ANOMALOUS PLASMA RESISTANCE IN A STRONG MAGNETIC FIELD

A. M. DYKHNE

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Fluctuations in a plasma in a strong magnetic field lead to a significant change in the electrical characteristics of the plasma. When the inhomogeneities are highly elongated along the magnetic field and are randomly moving regions, with high or low plasma densities, it is possible to obtain exact expressions for the effective conductivity and the effective Hall parameter. It is shown that in a strong magnetic field the conductivity is inversely proportional to the field while the Hall parameter is independent of the field (saturation).

1. It is well known that a conducting medium in a magnetic field exhibits an anisotropic conductivity. An especially large anisotropy arises in strong magnetic fields in which the Larmor frequency of the current carriers (Ω) is higher than the collision frequency ($1/\tau$). In this case the conductivity tensor becomes (magnetic field parallel to the z axis)

$$\hat{\sigma}=\begin{pmatrix} \sigma_2 & \sigma_1 & 0 \\ -\sigma_1 & \sigma_2 & 0 \\ 0 & 0 & \sigma_0 \end{pmatrix}, \tag{1}$$

where $\sigma_k \sim H^{-k}$ in strong magnetic fields.

Herring[1] has shown that the electric properties of a medium of this kind are substantially changed if the medium is inhomogeneous, even if only to a small degree. The work carried out by Herring refers to solids. The analogous problem for a plasma has been considered in[2,3]. In all of the work cited the inhomogeneity of the medium is assumed to be small and the first term in an expansion in the relative fluctuation of the conductivity $\Delta^2 = \langle \delta\sigma^2 \rangle / \langle \sigma \rangle^2$ was calculated. The authors of[1-3] reached the conclusion that even when $\Delta \ll 1$ (but $\beta\Delta \gg 1$, where $\beta = \Omega\tau$ is the Hall parameter) there is a large change in the conductivity tensor. However, as is shown by an estimate of the next approximation, the parameter in which the expansion is carried out is $\beta^2\Delta^2 \ll 1$. Thus, in[1-3] they obtained in essence small corrections to the "uniform" value of the tensor (1), and the extension of the results into the region $\beta\Delta \gg 1$ is an inadmissible excess of accuracy. This circumstance was noted in[4]. Thus, the question of the electrical characteristics of a weakly inhomogeneous medium in a strong magnetic field is essentially still open. Nonetheless, when $\beta\Delta \gg 1$ one should apparently expect a strong change in the electrical characteristics of the medium (when $\Delta \ll 1$). A number of considerations support this statement. We note some of these below.

In the first place we cite the solution of the corresponding problem for a medium with one-dimensional inhomogeneities (stratified medium), where the problem admits of an exact solution.[5,6] Second, the results of measurements of the conductivity of a turbulent, weakly ionized plasma in a strong magnetic field.[7,8] Finally, we note that in[4] a marked change in the conductivity was qualitatively demonstrated in the case of a two-dimensional inhomogeneity (in the plane perpendicular to the magnetic field).

In solving the problem of anomalous conductivity of a plasma we shall make a number of assumptions. First, we assume a two-dimensional pattern of fluctuations arising rather frequently as a result of instabilities in a plasma located in a strong magnetic field. This picture is conventional for the development of the ionization instability in a cold plasma.[7,8] A second assumption is of a model nature, namely that the plasma parameters (in particular the density) can assume two values, that is to say, there are regions with higher and lower density. A similar (two-level) model for a conducting system without a magnetic field was used in[9]. Under these assumptions, if the volumes occupied by the phases are equal and their locations are random, we can carry out an exact calculation of all components of the conductivity tensor. Using the weaker assumptions given below, we can find one relation which satisfies two different (transverse) components of the tensor $\hat{\sigma}$. This relation has been measured experimentally by a number of authors.[7,10] A comparison shows satisfactory agreement between theory and experiment.

The possibility of finding exact solutions with these assumptions is due, as in[9], to the existence of symmetry in the equations that describe the detailed behavior of the system. It turns out there are two independent symmetry transformations, the use of which can determine two independent components of the effective conductivity tensor. To describe the properties of the turbulent plasma it is necessary to study simultaneously the transport processes and the instabilities causing the plasma inhomogeneities. We shall consider separately the problem of a medium with specified noise input. This approach does not give a complete solution of the problem, but does allow us to examine the physical situation; in particular, it enables us to understand the noise characteristics that affect the properties of the medium most adversely.

2. The system of equations that describe an inhomogeneous conducting medium consists of Ohm's law which, for the tensor $\hat{\sigma}$ given by (1), can be written in vector form

$$\mathbf{j} + [\mathbf{j}\boldsymbol{\beta}] = \sigma\mathbf{e}, \quad \boldsymbol{\beta} \parallel \mathbf{H}, \tag{2}$$ *

and the equations for the constant current

$$\operatorname{div} \mathbf{j} = 0, \quad \operatorname{rot} \mathbf{e} = 0. \tag{3}$$

*$[\mathbf{j}\boldsymbol{\beta}] \equiv \mathbf{j} \times \boldsymbol{\beta}$.

The lower-case symbols (j; e) denote local values of the corresponding quantities while the upper case symbols (J; E) denote mean values averaged over the volume.

To simplify the calculations we first consider the case in which the fluctuations occur in the quantity σ in (2), whereas β remains constant along the system. An example of this situation appears in a partially ionized gas with hot electrons if the Coulomb collisions of the electrons can be neglected. In fact, in this case β is a smooth function of electron temperature while σ is a sharp (exponential) function; as a result the fluctuations in conductivity are much greater than the fluctuations in the Hall parameter. The fluctuations of β will be taken into account below whenever the situation calls for it.

We now subject the system in (2) and (3) to two independent symmetry transformations. The first is a simple rotational transformation such as used in[9] in the absence of a magnetic field. We introduce a new current density j′ and a new field e′:

$$\mathbf{j}' = \left(\frac{\sigma_1\sigma_2}{1+\beta^2}\right)^{1/2}[\mathbf{en}],$$
$$\mathbf{e}' = \left(\frac{\sigma_1\sigma_2}{1+\beta^2}\right)^{-1/2}[\mathbf{jn}]. \tag{4}$$

Here σ_1 and σ_2 ($<\sigma_1$) are the values of the conductivity in the hot and cold regions and n is the unit vector for the magnetic field. The retention of the unity term in the expression $1+\beta^2$ increases the accuracy since (2) and (1) correspond to $\beta \gg 1$. Writing the expressions in this form still allows us to take the limit $\beta = 0$. The transformed quantities (j′, e′) satisfy the same equations

$$\operatorname{div}\mathbf{j}' = 0, \quad \operatorname{rot}\mathbf{e}' = 0, \tag{5}$$

as the original quantities. In the new variables Ohm's law becomes

$$\mathbf{j}' + [\mathbf{j}'\boldsymbol{\beta}'] = \sigma'\mathbf{e}', \tag{6}$$

where

$$\sigma' = \sigma_1\sigma_2/\sigma, \quad \beta' = -\beta. \tag{7}$$

Our problem lies in finding a solution of (2) and (3) by successive averaging of the expressions and the determination of a linear (by virtue of the linear equations) relation between the mean quantities. This relation can be written in the form of an effective Ohm's law:

$$\mathbf{J} + [\mathbf{J}\boldsymbol{\beta}_e] = \sigma_e\mathbf{E}. \tag{8}$$

Let us assume that the problem has been solved and the required relation has been found. An analogous relation between the primed variables can be found by solving (5) and (6). If the regions with conductivities σ_1 and σ_2 occupy equal volumes, but are in other respects arbitrary and randomly distributed, then by virtue of (6) and (7) the primed system cannot be distinguished macroscopically from the original system (with the exception of the unimportant change in the sign of the magnetic field); it therefore leads to Ohm's law with the same effective parameters that are the macroscopic characteristics of the medium. Thus

$$\mathbf{J}' + [\boldsymbol{\beta}_e\mathbf{J}'] = \sigma_e\mathbf{E}'. \tag{9}$$

Averaging the relations in (4) we have

$$\mathbf{J}' = \left(\frac{\sigma_1\sigma_2}{1+\beta^2}\right)^{1/2}[\mathbf{En}], \quad \mathbf{E}' = \left(\frac{\sigma_1\sigma_2}{1+\beta^2}\right)^{-1/2}[\mathbf{Jn}]. \tag{10}$$

Substituting (10) in (9) we have

$$\mathbf{J} + [\mathbf{J}\boldsymbol{\beta}_e] = \frac{\sigma_1\sigma_2}{\sigma_e}\frac{1+\beta_e^2}{1+\beta^2}\mathbf{E}. \tag{11}$$

Comparing (8) and (11) we obtain a relation between σ_e and β_e, which can be rewritten in the form

$$\frac{\sigma_e}{\sqrt{1+\beta_e^2}} = \left(\frac{\sigma_1\sigma_2}{1+\beta^2}\right)^{1/2}. \tag{12}$$

The relation (12) can be generalized easily to the case of smooth (not necessarily two-phase) fluctuations of σ, the distribution of which is an even function of the variables $\chi = \ln\sigma - \langle\ln\sigma\rangle$. The symbol $\langle\;\rangle$ denotes an average over the system. In this case

$$\frac{\sigma_e}{\sqrt{1+\beta_e^2}} = \frac{\exp\langle\ln\sigma\rangle}{\sqrt{1+\beta^2}}. \tag{13}$$

In the absence of a magnetic field (12) and (13) become the expressions obtained in[9] $\sigma_e = \exp\langle\ln\sigma\rangle$; for a two-level system $\sigma_e = \sqrt{\sigma_1\sigma_2}$.

In the absence of the magnetic field these relations solve the problem. In the presence of a magnetic field, in order to obtain a complete solution it is necessary to find one additional independent relation between σ_e and β_e. To obtain this relation we consider a more general linear transformation that does not change the differential equations in (3):

$$\mathbf{j}' = a\mathbf{j}'' + b[\mathbf{ne}''], \quad \mathbf{e} = c\mathbf{e}'' + d[\mathbf{nj}'']. \tag{14}$$

Without losing generality we can write $a = 1$. Substituting (14) in (2) we have

$$\mathbf{j}'' + [\mathbf{j}''\boldsymbol{\beta}''] = \sigma''\mathbf{e}'', \tag{15}$$

where

$$\beta'' = \frac{\beta\sigma(c-bd) - b(1+\beta^2)}{(c+bd)\sigma}, \quad \sigma'' = \frac{b^2 + (c\sigma - b\beta)^2}{(c+bd)\sigma}.$$

Requiring that the following relations be satisfied when $\sigma = \sigma_{1,2}$:

$$\sigma'' = \sigma, \quad \beta'' = -\beta, \tag{16}$$

we have

$$c = 1, \quad d = -\frac{\beta}{\langle\sigma\rangle}, \quad b = \frac{\beta}{1+\beta^2}\left\langle\frac{1}{\sigma}\right\rangle^{-1}. \tag{17}$$

In terms of the double-primed variables the effective Ohm's law becomes

$$\mathbf{J}'' - [\mathbf{J}''\boldsymbol{\beta}_e] = \sigma_e\mathbf{E}''. \tag{18}$$

Averaging (14) and taking account of (17) we find

$$\mathbf{J} = \mathbf{J}'' + \frac{\beta}{1+\beta^2}\left\langle\frac{1}{\sigma}\right\rangle^{-1}[\mathbf{nE}''],$$
$$\mathbf{E} = \mathbf{E}'' - \frac{\beta}{\langle\sigma\rangle}[\mathbf{nJ}'']. \tag{19}$$

Substituting (19) in (18) and comparing with (2) we obtain a second relation between σ_e and β_e:

$$\sigma_e^2\beta(1+\beta^2) - 2\beta_e\sigma_e(1+\beta^2)\langle\sigma\rangle + \beta(1+\beta_e^2)\langle\sigma\rangle\left\langle\frac{1}{\sigma}\right\rangle^{-1} = 0. \tag{20}$$

We note that the relation in (20) applies, as is evident from (16), for a "two-phase" plasma for any (not necessarily equally divided) composition. (In this case

⟨ ⟩ denotes half of the sum.) From (12) and (20) we have

$$\sigma_e / \beta_e = \langle\sigma\rangle / \beta, \qquad (21)$$

whence

$$\sigma_e = \langle\sigma\rangle \left\{ \langle\sigma\rangle \left\langle \frac{1}{\sigma} \right\rangle + \beta^2 \left[\langle\sigma\rangle \left\langle \frac{1}{\sigma} \right\rangle - 1 \right] \right\}^{-1/2}, \qquad (22)$$

$$\beta_e = \beta \left\{ \langle\sigma\rangle \left\langle \frac{1}{\sigma} \right\rangle + \beta^2 \left[\langle\sigma\rangle \left\langle \frac{1}{\sigma} \right\rangle - 1 \right] \right\}^{-1/2} \qquad (23)$$

When $\beta \gg 1$ and $\Delta \ll 1$ [Δ is the mean-square fluctuation of σ, $\Delta = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$], (22) and (23) assume the form

$$\sigma_e = \frac{\langle\sigma\rangle}{(1 + \beta^2\Delta^2)^{1/2}}, \quad \beta_e = \frac{\beta}{(1 + \beta^2\Delta^2)^{1/2}}. \qquad (24)$$

Thus, the parameter that determines the behavior of the electrical characteristics of the plasma is the product $\beta\Delta$. When $\beta\Delta \gg 1$ we have

$$\sigma_e = \langle\sigma\rangle/\beta\Delta, \qquad \beta_e = 1/\Delta. \qquad (25)$$

It is interesting to note that the correlation properties of the fluctuations, the characteristic dimensions, and so on do not appear in the expression for the effective plasma parameters; these are determined only by the mean-square fluctuations. The results of an exact analysis with this model are considerably different from the approximate calculations.[1,2] Using the method applied in[1,2] for we find the two-dimensional case

$$\sigma_e = \frac{\langle\sigma\rangle}{1 + \beta^2\Delta^2/2}, \quad \beta_e = \frac{\beta}{1 + \beta^2\Delta^2/2}. \qquad (26)$$

As expected, these expressions agree with the correct expressions (24) only to first order in the parameter $\beta^2\Delta^2$. For large values of $\beta\Delta$ the results obtained given in[1,2] are not correct. This same can be said regarding the case of isotropic three-dimensional fluctuations, considered by the same method in[1,2].

3. We now consider the other limiting case, in which σ is constant and only β fluctuates. This situation occurs in a partially ionized gas when the electron behavior is dominated by electron-ion Coulomb collisions. In this case the conductivity σ is a smooth function of the temperature while β, because of the factor $1/n_e$, is a sharp (exponential) function. Hence the fluctuations of the Hall parameter are much larger than those of the conductivity. Making the substitutions

$$\tilde{\mathbf{j}} = \frac{\sigma}{2\langle\beta\rangle} [\mathbf{n}\mathbf{e}], \quad \tilde{\mathbf{e}} = \frac{1}{2\langle\beta\rangle} \mathbf{e} + \frac{1}{\sigma} [\mathbf{n}\mathbf{j}] \qquad (27)$$

we obtain the relation

$$\tilde{\mathbf{j}} + [\tilde{\mathbf{j}}\tilde{\boldsymbol{\beta}}] = \tilde{\sigma}\tilde{\mathbf{e}}, \qquad \tilde{\boldsymbol{\beta}} \| \boldsymbol{\beta}, \qquad (28)$$

where

$$\tilde{\sigma} = \frac{\sigma}{2\langle\beta\rangle} (1 + \beta^2),$$

$$\tilde{\beta} = \frac{\langle\beta\rangle^2 - 1 - (\beta - \langle\beta\rangle)^2}{2\langle\beta\rangle}. \qquad (29)$$

Thus, in the new system the quantity $\tilde{\beta}$ remains fixed $\tilde{\sigma}$ fluctuates. Using (22) and (23), which apply to this case, and using (29) to convert to the original variables, we obtain the expressions

$$\sigma_e = \frac{\sigma^2 + (\sigma\tilde{\beta}_e + \tilde{\sigma}_e)^2}{2\langle\beta\rangle\tilde{\sigma}_e}, \quad \beta_e = \tilde{\beta}_e + \frac{\tilde{\sigma}_e}{\sigma}, \qquad (30)$$

where $\tilde{\sigma}_e$ and $\tilde{\beta}_e$ are determined from (22) and (23) in which σ and β are replaced by $\tilde{\sigma}$ and $\tilde{\beta}$ from (29). When $\beta\Delta \gg 1$ ($\Delta = |\beta_2 - \beta_1|/(\beta_2 + \beta_1)$ is the fluctuation of the Hall parameter) these relations assume the form

$$\sigma_e = \frac{\sigma}{\beta\Delta}, \quad \beta_e = \frac{1}{\Delta}. \qquad (31)$$

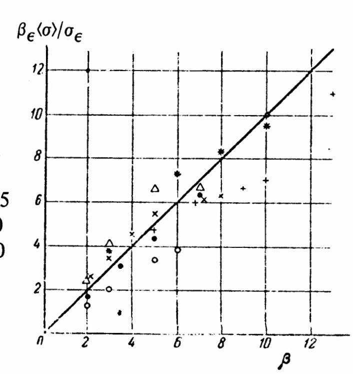
We see that in the case in which only the Hall parameter fluctuates the expressions for the effective plasma characteristics have the same asymptotic form (for large values of $\beta\Delta$) as for the fluctuation in conductivity. For intermediate values of $\beta\Delta$ the functional dependences are different in the two cases.

The same can be said regarding joint fluctuations of σ and β. We shall not give here the appropriate formulas for the intermediate region, which are cumbersome.

4. In addition to computing the conductivity and the Hall parameter it is also possible to compute certain other quantities that characterize the distribution of the currents and fields in the system. The strong drop when $\beta\Delta \gg 1$) in the electrical characteristics of the system as compared with the uniform case is obviously caused by the strong current fluctuations. (In the absence of a magnetic field the current fluctuations are of order unity when $\Delta \sim 1$.) Omitting calculations similar to those carried out in[9], we present the results for the mean-square fluctuations of the current (field)

$$\delta = \frac{\langle j^2\rangle - J^2}{J^2} = \frac{\langle e^2\rangle - E^2}{E^2} = \beta\Delta, \qquad \beta\Delta \gg 1.$$

5. A comparison of the obtained expressions with the experimental data can serve to verify the model used above. In this case, as we have already noted, we consider only one aspect of the problem, namely the dependence of the electrical characteristics on the magnetic field at a specified noise level. The dependence of the noise itself on the magnetic field, which cannot be eliminated from the measurements, is determined by the actual form of the instability that causes the noise. Hence we limit our comparison with experiment to the relation (21), which is universal for any noise (or type of instability). The results of the comparison are given in the figure (the experimental points are taken from Shipuk[10]). Similar measurements have been carried out in[11]. Satisfactory agreement is observed for various mixtures at various gas pressures.



Comparison of the relation in (21) with experimental data obtained by Shipuk. [10] O—100 mm Hg Ar + 0.02 mm Hg Cs; Δ, ●— 50 mm Hg Ar – 0.02 mm Hg Cs; X—25 mm Hg Ar + 0.02 mm Hg Cs; +—10 mm Hg He + 0.02 mm Hg Cs; *—80 mm Hg Ar + 0.02 mm Hg Hg.

Among the other data, notice should be taken of a saturation effect of the effective Hall parameter, observed experimentally by a number of authors[8,12] (see also the review[13]). Simultaneous measurements of the noise level, carried out in these investigations, show a weak dependence of the density fluctuation on magnetic field. The level at which saturation of the Hall parameter occurs varies from 2 to 4 in various measurements and the relative fluctuation in conductivity is of the order of 0.5. These results are also in qualitative agreement with (25). The detailed data on the measurements of the Hall parameter are given in the review[13].

In the comparison, it should be noted that the quantity β_e for a system with an arbitrary (not necessarily equally divided) composition the relation (20) leads to the inequality

$$\beta_e \geqslant \beta\left\{\langle\sigma\rangle\left\langle\frac{1}{\sigma}\right\rangle+\beta^2\left[\langle\sigma\rangle\left\langle\frac{1}{\sigma}\right\rangle-1\right]\right\}^{-1/2}.$$

Hence, for a system of this kind the expressions in (23)—(25) give the lower bound of the Hall parameter.

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