INTERNAL WAVES IN A FIELD OF CENTRIFUGAL FORCES AND THEIR USE FOR THE INVESTIGATION OF THE CRITICAL STATE OF A MEDIUM

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The frequency spectrum of internal gravitational waves is bounded from above by a certain frequency ω_0, a low value of which complicates the experimental study of the internal waves. It is shown in this paper that the limiting frequency of the internal waves produced in the field of centrifugal forces when the medium rotates is not lower than double the rotation frequency, and can be made sufficiently high. The presence of Coriolis forces, which cause an increase of the limiting frequency, does not, however, change the character of the waves. The problem of the propagation of internal waves in a medium under critical conditions of the vapor-liquid transition is solved. The equation of state, which determines the distribution of the unperturbed density ρ_0, is taken in the form $p - p_{cr} \sim \beta|\rho - \rho_{cr}|^{\delta}$ (p_{cr} and ρ_{cr} are the critical pressure and density). The experimental determination of the dependence of the wavelength on the medium's rotation frequency makes it possible to determine the parameters β and δ, which are important for the theory of the critical state.

1. The frequency of propagation of internal waves in a gravitational field, as is well known[1], is bounded from above by the frequency $\omega_0 = [g \max|\nabla(\ln\rho_0)|]^{1/2}$ (g is the acceleration of free fall and ρ_0 is the unperturbed density of the medium). For the experimentally-investigated internal waves in the ocean and in the atmosphere, this quantity does not exceed 10^{-1} sec^{-1} [2]. Even when the medium is close to the critical point of the vapor-liquid transition and the density gradient is therefore relatively large, the frequency ω_0 does not exceed 10 sec^{-1} [3]. Such a low value of the internal-wave frequency greatly complicates both the experimental verification of the theory of these waves, and their use for the investigation of the critical state of the medium, as was proposed in[3].

It is possible, however, to raise the frequency ω_0 appreciably by using in lieu of the gravitational forces the centrifugal forces that occur when the medium is rotated. In addition, the Coriolis forces occurring in rotating media also raise the limiting frequency of the propagation of the internal waves.

2. Let a liquid be placed in a space between two coaxial cylinders of radii R_1 and R_2 ($R_2 > R_1$), rotating about a common axis with angular velocity Ω. The linearized equations of hydrodynamics describing the internal waves have the following form under the usual conditions for the theory of these waves, namely the approximations of an incompressible and ideal liquid:

$$\rho_0\frac{\partial \mathbf{V}}{\partial t} = -\nabla p' - 2\rho_0[\boldsymbol{\Omega}\mathbf{V}] + [[\boldsymbol{\Omega}\mathbf{r}]\boldsymbol{\Omega}]\rho',$$
$$\frac{\partial \rho'}{\partial t} + (\mathbf{V}\nabla)\rho_0 = 0, \quad \operatorname{div}\mathbf{V} = 0. \qquad (1)^*$$

Here $\mathbf{V}$ is the velocity of the liquid, ρ_0 its unperturbed density, $\rho' = \rho - \rho_0$ and $p' = p - p_0$ are respectively the deviations of the density and of the pressure from their unperturbed values, $\boldsymbol{\Omega}$ is the vector of the angular velocity of rotation of the cylinders, and $\mathbf{r}$ is the radius vector (the origin is on the rotation axis).

Let r, φ, and z be the coordinates of a cylindrical system with z axis along the rotation axis. The unperturbed density (the unperturbed state is assumed to be isothermal) will obviously be a function only of r and is the solution of the hydrostatic equation

$$\left(\frac{\partial p}{\partial \rho}\right)_T \frac{d\rho_0}{dr} = \rho_0\Omega^2 r. \qquad (2)$$

We are interested in axially-symmetrical solutions (independent of φ) of the system (1), which in this case takes the form

$$\rho_0\frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial r} + 2\rho_0\Omega v + \Omega^2 r\rho',$$
$$\frac{\partial v}{\partial t} = -2\Omega u, \quad \rho_0\frac{\partial w}{\partial t} = -\frac{\partial p'}{\partial z}, \qquad (3)$$
$$\frac{\partial \rho'}{\partial t} + u\frac{d\rho_0}{dr} = 0, \quad \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0,$$

where u, v, and w are respectively the radial, transverse, and axial components of the velocity. Eliminating all the unknowns except u from (3), we get

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial}{\partial r}\left[\frac{\rho_0}{r}\frac{\partial}{\partial r}(ru)\right] + \rho_0\frac{\partial^2 u}{\partial z^2}\right\} + \left(4\Omega^2\rho_0 + \Omega^2 r\frac{d\rho_0}{dr}\right)\frac{\partial^2 u}{\partial z^2} = 0. \qquad (4)$$

It is necessary to add to this equation the boundary conditions on the solid surfaces of the cylinders

$$u = 0 \quad \text{at} \quad r = R_1 \text{ and } r = R_2. \qquad (5)$$

We shall consider monochromatic waves propagating along the z axis. Thus,

$$u = u(r)\exp[i(\omega t - kz)].$$

The velocity amplitude $u(r)$ will satisfy the equation

$$\frac{d}{dr}\left[\frac{\rho_0}{r}\frac{d}{dr}(ru)\right] - k^2\left[\rho_0\left(1 - 4\frac{\Omega^2}{\omega^2}\right) - \frac{\Omega^2}{\omega^2}r\frac{d\rho_0}{dr}\right]u = 0. \qquad (6)$$

The difference between this equation and the usual equation describing the internal gravitational waves[1] consists in the fact that it contains the centrifugal ac-

*$[\boldsymbol{\Omega}\mathbf{V}] \equiv \boldsymbol{\Omega}\times\mathbf{V}$.

celeration $\Omega^2 r$ in lieu of the gravitational acceleration g, and also in the insignificant difference brought about by the cylindrical geometry and in the term $4\Omega^2/\omega^2$ due to the Coriolis forces.

3. The frequency of propagation of the internal waves is also limited in this case. To demonstrate this, we multiply (6) by ru and integrate from R_1 to R_2. Taking the boundary conditions (5) into account, we obtain

$$\omega^2 = \Omega^2 k^2 \left\{ 4 \int_{R_1}^{R_2} \rho_0 r u^2 \, dr + \int_{R_1}^{R_2} r^2 \frac{d\rho_0}{dr} u^2 dr \right\} \times \left\{ \int_{R_1} \frac{\rho_0}{r} \left[\frac{d}{dr}(ru) \right]^2 dr + k^2 \int_{R_1}^{R_2} \rho_0 r u^2 dr \right\}^{-1}.$$

From this, obviously,

$$\omega^2 < 4\Omega^2 + \Omega^2 \int_{R_1}^{R_2} r^2 \frac{d\rho_0}{dr} u^2 dr \Big/ \int_{R_1}^{R_2} \rho_0 r u^2 dr.$$

It is easy to see that

$$\int_{R_1}^{R_2} r^2 \frac{d\rho_0}{dr} u^2 dr = \int_{R_1}^{R_2} \left(\frac{r}{\rho_0} \frac{d\rho_0}{dr} \right) \rho_0 r u^2 dr < \max \left(\frac{r}{\rho_0} \frac{d\rho_0}{dr} \right) \int_{R_1}^{R_2} \rho_0 r u^2 dr.$$

Thus,

$$\omega^2 < 4\Omega^2 + \Omega^2 \max \left(\frac{r}{\rho_0} \frac{d\rho_0}{dr} \right).$$

The limiting frequency of the internal waves turns out to be not smaller than double the frequency of rotation of the cylinders, which can be made sufficiently high.

4. We now solve Eq. (6) for the case of a constant $\rho_0^{-1} d\rho_0/dr$ and a gap between the cylinders that is small compared with the cylinder radii. Equation (6) in this case takes the form

$$\frac{d^2u}{dr^2} + \frac{1}{r_0}\frac{du}{dr} - k^2 \left[1 - 4\frac{\Omega^2}{\omega^2} - \frac{\Omega^2}{\omega^2}\frac{R_0}{r_0} \right] u = 0, \tag{7}$$

where

$$r_0^{-1} = \rho_0^{-1} d\rho_0 / dr, \text{and } R_0 = (R_1 + R_2)/2.$$

Equation (7) can be rewritten in the form

$$\frac{d^2u}{dr^2} + \frac{1}{r_0}\frac{du}{dr} - k^2 \left[1 - \frac{g}{\omega'^2}\frac{1}{r_0} \right] u = 0,$$

where

$$\omega'^2 = \omega^2 \left(\frac{g}{\Omega^2 R_0} \right) \frac{R_0/r_0}{4 + R_0/r_0}.$$

This equation coincides formally with the usual equation for gravitational waves. Thus, the considered case of internal waves with frequency ω in rotating liquid corresponds to gravitational waves of frequency ω'.

The solution of (7) is

$$u = Ae^{-r/2r_0} \sin(\lambda r + \alpha),$$

where

$$\lambda^2 = k^2 \left[\frac{\Omega^2}{\omega^2}\left(4 + \frac{R_0}{r_0} \right) - 1 \right] - \frac{1}{4r_0^2},$$

and A and α are the integration constants.

Satisfying the boundary conditions (5), we get

$$\lambda_n = \pi n / (R_2 - R_1), \quad \alpha_n = -\lambda_n R_1, \quad n = 1, 2, \ldots,$$

from which we get

$$\omega^2 = \left(4 + \frac{R_0}{r_0} \right) \Omega^2 \Big/ \left[1 + \frac{1}{4k^2 r_0^2} + \frac{\pi^2 n^2}{(R_2 - R_1)^2 k^2} \right].$$

The following fact is of interest: when the sign of r_0 (i.e., the sign of the density gradient) is reversed, the sign of ω^2 does not change, provided $1/r_0 > -4/R_0$. This means that stratification with negative density gradient is stable. This fact is obviously due to the Coriolis forces. It is known that in a non-rotating medium the stratification with arbitrarily small negative density gradient (relative to the direction of the gravitational forces) is unstable. It must be noted, however, that this stability occurs only relative to the perturbations considered here (which do not depend on φ).

Let us find the value of r_0 for an ideal gas with an isothermal unperturbed state. From (2) we get (assuming a small gap)

$$\frac{1}{\rho_0}\frac{d\rho_0}{dr} = \frac{\Omega^2 R_0}{(\partial p/\partial\rho)_T} = \frac{\Omega^2 R_0 \mu}{T},$$

where μ is the gram-molecular weight and T is the temperature in energy units. From this we get $r_0 = r_g g/\Omega^2 R_0$, where $r_g = T/\mu g \sim 10^6$ cm is the characteristic dimension of density variation in the gravitational field. The limiting frequency is in this case

$$\omega_0 = 2\Omega \left[1 + \frac{R_0}{4r_g}\left(\frac{\Omega^2 R_0}{g} \right) \right]^{1/2}.$$

5. We now consider internal waves in a liquid under the critical conditions of the liquid-vapor transition. The possibility of applying the considered theory in such a medium is justified in[3].

Thus, let the liquid in the unperturbed state be maintained at a critical temperature T_{cr} and let the critical pressure and critical density be reached at $r = R_0$. Let the liquid layer be sufficiently thin to make the conditions close to critical in the entire volume.

The equation of the critical isotherm is taken in the form

$$\frac{p - p_{cr}}{p_{cr}} = \beta \left| \frac{\rho - \rho_{cr}}{\rho_{cr}} \right|^{\delta} \frac{\rho - \rho_{cr}}{|\rho - \rho_{cr}|},$$

where β and δ have arbitrary positive values, p_{cr} and ρ_{cr} are the critical pressure and density.

Equation (2) then yields, assuming a small gap:

$$\frac{\rho - \rho_{cr}}{\rho_{cr}} = |\eta|^{1/\delta} \frac{\eta}{|\eta|}, \quad \eta = \frac{r - R_0}{r_0}, \tag{8}$$

where

$$r_0 = \frac{p_{cr}\beta}{\rho_{cr}\Omega^2 R_0} = r_g \frac{g}{\Omega^2 R_0},$$

$$r_g = \beta p_{cr} / \rho_{cr} g \sim T_{cr}/\mu g \sim 10^3 \text{ cm}. \tag{9}$$

Let us consider Eq. (6) with $\omega = 2\Omega$. We have

$$\frac{d}{dr}\left[\frac{\rho_0}{r}\frac{d}{dr}(ru) \right] + \frac{1}{4}k^2 \frac{d\rho_0}{dr} ru = 0.$$

We replace the sought function and the independent variable in this equation

$$ru = \frac{\psi(t)}{f(t)}\sqrt{t}, \quad f = \left(\frac{\rho_0}{r}\frac{d\rho_0}{dr} \right)^{1/4},$$

$$t = \int_{R_0}^{r} \left(\frac{r}{\rho_0}\frac{d\rho_0}{dr} \right)^{1/2} dr. \tag{10}$$

Then $\psi(t)$ satisfies the equation

$$\frac{d^2\psi}{dt^2} + \frac{1}{t}\frac{d\psi}{dt} + \left[\frac{1}{4}k^2 - \left(\frac{1}{4t^2} + \frac{1}{f}\frac{d^2 f}{dt^2} \right) \right] \psi = 0. \tag{11}$$

Taking (8) into account and using the smallness of the gap, we get from (10)

$$t = \frac{2\sqrt{\delta r_0 R_0}}{1+\delta} |\eta|^{1/2\Delta}, \quad \Delta \equiv \frac{\delta}{1+\delta}.$$

Thus, $f(t) \sim |t|^{1/2-\Delta}$, from which we get

$$\frac{f''}{f} = \frac{(\delta-1)(3\delta+1)}{4(1+\delta)^2} \frac{1}{t^2}.$$

Substituting in (11) $\xi = kt/2$, we get for $\psi(\xi)$ the Bessel equation

$$\frac{d^2\psi}{d\xi^2} + \frac{1}{\xi}\frac{d\psi}{d\xi} + \left[1 - \frac{\Delta^2}{\xi^2}\right]\psi = 0.$$

We have a general solution for $u(\eta)$:

$$ru = C_1\left[\frac{J_\Delta(\lambda|\eta|^{1/2\Delta})}{|\eta|^{1/2}}\right]\eta + C_2\left[\frac{J_{-\Delta}(\lambda|\eta|^{1/2\Delta})}{|\eta|^{-1/2}}\right],$$

where $J_{\pm\Delta}(\xi)$—is a Bessel function of the first kind, and $\lambda^2 = k^2\delta r_0 R_0/(1+\delta)^2$.

The function $u(\eta)$ is even if $C_1 = 0$. From the boundary conditions (5) we then get

$$\lambda_{2n} = (r_0/d)^{1/2\Delta}\xi_{2n}(-\Delta), \quad n = 0, 1, 2, \ldots,$$

where $d = (R_2 - R_1)/2$, and $\xi_{2n}(-\Delta)$ is the n-th zero of the function $J_{-\Delta}(\xi)$.

Odd solutions are obtained when $C_2 = 0$. In this case we get

$$\lambda_{2n+1} = (r_0/d)^{1/2\Delta}\xi_{2n+1}(\Delta),$$

where $\xi_{2n+1}(\Delta)$ is the n-th zero of the function $J_\Delta(\xi)$.

From this we get

$$k_n^2 = \frac{(1+\delta)^2}{\delta d R_0}\left(\frac{r_0}{d}\right)^{1/\delta} \xi_n^2\left((-1)^{n+1}\frac{\delta}{1+\delta}\right),$$

or, taking (9) into account, $k_n \sim \Omega^{-1/\delta}$, the proportionality coefficient being

$$\frac{1+\delta}{\sqrt{\delta d R_0}}\left(\frac{\beta p_{cr}}{\rho_{cr} R_0 d}\right)^{1/2\delta} \xi_n\left((-1)^{n+1}\frac{\delta}{1+\delta}\right).$$

Thus, from the character of the dependence of k on Ω we can determine δ, and from the value of the proportionality coefficient we can determine β. The values of δ and β are of interest for the theory of the critical state.

We must note the following. Gravitational waves are due to the deviation of the density of the medium from its hydrostatic-equilibrium distribution. In the present paper it is assumed that in the unperturbed state the temperature in the entire medium is constant and is equal to the critical value. It is precisely under these conditions that the wave parameters are connected with the critical indices of the medium by the relations derived in this paper. As is well known[4], near the critical point the thermal equilibrium (and consequently also the here-considered hydrostatic-equilibrium density distribution) is established very slowly. This circumstance must be borne in mind during the performance of the experiment.

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