# NONLINEAR ELECTRODYNAMICS OF THIN SUPERCONDUCTING FILMS

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The nonlinear electrodynamics of a thin superconducting film located in a nearly-critical stationary magnetic field is considered. At temperatures close to $T_c$, anomalous terms in the equations for superconductors become important. A very pronounced nonadiabatic pattern of dependence of the gap $\Delta$ on the field strength is observed in this case throughout the entire frequency range under consideration $(1/\tau_\epsilon \ll \omega \ll \Delta_0)$. This circumstance permits one to derive the quantitative dependence of the penetration factor D on the microwave power. The hysteresis nature of the dependence of D on the microwave power, which arises in this case, is considered. Current states of the film are also investigated briefly.

## 1. GENERAL FORMULAS

THE nonlinear electrodynamics of thin superconducting films was first considered by Kulik[1]. However, the nonstationary equations used in[1], as shown in[2], are not valid at finite temperatures. In general, in accordance with[3], there are no sufficiently general equations for superconductors in strong alternating fields. We shall consider below a thin $(d \ll \delta, \xi_0)$ superconducting film placed in a strong constant magnetic field $(1 - H/H_c \ll 1)$. This case, corresponding to the so-called "gapless superconductivity," is described by the equations derived in[4]:

$$\alpha\left[\left(1-\frac{H}{H_c}\right)-\frac{\langle \tilde{A}_x z\rangle}{{}^1/_3 H_c d^2}-\frac{\tilde{A}^2}{{}^2/_3 H_c^2 d^2}\right]\Delta-\beta\frac{\partial\Delta}{\partial t}-\gamma\Delta^3+2\pi TU\Delta=0, \tag{1}$$

$$\mathbf{j}=\sigma\left[\mathbf{E}-\frac{1}{\pi T\hbar c}\psi'\left(\frac{1}{2}+\rho_0\right)\tilde{\mathbf{A}}\Delta^2\right]. \tag{2}$$

Here $\tilde{\mathbf{A}}$ is the vector potential of the alternating field, and 2d is the thickness of the film. The constant magnetic field H is directed along the y axis parallel to the plane of the film (the xy plane), while the angle brackets $\langle\,\rangle$ denote averaging over z,

$$\alpha=4\pi T\psi'\left(\frac{1}{2}+\rho_0\right)\rho_0,\quad \beta=\frac{\hbar}{2}\psi'\left(\frac{1}{2}+\rho_0\right),$$

$$\gamma=\frac{1}{8\pi T}\left[-\psi''\left(\frac{1}{2}+\rho_0\right)-\frac{\rho_0}{3}\psi'''\left(\frac{1}{2}+\rho_0\right)\right],$$

$\psi(x)$ is the logarithmic derivative of the Gamma function,

$$\rho_0=\frac{1}{2\pi T}\frac{{}^2/_3 De^2H_c^2d^2}{\hbar c^2};$$

and $D = vl/3$ is the diffusion coefficient.

The anomalous term U satisfies the equation

$$\left(\frac{d}{dt}+\frac{1}{\tau_\epsilon}\right)U=-\frac{1}{8(\pi T)^2}f(\rho_0)\frac{d\Delta^2}{dt}, \tag{3}$$

where $f(\rho_0) = (2\rho_0)^{-1}\psi'(1/2+\rho_0) + 1/2\psi''(1/2+\rho_0)$ and $1/\tau_\epsilon$ is the reciprocal time of homogeneous relaxation.

In the absence of an alternating field, the gap $\Delta_0$ is equal to

$$\Delta_0^2=\frac{\alpha}{\gamma}\left(1-\frac{H}{H_c}\right).$$

We introduce the parameter $\nu = f(\rho_0)/4\pi T\gamma$ and change over to the dimensionless quantities

$$\Delta(t)=\eta(t)\Delta_0,\quad H=H_cH',\quad z=\sqrt{\frac{2}{3}}z'd,\quad \omega=\frac{8\pi T\rho_0}{\hbar}\omega',$$

$$A=H_c\sqrt{\frac{2}{3}}A'd,\quad E=\frac{8\pi T\rho_0}{\hbar c}\sqrt{\frac{2}{3}}H_cE'd,\quad j=\frac{8\pi T\rho_0}{\hbar c}\sqrt{\frac{2}{3}}H_c\sigma j'd.$$

Equations (1)–(3) take the form

$$[(1-H')-2\langle\tilde{A}_x'z'\rangle-\tilde{A}'^2]\eta^2-\frac{1}{2}\frac{d\eta^2}{dt'}-(1-H')\eta^4+U'\eta^2=0, \tag{4}$$

$$\left(\frac{d}{dt'}+\frac{1}{\tau_\epsilon'}\right)U'=-\nu(1-H')\frac{d\eta^2}{dt'}, \tag{5}$$

$$-j'=\frac{\partial\tilde{A}'}{\partial t'}+a(1-H')\eta^2\tilde{A}', \tag{6}$$

where

$$a=[\psi'(1/2+\rho_0)]^2/2\pi T\gamma.$$

The dependence of $\rho_0$ on T is described by the equation[5]

$$\ln\frac{T}{T_c}+\psi\left(\frac{1}{2}+\rho_0\right)-\psi\left(\frac{1}{2}\right)=0. \tag{7}$$

Near the critical temperature we have $\rho_0 = 2\pi^{-2}(1 - T/T_c) \ll 1$ and

$$f(\rho_0)=\frac{\pi^2}{4\rho_0},\quad \gamma=\frac{7\zeta(3)}{4\pi T_c},\quad a=\frac{\pi^4}{14\zeta(3)}\approx 5.8,$$

$$\nu=\frac{\pi^2}{28\zeta(3)}\frac{1}{\rho_0}\gg 1.$$

When $T \to 0$, to the contrary,

$$\rho_0\gg 1,\quad a=12,\quad \nu=3/(2\rho_0^2)\ll 1.$$

Thus, the anomalous term becomes significant when $T \to T_c$.

In the units employed here, Maxwell's equation takes the form

$$-j'=\frac{b}{4\pi}\frac{\partial^2A'}{\partial z'^2},\quad b^{-1}=\frac{{}^{16}/_3\pi T\rho_0\sigma d^2}{\hbar c^2}.$$

Equations (4)–(6) are analogous to those obtained in[2]. We shall henceforth omit the primes from the various quantities.

Let us make one preliminary remark. It can be easily verified that the ratio of the term $\langle\tilde{A}_x z\rangle$ to $\tilde{A}^2$ in (4) is of the order of $\tilde{H}^{-1}(d/\delta)^4(1 - H)^{-1/2}$. Thus, terms linear in the field are significant only in weak fields, $\tilde{H} \ll (d/\delta)^4(1 - H)^{-1/2}$. We shall henceforth neglect the linear terms in (1) and (4).

Assume that an external microwave is incident on the film. Let the frequency of the wave be much larger than $1/\tau_\epsilon$. In Eqs. (4) and (6), the vector potential $\underset{\sim}{A}$ describes the field inside the film, and its connection with the microwave field will be established later. When $\tau_\epsilon\omega \gg 1$, just as in[2], we obtain from (5)

$$U = -\nu(1-H)(\eta^2 - \overline{\eta^2}). \tag{8}$$

The bar denotes averaging with respect to time. Taking (8) into account, Eq. (4) has an exact solution (see[3]):

$$\eta^2 = \eta^2(0)F(t)\left[1 + 2\eta^2(0)(1-H)(\nu+1)\int_0^t F(t')dt'\right]^{-1}, \tag{9}$$

where

$$F(t) = \exp\left\{2\int_0^t [(1-H)(\nu\overline{\eta^2}+1) - \overline{A^2}]dt'\right\}.$$

The condition for the existence of a stationary solution is obviously

$$(1-H)(\nu\overline{\eta^2}+1) - \overline{A^2} > 0. \tag{10}$$

In this case we have when $t\to\infty$

$$\eta^2(t) = F(t)/2(1-H)(\nu+1)\int_0^t F(t')dt'. \tag{11}$$

Let the external field be harmonic, $E = E_0 \cos \omega t$. Then the vector potential is

$$\tilde{A} = -\frac{E_0}{\omega}\sin\omega t - q,$$

where q corresponds to direct current. The inequality (10) determines the critical ($\overline{\eta^2} = 0$) value $E_0 = E_{cr}$ (see[1]) at which a nonzero solution of (4) appears for the first time. When $q = 0$ we have

$$E_{cr}^2 = 2\omega^2(1-H). \tag{12}$$

Let us calculate $\overline{\eta^2}$. We note to this end that (11) can be written in the form

$$\eta^2(t) = \frac{1}{2(1-H)(\nu+1)}\left\{\ln\left[\int_0^t F(t')dt'\right]\right\}', \quad t\to\infty.$$

For a harmonic field

$$F(t) = \exp\left\{2\left[(1-H)(\nu\overline{\eta^2}+1) - \frac{E_0^2}{2\omega^2} - q^2\right]t + f(t)\right\},$$

where f(t) is a periodic function. Transforming, just as in[2], the obtained expressions for $t\to\infty$, we get

$$\overline{\eta^2} = 1 - \frac{E_0^2}{E_{cr}^2} - \frac{q^2}{1-H}. \tag{13}$$

Substituting once more (8) in (4) we obtain, with allowance for (13)

$$-\frac{1}{2}\dot{\eta}^2 + (1-H)\left[(\nu+1)\overline{\eta^2} - \frac{\overline{A^2} - A^2}{1-H}\right]\eta^2 - (1-H)(\nu+1)\eta^4 = 0. \tag{14}$$

We represent $\eta^2$ in the form $\eta^2 = \overline{\eta^2} + (\eta^2)_1$, where $(\eta^2)_1$ describes rapid variation of $\eta^2$ with frequency $\sim\omega(\overline{\eta^2})_1 = 0$). From (14) we obtain for the alternating part of $(\eta^2)_1$ the estimate

$$\left[O\left(\frac{\omega}{1-H}\right) + (\nu+1)(\overline{\eta^2} + (\eta^2)_1)\right](\eta^2)_1 \sim \frac{\overline{A^2} - A^2}{1-H}(\overline{\eta^2} + (\eta^2)_1).$$

We see therefore that when $\nu \gg 1$ and $\nu\overline{\eta^2} \gg 1$, and for all frequencies ($\omega\tau_\epsilon \gg 1$), we have $(\eta^2)_1 \ll \overline{\eta^2}$, and in particular when $\omega \ll \nu(1-H)$ we have $(\eta^2)_1 \sim 1/\nu$.

It is interesting to note that in[1] the condition $\omega \ll 1 - H$ would correspond to the fully adiabatic picture. However, as follows from the foregoing, even in this frequency region, when $T_c - T \ll T_c$, the presence of the anomalous term $\nu \gg 1$ leads in our case to the opposite situation, $\eta \approx \overline{\eta}$. Thus, when $T_c - T \ll T_c$, the interaction of the microwave power with the superconducting film can be described in a wide range of frequencies in the language of average quantities, which greatly simplifies the mathematical formalism and makes it possible to cope with the entire picture.

Assuming that the temperature satisfies the condition $\nu \approx 0.3(1 - T/T_c)^{-1} \gg 1$, let us turn to the equations describing the behavior of $\overline{\eta}(t)$ in the case of slow variation of the applied microwave power in the absence of a dc transport current through the film. We note first that, in the ordinary units, $1/\tau_\epsilon \sim 10^8$–$10^9$ sec$^{-1}$, and therefore the limitations on the frequency ($1/\tau_\epsilon \ll \omega \ll \Delta_0$) yield

$$10^8 \div 10^9\,\text{sec}^{-1} \ll \omega \ll 5\cdot 10^{11} T^0 (1 - T/T_c)^{1/2}(1 - H/H_c)^{1/2}\ \text{sec}^{-1}.$$

The condition $\omega \ll \nu(1-H)$ takes the form $\omega \ll 2 \times 10^{11}\,T^0(1 - H/H_c)\,\text{sec}^{-1}$. Consequently, there is the possibility of experimentally investigating the phenomena considered here even in the centimeter band.

Let us separate again $\eta^2$ in (4) into two parts: $\eta^2 = \overline{\eta}^2(t) + (\eta^2)_1$, where $(\eta^2)_1$ is a rapidly oscillating increment ($|(\eta^2)_1| \gg \eta^2$, and by virtue of this inequality it is possible to disregard the difference between $\overline{\eta^2}$ and $\overline{\overline{\eta^2}}$), whereas $\overline{\eta}(t)$ varies slowly. Retaining the principal terms and averaging over the period, we obtain from (4) and (5)

$$\overline{U} = (1-H)[\overline{\eta}^2 - 1 + E_0^2(t)/E_{cr}^2],$$
$$\dot{\overline{U}} + \tau_\epsilon^{-1}\overline{U} = -\nu(1-H)\dot{\overline{\eta}}^2,$$

from which we finally get

$$\nu\frac{d}{dt}\overline{\eta^2} = \frac{1}{\tau_\epsilon}\left(1 - \overline{\eta^2} - \frac{E_0^2(t)}{E_{cr}^2}\right) - \frac{d}{dt}\left(\frac{E_0^2}{E_{cr}^2}\right). \tag{15}$$

In accordance with the remarks made above, this equation is applicable only when $\overline{\eta}^2 \gg 1/\nu$. Of course, when $E_0^2/E_{cr}^2 > 1$ we have, according to (10), $\overline{\eta} = 0$.

## 2. TRANSMISSION COEFFICIENT OF MICROWAVE RADIATION

Let us establish now a connection between the field of the incident wave, the field of the wave passing through the film, and the field $E_0$ inside the film. We shall assume the transmission coefficient $D \sim (\delta^2)/\lambda d)^2$, where $\delta$ is the depth of penetration of the field, to be small. The field $E_0(z)\exp(-i\omega t)$ inside the film ($-d < z < d$) satisfies the equation

$$\frac{b}{4\pi}\frac{\partial^2 E}{\partial^2 z} = \frac{\partial E}{\partial t} + a\overline{\eta^2}(1-H)E.$$

Expanding, as in[1], $E_0(z)$ in powers of $z/\delta$, using the continuity of E and H on the boundary of the film, and neglecting terms of order $\delta^2/\lambda d$ and $d/\lambda$ ($\lambda$ is the wavelength), we find that the amplitude of the transmitted wave is $E_2 = E_0$, and the amplitude of the incident wave E is connected with $E_0$ by the relation

$$(kd)^2E^2 = \frac{9}{4}\left[\left(\frac{4\pi\omega}{b}\right)^2 + \left(\frac{4\pi a}{b}\right)^2(1-H)^2\overline{\eta^4}\right]E_0^2. \tag{16}$$

Equations (15) and (16) determine the transmission coefficient $D = (E_2/E)^2 = (E_0/E)^2$ and the behavior of the reduced gap $\eta$ as a function of the microwave power.

Let $\tilde{\omega} = \omega/a(1-H)$. We then get from (16)

$$E_0^2 = \frac{(kd)^2}{{}^9/_4(4\pi a/b)^2(1-H)^2}\frac{E^2}{\bar{\eta}^4+\tilde{\omega}^2}. \tag{17}$$

The transmission coefficient is

$$D = \frac{(kd)^2}{{}^9/_4(4\pi a/b)^2(1-H)^2(\bar{\eta}^4+\tilde{\omega}^2)}.$$

The limiting values of the transmission coefficient are: in the normal state

$$D_N = \frac{(kd)^2}{{}^9/_4(4\pi a/b)^2(1-H)^2\tilde{\omega}^2} = \pi^2\left(\frac{\delta_{cr}^2}{\lambda d}\right)^2, \quad \delta_{cr}^2 = \frac{c^2}{2\pi\sigma\omega}$$

(for films $d \sim l \sim 10^{-6}$ cm we have $D_N \sim 10^{-3}-10^{-4}$)); in the superconducting state at low frequencies $\tilde{\omega} \ll 1$ and in a weak external field

$$D_{S0} = \frac{(kd)^2}{{}^9/_4(4\pi a/b)^2(1-H)^2}, \quad \frac{D_{S0}}{D_N} = \tilde{\omega}^2. \tag{18}$$

Introducing the wave intensity $I = ScE^2/8\pi$, where S is the area of the surface of the film, we rewrite (17) in the form

$$\frac{E_0^2}{E_{cr}^2} = \frac{D_{S0}}{\bar{\eta}^4+\tilde{\omega}^2}\frac{E^2}{E_{cr}^2} = \frac{\tilde{\omega}^2}{\bar{\eta}^4+\tilde{\omega}^2}\tilde{I}. \tag{19}$$

Here $\tilde{I} = I/I_{cr}$, where the critical intensity $I_{cr} = ScE_{cr}^2/8\pi D_N$, according to (10) and (12), determines the instant of occurrence of $\bar{\eta} \neq 0$ when I tends to $I_{cr}$ from the direction of the normal metal. In ordinary units

$$E_{cr}^2 = \frac{\omega^2}{c^2}\frac{(2dH_c)^2}{3}\left(1-\frac{H}{H_c}\right),$$

and therefore (see[6]) where $T \to T_c$ and $\omega = 10^{10}$ sec$^{-1}$

$$I_{cr} \sim \frac{10^{-4}T_c^0(1-T/T_c)(1-H/H_c)S(\mathrm{cm}^2)}{D_N}[\mathrm{W}].$$

The solutions corresponding to $\bar{\eta} \neq 0$ are determined from (15), in which we can omit the last term if we confine ourselves to slow variation of I (and $\nu \gg 1$). With allowance for (19) we get, finally,

$$\dot{\eta} = \frac{1}{2\nu\tau_\varepsilon\bar{\eta}}\left(1-\bar{\eta}^2-\frac{\tilde{\omega}^2}{\bar{\eta}^4+\tilde{\omega}^2}\tilde{I}\right). \tag{20}$$

Let us consider first the steady state regime. Equation (20) yields

$$(\bar{\eta}^4+\tilde{\omega}^2)(1-\bar{\eta}^2) = \tilde{\omega}^2\tilde{I}. \tag{21}$$

Expressing $\bar{\eta}$ in terms of the ratio $D/D_N = x$, $x = \tilde{\omega}^2/(\bar{\eta}^4+\tilde{\omega}^2)$, we obtain from (21) an equation for x:

$$\tilde{\omega}\sqrt{(1-x)/x} = 1 - x\tilde{I}. \tag{21'}$$

Figure 1 shows schematically the behavior of x as a function of $I/I_{ci}$. The dashed lines correspond to unstable sections of the curve (see the Appendix). Curve 1 is obtained when $\tilde{\omega}^2 < {}^1/_4$, curve 2 corresponds to $\tilde{\omega}^2 = {}^1/_3$, whereas curve 4 takes place when $\tilde{\omega}^2 > {}^1/_3$. The limits of the stable sections correspond to the points B and C. For the point B we have

$$\tilde{I}_B = \frac{2}{27}\frac{(1+\sqrt{1-3\tilde{\omega}^2})(1-3\tilde{\omega}^2)+12\tilde{\omega}^2}{\tilde{\omega}^2},$$

$$x_B = \frac{9\tilde{\omega}^2}{2(1+\sqrt{1-3\tilde{\omega}^2}+3\tilde{\omega}^2)} \quad \eta_B^2 = \frac{1}{3}(1+\sqrt{1-3\omega^2}). \tag{22}$$

The points C correspond to

$$\tilde{I}_C = \frac{2}{27}\frac{(1-\sqrt{1-3\tilde{\omega}^2})(1-3\tilde{\omega}^2)+12\tilde{\omega}^2}{\tilde{\omega}^2},$$

$$x_C = \frac{9\tilde{\omega}^2}{2(1-\sqrt{1-3\tilde{\omega}^2}+3\tilde{\omega}^2)}, \quad \eta_C^2 = \frac{1}{3}(1-\sqrt{1-3\tilde{\omega}^2}). \tag{22'}$$

As is seen from (18) and Fig. 1, the hysteresis loop is large when $\tilde{\omega}^2 < {}^1/_4$, particularly when $\tilde{\omega}^2 \ll 1$. The latter case corresponds to $\tilde{I}_B \approx {}^4/_{27}\tilde{\omega}^2$, $\tilde{I}_C \approx 1$, and $D_B = {}^9/_4 D_{S0}$ and $D_C = D_N$. In the frequency region ${}^1/_4 < \tilde{\omega}^2 < {}^1/_3$, the hysteresis section lies at $I < I_{cr}$, and finally at $\tilde{\omega}^2 > {}^1/_3$ there is no hysteresis.

The existence of hysteresis of the transmission coefficient was pointed out by Kulik[1]. The condition $\nu \gg 1$ makes it possible, as we see, to obtain, unlike in[1], the quantitative picture of the phenomenon.

One must assume that a stationary regime is possible at a given intensity, but only on one of the sections of the curve of Fig. 1. In other words, when the power changes the regime will change jumpwise from the upper to the lower section of the curve and back. The question of the choice between the two stable roots of Eq. (21) is quite complicated. It seems to us, however, that in this case it is possible to use Langevin's general method of random forces[6,7]. To this end, we note that it is necessary to insert in the right side of the initial equation (1) the fluctuation force f(r, t), which has a random character[8]. The correlator of the forces (the averaging is over the statistical distribution)

$$\langle f(\mathbf{r},t)f(\mathbf{r}',t')\rangle = A\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

is chosen in such a way that the mean value $\langle\Delta(\mathbf{r},t)\Delta^*(\mathbf{r}',t')\rangle$ above the transition temperature coincide with the thermodynamic mean[8]. Near $T_c$ we have $A = (1/2)V\hbar\pi^2 T_c$.

The next step is to write a Fokker-Planck equation for the function $W(\bar{\eta}, t)$, which has the meaning of the probability of the given value $\bar{\eta}$ the instant t[7]. Recognizing that the characteristic frequencies of the fluctuation forces are large, we can rewrite (20) in the form

$$\dot{\eta} = \frac{1}{2\nu\tau_\varepsilon\bar{\eta}}\left(1-\bar{\eta}^2-\frac{\tilde{\omega}^2}{\bar{\eta}^4+\tilde{\omega}^2}\tilde{I}\right) - \frac{\bar{f}(t)}{\beta\Delta_0},$$

where f(t) are averaged over the volume of the film V:

$$\langle\bar{f}(t)\bar{f}(t')\rangle = \frac{A}{V}\delta(t-t').$$

For $W(\bar{\eta}, t)$ we have

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial\bar{\eta}}(F(\bar{\eta})W) + \frac{A}{2(\beta\Delta_0)^2 V}\frac{\partial^2 W}{\partial\bar{\eta}^2},$$

where

$$F(\bar{\eta}) = \frac{1}{2\nu\tau_\varepsilon\bar{\eta}}\left(1-\bar{\eta}^2-\frac{\tilde{\omega}^2}{\bar{\eta}^4+\tilde{\omega}^2}\tilde{I}\right).$$

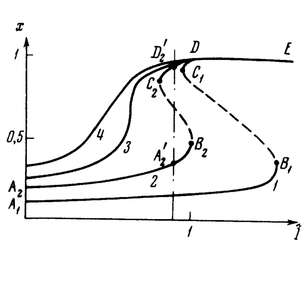


FIG. 1. Dependence of $D/D_N$ on I (shown schematically; the abscissa scale is stretched). The dash-dot line corresponds to I* for $\tilde{\omega}^2 = 1/4$.

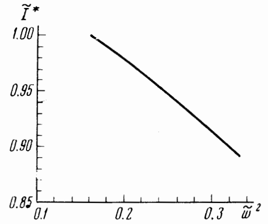


FIG. 2. Dependence of I* on $\tilde{\omega}^2$.

The stationary solution $W(\bar{\eta})$ has the form of $\exp\{S(\bar{\eta})\}$, where

$$S(\bar{\eta})=\frac{2(\beta\Delta_0)^2V}{A}\int F(\bar{\eta})\,d\bar{\eta}.$$

The stable roots of Eq. (21) correspond to the maxima of the distribution function. The transition from one branch to the other takes place thus at an intensity $I^*$ determined by the equality of the functionals $S(\bar{\eta}_2) = S(\bar{\eta}_2)$ for the two stable solutions of (21). The functional $S(\bar{\eta})$, which plays the role of the "effective energy," is proportional to the quantity

$$(1-\tilde{I})\ln\bar{\eta}-\frac{1}{2}\bar{\eta}^2+\frac{\tilde{I}}{4}\ln(\bar{\eta}^4+\tilde{\omega}^2). \tag{23}$$

Figure 2 shows a result of the numerical calculation of $I^*$ as a function of $\tilde{\omega}^2$. At $\tilde{\omega}^2 \approx 0.16$ we have $I^* = 1$. This raises the question of what happens with further decrease of the frequency. Expression (23) for $S(\bar{\eta})$ diverges logarithmically as $\bar{\eta} \to 0$. This means that the method used above ceases to be applicable (it is impossible to use the mean values near $\bar{\eta} = 0$). Thus, this question still remains open.

## 3. FILM UNDER NONSTATIONARY CONDITIONS

We now proceed to the question of the behavior of $\bar{\eta}$ and of the transmission coefficient D in the case of slow variation of the external microwave power. In the case it is necessary to use the general expression (20). In the case of sufficiently slow variation of I far from the critical points B and C of Fig. 1, the value of $\bar{\eta}$, in the main, follows adiabatically the variation of the power. Let us assume that we have succeeded in this manner in entering into the metastable regions A′B or D′C. The adiabatic behavior of $\bar{\eta}$ will take place up to the critical points B and C, but the vicinities of these points already call for a special analysis.

Let us examine, for example, the behavior of $\bar{\eta}$ near the point B. We put $\tilde{I} = \tilde{I}_B + \tilde{I}_1 + \tilde{I}_B - \tilde{I}_{10} + \tilde{I}'t$, where $\tilde{I}_B$ is defined by (22), $\tilde{I}_{10} \ll 1$, and $\tilde{I}'$ is the derivative of $\tilde{I}$ with respect to t. If $\tilde{I}' < 0$, the value of $\bar{\eta}$ is close to $\eta_B$, viz., $\bar{\eta}^2 = \eta_B^2 + \delta$. So long as $|\delta| \ll 1$, we obtain from (20) for $\delta$ the Ricatti equation

$$(\eta_B^4+\tilde{\omega}^2)\dot{\delta}=-\frac{1}{\tau_\varepsilon\nu}[(-\tilde{I}_{10}+\tilde{I}'t)\tilde{\omega}^2+(3\eta_B^2-1)\delta^2]. \tag{24}$$

When $t = 0$, $\delta$ is equal to $I_{10}\tilde{\omega}^2/(3\eta_B^2 - 1)^{1/2}$. A solution of (24), satisfying such an initial condition, is

$$\delta=-\frac{\nu\tau_\varepsilon(\eta_B^4+\tilde{\omega}^2)}{3\eta_B^2-1}\,\frac{[t_1^{1/2}K_{1/3}(Ct_1^{3/2})]_{t_1}'}{t_1^{1/2}K_{1/3}(Ct_1^{3/2})}. \tag{25}$$

Here

$$C=\frac{2}{3}\,\frac{\tilde{\omega}(3\eta_B^2-1)^{1/2}}{\nu\tau_\varepsilon(\eta_B^4+\tilde{\omega}^2)}(\tilde{I}')^{1/2},$$

and $t_1$ is determined by the relation $-\tilde{I}_{10} + \tilde{I}'t = -\tilde{I}'t_1$. When $t_1 = 0$ we have $\tilde{I} = \tilde{I}_B$, and $t_1$ reverses sign with further increase of t. The solution (25) is transformed into

$$\delta=\left(\frac{\tilde{\omega}^2\tilde{I}'t}{3\eta_B^2-1}\right)^{1/2}\frac{J_{-2/3}(Ct^{3/2})-J_{2/3}(Ct^{3/2})}{J_{-1/3}(Ct^{3/2})+J_{1/3}(Ct^{3/2})} \tag{26}$$

(we now reckon the time t from the instant that I goes through $I_B$). With increasing t, the numerator (26) vanishes, after which $\delta$ becomes negative. Then the denominator tends to zero. In this region, using the relation

$$J_{-2/3}(x)-J_{2/3}(x)=\frac{1}{3x}[J_{-1/3}(x)+J_{1/3}(x)]+\frac{d}{dx}[J_{-1/3}(x)+J_{1/3}(x)],$$

we can write (26) in the form

$$\delta=\frac{\nu\tau_\varepsilon(\eta_B^4+\tilde{\omega}^2)}{3\eta_B^2-1}\,\frac{1}{t_B-t}, \tag{27}$$

where $Ct_B^{3/2} = x_0$ is the root of the equation $J_{-1/3}(x) + J_{1/3}(x) = 0$ ($x_0 \approx 2.3$);

$$t_B\approx 2{,}7\left[\frac{\nu\tau_\varepsilon(\eta_B^4+\tilde{\omega}^2)}{(1-3\tilde{\omega}^2)^{1/4}\tilde{\omega}(\tilde{I}')^{1/2}}\right]^{2/3}.$$

When t approaches $t_B$, the value of $\delta$ increases without limit in absolute magnitude, so that we must turn to Eq. (20). Neglecting in it the quantity $\tilde{I}_1 \approx \tilde{I}'t_B \sim (\nu\tau_\epsilon\tilde{\omega}^2\tilde{I}')^{2/3} \ll 1$, we obtain for $\bar{\eta}$ the quadrature

$$\int_{\eta_B^2+\delta}^{\bar{\eta}^2}\frac{(\bar{\eta}^4+\tilde{\omega}^2)\,d\bar{\eta}^2}{\tilde{\omega}^2\tilde{I}_B-(1-\bar{\eta}^2)(\bar{\eta}^4+\tilde{\omega}^2)}=-\frac{1}{\nu\tau_\varepsilon}(t+\mathrm{const}). \tag{28}$$

Expression (28) must be "joined together" with the solution (27).

The denominator of the integrand in (28) has a double root $\eta_B^2$ and a simple root $\eta_0^2 = 1 - 2\eta_B^2$. Calculations yield

$$\frac{1}{\nu\tau_\varepsilon}(t_B-t)=\frac{\eta_B^4+\tilde{\omega}^2}{(1-3\tilde{\omega}^2)^{1/2}}\,\frac{1}{\eta_B^2-\bar{\eta}^2}+\left(1-\frac{\eta_0^4+\tilde{\omega}^2}{1-3\tilde{\omega}^2}\right)\ln(\eta_B^2-\bar{\eta}^2)$$
$$+\frac{\eta_0^4+\tilde{\omega}^2}{1-3\tilde{\omega}^2}\ln(\bar{\eta}^2-\eta_0^2). \tag{29}$$

If $1/4 < \tilde{\omega}^2 < 1/3$, then $\eta_0^2$ lies on the section CD (Fig. 1), and when $t \to \infty$ we have $\bar{\eta}^2 \to \eta_0^2$, i.e., the last term of (29) becomes important, and we obtain

$$\eta^2=\eta_0^2+\exp\left\{\frac{1-3\tilde{\omega}^2}{\eta_0^4+\tilde{\omega}^2}\,\frac{t_B-t}{\nu\tau_\varepsilon}\right\}.$$

If $\tilde{\omega}^2 < 1/4$, then $\bar{\eta}^2$ vanishes (accurate to $\sim 1/\nu$) at $t = t_*$, where

$$t_*=t_B-\nu\tau_\varepsilon\left\{\frac{\eta_B^4+\tilde{\omega}^2}{(1-3\tilde{\omega}^2)^{1/2}}\,\frac{1}{\eta_B^2}+\left(1-\frac{\eta_0^4+\tilde{\omega}^2}{1-3\tilde{\omega}^2}\right)\ln\eta_B^2\right.$$
$$\left.+\frac{\eta_0^4+\tilde{\omega}^2}{1-3\tilde{\omega}^2}\ln|\eta_0^2|\right\}.$$

The last term here represents a small correction, since $t_B \gg \nu\tau_\epsilon$, therefore the principal time of variation of $\eta$ reduces to $t_B$. The case of decreasing power is considered analogously.

## 4. CURRENT STATE OF THE FILM

Assume now that a constant transport current $\bar{j}$ flows through a film situated in an external microwave field. In this case we obtain in place of (15)

$$\dot{\bar{\eta}^2}=\frac{1}{\nu\tau_\varepsilon}\left(1-\bar{\eta}^2-\frac{E_0^2}{E_{\text{кр}}^2}-\frac{q^2}{1-H}\right),$$
$$\bar{j}=a(1-H)\bar{\eta}^2q.$$

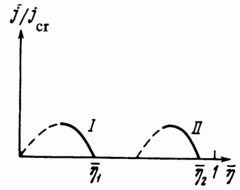


FIG. 3. Schematic dependence of current on $\bar\eta$ at a given frequency $1/4 < \tilde\omega^2 < 1/3$ intensity $I_C < I < I_B$.

As a result (see (19))

$$\dot{\bar\eta}=\frac{1}{2\nu\tau_\varepsilon\eta}\left(1-\eta^2-\frac{\tilde\omega^2\tilde I}{\eta^4+\tilde\omega^2}-\frac{4}{27}\left(\frac{\bar j}{j_{cr}}\right)^2\frac{1}{\eta^4}\right) \qquad (30)$$

where $j_{cr} = (\frac{2}{3}\sqrt{3})a(1-H)^{3/2}$ is the critical current at $I = 0$. The stationary solutions of (30) have the form

$$\frac{4}{27}\left(\frac{\bar j}{j_{cr}}\right)^2=\bar\eta^4\left[1-\bar\eta^2-\frac{\tilde\omega^2\tilde I}{\bar\eta^4+\tilde\omega^2}\right].$$

The expression in the square brackets in (31) vanishes if $\bar\eta$ and $\tilde I$ at a given frequency are connected by the relation (21), i.e., if they are determined by the curve of Fig. 1. As seen from Fig. 1, at different $\tilde\omega$ and $\tilde I$ the expression in the square brackets in (31) can have one, two, or three roots. The dependence of the current on $\bar\eta$ is quite complicated in this case. Figure 3 shows, for example, the most interesting case, in our opinion, of three roots of the expression in the square brackets in (31), corresponding to $I_C < I < I_B$ at $\frac14 < \tilde\omega^2 < \frac13$.

The maxima of the right side of (31) are determined by the equation $3\bar\eta^2 - 2 + 2\tilde\omega^4\tilde I/(\bar\eta^4+\tilde\omega^2)^2 = 0$. We see that this equation coincides with the equation that determines the limits of stability of the stationary solutions of Eq. (30) with respect to infinitesimally small perturbations (see the Appendix). Thus, the values of $\eta$ corresponding to the extrema of the right side of (31) coincides with the values of $\eta$ which determine the limits of the instability. An analysis of different possible curves shows that only the decreasing sections of the dependence of the current on $\eta$ are always stable.

The question of which of the decreasing sections of the current (I or II in Fig. 3) is actually realized can be answered in the same manner as in Sec. 2 (the random-force method). In this case we have for the stationary functional $S(\bar\eta)$ (see Sec. 2) at specified $\tilde I$ and $\bar j$

$$S\propto(1-\tilde I)\ln\bar\eta+\frac{\tilde I}{4}\ln(\bar\eta^4+\tilde\omega^2)-\frac{1}{2}\bar\eta^2+\frac{1}{27}\left(\frac{\bar j}{j_{cr}}\right)^2\frac{1}{\bar\eta^4}. \qquad (32)$$

Section I is realized if $S(I) > S(II)$, and vice versa.

Let us turn to the case shown in Fig. 3 at $\bar j = 0$, if $I > I^*$, the point $\eta_1$ will be realized, in accordance with the result of Sec. 2 ($S(\bar\eta_1) > S(\bar\eta_2)$). Therefore the section II is not possible at all, for when the direct current is introduced we can only decrease the parameter $\eta$ of the superconductor. However, if $I < I^*$, then the section 2 is initially realized at low current, and then, when the current increases, the transition from curve II to curve I is possible. With further increase of $\bar j$, the superconductivity is destroyed by the current These transitions are revealed by the jumps of the transmission coefficient.

Let us dwell briefly on the question of the generation of a difference frequency by the film in the presence of a microwave containing two close harmonics. Let the electric field inside the film be equal to $E_{10}\cos\omega_1 t + E_{20}\cos\omega_2 t$. We assume that $\tau_e\omega_{1,2} \gg 1$ and $\Omega = \omega_1 - \omega_2 \ll \omega_{1,2}$. For arbitrary temperatures we obtain for the amplitude of the harmonic $\eta^2$ of the frequency $\Omega$, in accord with[4],

$$(\eta^2)_\Omega=\frac{2E_{10}E_{20}\mu}{E_{cr}^2}, \quad \mu=\frac{\sqrt{1+\Omega^2\tau_\varepsilon^2}}{\sqrt{1+(\nu+1)^2\Omega^2\tau_\varepsilon^2}}.$$

This expression is valid also for arbitrary intensities $I_1 \propto E_1^2$ and $I_2 \propto E_2^2$, provided $\omega \ll 1 - H$.

The field generated by the film in the space outside the film is connected with the current flowing through the film by the relation

$$-j=\frac{b}{4\pi}\frac{\partial H}{\partial z}.$$

Therefore the amplitude $H_\Omega$ of the magnetic field generated by the film, in accordance with[7], is

$$H_\Omega=\frac{4\pi}{b}d\bar j\frac{\eta_\Omega^2}{\bar\eta^2}.$$

We put $H_{cr} \equiv 4\pi d j_{cr}/b$, and then

$$\frac{H_\Omega}{H_{cr}}=\frac{\bar j}{j_{cr}}\frac{\eta_\Omega^2}{\bar\eta^2}.$$

At a temperature close to $T_c$, Eq. (19) is valid, and therefore

$$\frac{H_\Omega}{H_{cr}}=\mu\frac{\bar j}{j_{cr}}\frac{2\tilde\omega^2}{\eta^2(\bar\eta^4+\tilde\omega^2)}\sqrt{\tilde I_1\tilde I_2}, \qquad (33)$$

and $\bar\eta$ and $\bar j$ are connected by Eq. (31), where $\tilde I = \tilde I_1 + \tilde I_2$. In weak fields and for a weak current, we have

$$\frac{H_\Omega}{H_{cr}}=\mu\frac{\bar j}{j_{cr}}\frac{2\tilde\omega^2}{1+\tilde\omega^2}\sqrt{\tilde I_1\tilde I_2}.$$

In[9] the plot of $H_\Omega$ against $\tilde I_1/\tilde I_2$ has a maximum. It is seen from (31) and (33) that $H_\Omega$ actually has a maximum at $\tilde I_1 = \tilde I_2$, if $\tilde I_1 + \tilde I_2 = \text{const}$. However, the maximum observed in[9] should not be observed in our model. (The authors of[9]) investigated films without a magnetic field. In fact, inasmuch as only the decreasing sections of the plot of $\bar j$ against $\bar\eta$ are stable, the first part in (33) will increase monotonically with the current on each stable section. From the point of view of generation of $H_\Omega$, transitions from one branch to the other, referred to above, will be manifest in jumps of $H_\Omega$ at certain values of the current, with further increase of $H_\Omega$ up to total destruction of the superconductivity.

It would be of interest to perform similar experiments.

In conclusion, the authors consider it their pleasant duty to thank G. M. Eliashberg and A. P. Kazantsev for useful discussions.

## APPENDIX

### 1. Stability of Stationary Solutions of Eq. (20)

For a small increment $\delta$, $\eta^2 = \eta_0^2 + \delta$, where $\eta_0$ satisfies Eq. (21), we get from (20)

$$\dot\delta=-\frac{1}{\nu\tau_\varepsilon}\left[\frac{3\eta_0^4-2\eta_0^2+\tilde\omega^2}{\eta_0^4+\tilde\omega^2}\right]\delta.$$

The numerator in the square brackets is positive if $\eta_0 < \eta_C$ or $\eta_0 > \eta_B$, in other words, the upper and lower solid branches in Fig. 1 are stable, and the central dashed branch is unstable.

## 2. Stability of Stationary Solutions of Eq. (30)

For a small increment $\delta$ to the stationary solution $\eta_0^2$, satisfying Eq. (31), we get

$$\dot{\delta}=-\frac{1}{\nu\tau_\varepsilon}\left[1-\frac{2\eta_0^2\tilde{\omega}^2\bar{I}}{(\eta_0^4+\tilde{\omega}^2)^2}-\frac{8}{27}\frac{1}{\eta_0^6}\left(\frac{\bar{j}}{j_{\text{кр}}}\right)^2\right]\delta.$$

Expressing the current in terms of $\eta_0$ in accordance with (31), we obtain the stability condition in the form

$$3\eta_0^2-2+\frac{2\tilde{\omega}^4\bar{I}}{(\eta_0^4+\tilde{\omega}^2)^2}>0.$$


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