gamma\gamma_1 + \gamma\gamma_0}{(\gamma + \gamma_1 - x_1 + i/\tau)(\gamma + \gamma_0 - x + i/\tau)} + \begin{pmatrix}\gamma \leftrightarrow \gamma_0\\ x_1 \leftrightarrow x_2\end{pmatrix}\right.\right.$$
$$\left.+\frac{\alpha - \gamma_0\gamma_1 - \gamma\gamma_1 - \gamma\gamma_0}{(\gamma_1 + \gamma - x_1 + i/\tau)(\gamma_1 + \gamma_0 + x_2 + i/\tau)}\right]$$
$$-\frac{\operatorname{th}(\varepsilon/2T) - \operatorname{th}\left(\frac{\varepsilon-\omega_1}{2T}\right)}{\tilde\gamma\gamma_1\gamma_0}\left[\frac{\alpha + \gamma_0\gamma_1 - \tilde\gamma\gamma_1 + \tilde\gamma\gamma_0}{(\tilde\gamma + \gamma_1 - x_1 + i/\tau)(\tilde\gamma + \gamma_0 - x + i/\tau)}\right.$$
$$\left.\left.+\begin{pmatrix}\tilde\gamma \leftrightarrow \gamma_0\\ x_1 \leftrightarrow x_2\end{pmatrix} + \frac{\alpha - \gamma_0\gamma_1 - \tilde\gamma\gamma_1 - \tilde\gamma\gamma_0}{(\gamma_1 + \tilde\gamma - x_1 + i/\tau)(\gamma_1 + \gamma_0 + x_2 + i/\tau)}\right]\right\} + (k_1 \leftrightarrow k_2).$$

Now the roots γ are already defined as indicated in the text. The first term in the curly brackets has no singularities in the lower ϵ half-plane other than the simple poles of $\tanh(\epsilon/2T)$. The second term has, besides the poles connected with $\tanh(\epsilon/2T) - \tanh(\epsilon - \omega_1)/2T$, also the cut $(-\infty, -\Delta)$ $(\Delta, +\infty)$ in accordance with the definition of γ. By closing the integration contour in the lower half plane, we can easily verify that the contribution from the poles of $\tanh(\epsilon/2T) - \tanh((\epsilon - \omega_1)/2T)$ cancels out the first term of the expansion of the first term in ω. On the other hand, the zeroth term of the expansion of the first term in ω yields the static result. Therefore, accurate to terms of order $(\omega/T)^2$, it is necessary to take into account only the path around the cut.

Thus, we arrive at the formula

$$L_2^{(1)}(kk_1k_2) = -\frac{\Delta}{32}\left(\frac{ev}{c}\right)^2\int_{-1}^{+1}(1-\mu^2)\,d\mu\int_c\frac{d\varepsilon}{\tilde\gamma\gamma_1\gamma_0}\left(\operatorname{th}\frac{\varepsilon}{2T} - \operatorname{th}\frac{\varepsilon-\omega_1}{2T}\right)$$
$$\times\left[\frac{\alpha + \gamma_0\gamma_1 - \tilde\gamma\gamma_1 + \tilde\gamma\gamma_0}{(\tilde\gamma + \gamma_1 - x_1 + i/\tau)(\tilde\gamma + \gamma_0 - x + i/\tau)} + \begin{pmatrix}\tilde\gamma \leftrightarrow \gamma_0\\ x_1 \leftrightarrow x_2\end{pmatrix}\right.$$
$$\left.+\frac{\alpha - \gamma_0\gamma_1 - \tilde\gamma\gamma_1 - \tilde\gamma\gamma_0}{(\gamma_1 + \tilde\gamma - x_1 + i/\tau)(\gamma_1 + \gamma_0 + x_2 + i/\tau)}\right] + (k_1 \leftrightarrow k_2).$$

If qv $\gg 1/\tau$, very small μ play the major role and we can extend the integration with respect to μ to infinity. After integration we obtain

$$L_2^{(1)}(kk_1k_2) = -\frac{i\pi\Delta^3}{8v}\left(\frac{ev}{c}\right)^2\int_c\frac{d\varepsilon}{\tilde\gamma\gamma_1\gamma_0}\left(\operatorname{th}\frac{\varepsilon}{2T} - \operatorname{th}\frac{\varepsilon-\omega_1}{2T}\right)$$
$$\times\left[\frac{1 - |qq_1|/qq_1}{\gamma_1|q| + \gamma_0|q_1| + \tilde\gamma|q_2| + i|q_2|/\tau} + \begin{pmatrix}\tilde\gamma \leftrightarrow \gamma_0\\ q_1 \leftrightarrow q_2\end{pmatrix}\right.$$
$$\left.+\frac{1 + |q_1q_2|/q_1q_2}{\gamma_1|q| + \gamma_0|q_1| + \tilde\gamma|q_2| + i|q|/\tau}\right] + (k_1 \leftrightarrow k_2). \tag{A2.2}$$

Recognizing that the values significant in the integration with respect to ϵ in (A2.2) are $|\epsilon| \sim \Delta$, and putting $\omega_2 = 0$, we arrive at formula (1.5)).

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