PROPAGATION OF ACOUSTIC WAVES IN A PIEZO-SEMICONDUCTOR LOCATED IN AN ALTERNATING ELECTRIC FIELD

V. M. LEVIN and L. A. CHERNOZATONSKIĬ

Institute of Physical Engineering and Radio Engineering Measurements, USSR Academy of Sciences

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Some features of ultrasonic propagation in a piezo-semiconductor to which an alternating electric field is applied are investigated. The acoustic wave in the crystal is found to be modulated due to the piezo-coupling. As a result, the natural oscillation of the lattice for the wave vector q is a set of waves whose frequencies differ from the fundamental frequency by an amount which is a multiple of the external field frequency. Dispersion relations are obtained for the fundamental frequency $\omega'(q)$ and the decrement (increment) γ of the sound, both under conditions of parametric resonance and for conditions far removed from resonance. In the nonresonance case, ω' depends weakly on the external field, whereas the increment in the presence of an external alternating field is an oscillating function of the constant dc of the electric field as well as of the amplitude of its alternating part. The sound oscillation spectra (i.e., the dependences of ω' and γ on q) are greatly modified under parametric resonance conditions as the result of parametric coupling between the various acoustic modes. Various possibilities of modification of the spectrum in the case of parametric coupling between two modes are investigated in detail.

IT is well known[1] that damping or growth of elastic waves in piezo-semiconductors is closely connected with the properties of the electronic subsystem. We have investigated the features of the propagation of sound waves in piezo-semiconductors following application of an external homogeneous electric field that contains both a constant (in time) and an oscillating (with frequency Ω) component:[1)]

$$\mathbf{E}_{ext} = \mathbf{E}_0 + \mathbf{E}_1 \cos \Omega t.$$

1. FUNDAMENTAL EQUATIONS

We shall describe the motion of the lattice by means of the ordinary equation of elasticity theory, which takes the piezo-effect into account:

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \lambda_{iklm} \frac{\partial u_{lm}}{\partial x_k} = \beta_{l,ik} \frac{\partial E_l}{\partial x_k}, \quad (1.1)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the lattice displacement vector; $\mathbf{E}(\mathbf{x}, t)$ is the electric field, which is due both to the lattice deformations and to the deviation of the electron density from its mean value; ρ is the density of the crystal, λ_{iklm} is the tensor of elastic constants, $\beta_{l,ik}$ the piezotensor.

The electric field $\mathbf{E}(\mathbf{x}, t)$ in the crystal obeys Poisson's equation

$$\varepsilon_0 \frac{\partial E_i}{\partial x_i} = 4\pi en + 4\pi \beta_{i,jl} \frac{\partial^2 u_j}{\partial x_i \partial x_l}, \quad (1.2)$$

where ϵ_0 is the dielectric permittivity of the lattice, e the electron charge, $n(\mathbf{x}, t)$ the deviation of the electron density from its mean value n_0. These equations must be supplemented by the continuity equation

$$e \frac{\partial n}{\partial t} + \frac{\partial j_k}{\partial x_k} = 0 \quad (1.3)$$

and by the expression for the electric current $\mathbf{j}(\mathbf{x}, t)$, which has the form

$$\mathbf{j} = e\mu(n_0 + n)(\mathbf{E}_{ext} + \mathbf{E}) - \mu T_e \frac{\partial n}{\partial \mathbf{x}}. \quad (1.4)$$

for the condition $\omega \ll \nu$. Here ω is the characteristic frequency of the sound waves, ν the frequency of carrier collisions, $\mu = e/m\nu$ the mobility of the electrons, m the effective mass, and T the effective temperature of the electrons.

In an electric field, the effective electron temperature is connected with the external electric field by the well-known formula:[5]

$$T_e = T_{at} + \frac{2}{3} \frac{e^2 E_{ext}^2}{m\delta\nu^2}, \quad (1.5)$$

where T_{lat} is the lattice temperature, and δ the mean fraction of the energy transferred to the scatterer in a single effective collision. In what follows, we shall consider only such scattering mechanisms for which the effective frequency does not depend on T_e and consequently does not depend on time. Since T_e is a function of time, the parametric effects are connected both with the conduction current and with the diffusion current. However, simple estimates show that the effects connected with the diffusion current are smaller by the factor $\delta\nu v_s/\omega v_d$. For reasonable values of the drift velocity, $\omega v_d/\delta\nu v_s \ll 1$, and dependence of T_e on time does not have to be taken into account. Here v_s is the characteristic velocity of the sound wave.

A characteristic property of piezo-semiconductors is the coupling between the lattice and plasma subsystem, which is characterized by the *electromechanical coupling constant* $\eta^2 = 4\pi\beta^2/\epsilon_0\rho v_s^2 \ll 1$ (β is the effective piezomodulus). The propagation of the sound wave in the crystalline medium leads to the appearance of an induced wave of electron density with the same frequency ω and wave vector q. Upon application to

[1)]The problem of the effect of an alternating electric field on the propagation of sound waves in piezo-semiconductors was first put forward in [2-4].

the specimen of an external periodic field $E_1 \cos \Omega t$, waves of electron density appear in the plasma subsystem in addition to the main wave, with combination frequencies $\omega + k\Omega$ $(k = \pm 1, \pm 2, \ldots)$ and vector $\mathbf{q}$. Such oscillations in turn produce (owing to the inverse piezoeffect) sound waves with the same frequencies. The usual picture of a parametric effect appears. Inasmuch as the corrections in the equation of motion of the lattice, necessitated by the allowance for the plasma subsystem, contain the small parameter η^2, the coupling between the acoustic oscillations with the same $\mathbf{q}$ is determined by this parameter. For an arbitrary value of the external periodic excitation, the propagation of the sound waves in the piezo-semiconductor can be considered with the help of the method of successive approximations in terms of the small coupling constant between the subsystems. Such a method is similar to the method which was used by Silin[6] for a collision-free electron-ion plasma in an external alternating electric field.

We shall assume that the quantities $u(x, t)$, $E(x, t)$, and $n(x, t)$ associated with the sound field are proportional to $e^{i\mathbf{q}\cdot\mathbf{x}}$. By linearizing the system (1.1)–(1.4) and eliminating the electric field $\mathbf{E}$, we get the equations which connect the lattice displacement with the nonequilibrium density of the electrons:

$$\rho \frac{\partial^2 u_i}{\partial t^2} + \lambda_{ij} q^2 u_j = \frac{4\pi e \beta_i}{\varepsilon_0} n, \tag{1.6}$$

$$\left[\frac{\partial}{\partial t} + i\mathbf{q}\mathbf{v}_0 + i\mathbf{q}\mathbf{v}_1 \cos \Omega t + a\right] n = \frac{4\pi\mu n_0 q^2}{\varepsilon_0} \beta_j u_j, \tag{1.7}$$

where

$$\lambda_{ij} = \lambda_{iljm} \frac{q_l q_m}{q^2} + \frac{4\pi\beta_i\beta_j}{\varepsilon_0}, \quad \beta_i = \beta_{l,ik} \frac{q_l q_k}{q^2},$$

$$\mathbf{v}_0 = \mu \mathbf{E}_0, \quad \mathbf{v}_1 = \mu \mathbf{E}_1, \quad a = \omega_M (1 + q^2 r_0^2),$$

$$\omega_M = 4\pi e^2 \mu n_0 / \varepsilon_0,$$

r_0 is the Debye radius.

We eliminate the external alternating field E_1 from Eq. (17), introducing a new unknown function $N(t) = n(t) e^{i\xi \sin \Omega t}$, where $\xi = \mathbf{q}\cdot\mathbf{v}_1/\Omega$. The function $N(t)$ has the meaning of an alternating electron density in the system of coordinates moving together with the volume charge under the action of the external oscillating field. We shall seek the quantities $\mathbf{u}$ and N in the form of a set of waves consisting of the fundamental oscillation with frequency ω and oscillations at the combination frequencies:

$$g(t) = \sum_{l=-\infty}^{\infty} g(l) e^{-i(\omega - l\Omega)t}, \quad g = \{\mathbf{u}, N\}. \tag{1.8}$$

Here the frequency ω is close to the frequency of one of the acoustic branches of the oscillations, corresponding to the chosen value of the wave vector $\mathbf{q}$. Using the well-known formula

$$e^{i\xi \sin \Omega t} = \sum_{l=-\infty}^{\infty} J_l(\xi) e^{il\Omega t},$$

where $J_l(\xi)$ is the Bessel function, one can express $N(t)$ in terms of the amplitude of the sound oscillations. Substituting $N(l)$ in the equation of motion of the lattice, we get an infinite set of interlocking equations for the displacement vectors $\mathbf{u}(k)$. In the system of coordinates in which the tensor λ_{ij} is diagonal, these equations have the following form:

$$[(\omega - k\Omega)^2 - \omega_\alpha^2] u_\alpha(k)$$

$$= -i \sum_{\alpha'} \eta^2_{\alpha\alpha'} \omega_\alpha \omega_{\alpha'} \omega_M \sum_{n,l=-\infty}^{\infty} (-1)^{n+k} P_l(\omega) J_{l+n}(\xi) J_{l+k}(\xi) u_{\alpha'}(n);$$

$$\eta^2_{\alpha\alpha'} = \frac{4\pi\beta_i\beta_j b_i^\alpha b_j^{\alpha'}}{\rho\varepsilon_0 v_\alpha v_{\alpha'}}, \quad P_l(\omega) = (\omega - \mathbf{q}\mathbf{v}_0 + l\Omega + ia)^{-1},$$

$$k = 0, \pm 1, \pm 2, \ldots, \quad \alpha = 1, 2, 3. \tag{1.9}$$

Here $\omega_\alpha = qv_\alpha$, $v_\alpha = \sqrt{\lambda_\alpha/\rho}$ is the sound velocity in the absence of the carriers, λ_α the eigenvalue of the tensor λ_{ij}, $\eta^2_{\alpha\alpha'}$ is the piezo-coupling constant, $\mathbf{b}^\alpha$ the basis vector corresponding to the α^{th} eigenvalue of the tensor λ_{ij}, and $u_\alpha(k)$ the component of the expansion of the vector $\mathbf{u}(k)$ in the basis vectors $\mathbf{b}^\alpha$.

For the solution of the set (1.9), one can use the standard perturbation theory. In zeroth approximation in η^2, the set (1.9) decomposes into a set of independent equations:

$$[(\omega - k\Omega)^2 - \omega_\alpha^2] u_\alpha(k) = 0.$$

Such equations have nonzero solutions only for those values of k for which the condition

$$\omega - k\Omega = \pm\omega_\alpha. \tag{1.10}$$

is satisfied. Inasmuch as ω is chosen close to some acoustic frequency, the condition (1.10) is known to be satisfied for $k = 0$.

In the nonresonance case, when Eq. (1.10) is valid only for $k = 0$, only the amplitude $u_\alpha(0)$ is different from zero. In the next approximation in η^2, the equation for $u_\alpha(0)$ determines the corrections to the frequency and the decrement (increment) of the sound wave. With account of such corrections, the dispersion relation for the α-th wave has the form

$$\omega^2 - \omega_\alpha^2 + i\eta^2_{\alpha\alpha}\omega_\alpha^2\omega_M \sum_{l=-\infty}^{\infty} P_l(\omega) J_l^2(\xi) = 0. \tag{1.11}$$

Under the conditions of acoustic resonance, the amplitudes of several waves in the expansion (1.8) are large and comparable with one another. These amplitudes are linearly coupled (through Eq. (1.9)) in η^2 and form a closed set of linear homogeneous equations. The condition for solvability of this set determines the spectra of oscillations which, as we depart from resonance conditions, transform into the spectra of the sound waves obtained in the nonresonance case. As a whole, the picture of the spectrum is identical with the usual intersection of oscillations branches and with the formation of coupled waves. Inasmuch as the redistribution of the spectrum here takes place because of the varying electric field, one can speak of the parametric coupling of the oscillations. However, the physical picture for parametric coupled oscillations is more complicated than the simple intersection of spectra, mainly because each solution of the secular equation (i.e., each value of the frequency for a fixed wave vector) corresponds to several waves, generally of different polarizations, whose frequencies differ by $r\Omega$, where r is an integer satisfying (1.10). These waves possess different phase velocities, close to the velocities of the corresponding modes of the sound waves, but with the same group velocity and decrement. The amplitudes of the waves which make up the characteristic oscillation are in strictly determined ratios, given by Eqs. (1.9).

The simplest case of acoustic resonance arises when two modes of characteristic waves (and, correspondingly, waves that are opposite to them) are coupled. For a fixed value of the wave vector, such a resonance takes place if the frequency of the external wave satisfies the condition

$$\omega_\alpha - r\Omega \approx \pm\omega_{\alpha'}, \quad r - \text{integer}. \tag{1.12}$$

In particular, when $\alpha' = \alpha$ and $\Omega = 2\omega_\alpha/r$, parametric coupling arises between the direct and reverse[2)] waves of the same polarization. The secular equation which determines the spectrum of sound waves under the conditions (1.12) has the form

$$[\omega^2 - \omega_\alpha^2 + i\omega_\alpha^2 A_0^\alpha(\omega)][(\omega - r\Omega)^2 - \omega_{\alpha'}^2 + i\omega_{\alpha'}^2 A_r^{\alpha'}(\omega)] + \omega_\alpha^2\omega_{\alpha'}^2 B_r^2(\omega) = 0. \tag{1.13}$$

Here

$$A_r^\alpha(\omega) = \eta_{\alpha\alpha}^2\omega_M \sum_{l=-\infty}^{\infty} P_l(\omega) J_{l+r}^2(\xi),$$

$$B_r(\omega) = \eta_{\alpha\alpha'}^2\omega_M \sum_{l=-\infty}^{\infty} (-1)^r P_l(\omega) J_l(\xi) J_{l+r}(\xi).$$

If the phase velocities of the two acoustic modes α and α' satisfy the condition

$$1 - v_{\alpha'}/v_\alpha \approx p/r, \tag{1.14}$$

where p and r are small integers, then, for $\Omega \approx 2\omega_\alpha/n$ (n an integer), four characteristic waves with the same value of the wave vector turn out to be coupled, viz., two direct and two reverse waves of modes α and α'. Further, if the phase velocities of three acoustic modes satisfy the condition

$$\frac{1 \pm v_{\alpha'}/v_\alpha}{1 \pm v_{\alpha''}/v_\alpha} \approx \frac{p}{r}, \tag{1.15}$$

while the frequency of the external field satisfies the condition (1.12), then for a given vector q there are two groups of coupled oscillations, in each of which there are three waves of different polarizations, and various combinations of direct and reverse waves are possible. To each such component there corresponds a definite selection of signs in (1.15). Finally, if there is a direction in the crystal for which the relation (1.14) is satisfied, both for the polarizations α' and for α'' (with different values of p and r), then for $\Omega = 2\omega_\alpha/n$, all six characteristic acoustic waves corresponding to a given value of the wave vector q are coupled (three direct and three reverse). The relations (1.13), (1.14) and (1.15) should be satisfied with accuracy to η^2.

It should be emphasized that acoustic resonance of waves with different polarizations is possible only for those directions of the vector q along which the piezoeffect exists for the given waves.

Inasmuch as

$$\frac{|\mathrm{Im}\,\omega|}{a} \leqslant \eta^2 \frac{v_s}{v_{Te}} \sqrt{\frac{\nu}{\omega_M}} \ll 1$$

close to the characteristic frequencies of the sound waves, for reasonable values of the electron density ($n_0 > 10^{12}$ cm^{-3}) and ν ($\nu \sim 10^{13}$–10^{14}), the frequencies considered are far from the poles $P_l(\omega)$. This means that in the consideration of the first approximation of the system (1.9) in $P_l(\omega)$ there is no necessity of taking into account corrections to the frequency of order η^2 and one can substitute the value ω in the zeroth approximation.

The result obtained corresponds to a well-known fact: the application of an electric field to a piezosemiconductor cannot lead to the intersection of the acoustic and plasma-drift branches.[1]

2. NONRESONANCE CASE

In the nonresonance case only the fundamental frequency corresponds to the characteristic oscillations of the lattice: the sound waves at the combination frequencies are induced. Their amplitudes are smaller by the factor η^{-2} than the amplitude of the fundamental oscillation.

Using Eq. (1.11), we get the equation for the characteristic frequency ω:

$$\omega = \omega' + i\gamma = \omega_\alpha - {}^1/_2 i\eta_{\alpha\alpha}^2\omega_\alpha\omega_M \sum_{\infty}^{\infty} P_l(\omega) J_l^2(\xi). \tag{2.1}$$

Each term of the sum in (2.1) corresponds to the account of an induced wave of the electron density, having the frequency $\omega + l\Omega$ in the reference frame that moves together with the carriers in the alternating electric field. The correction to ω associated with the l-th electron wave is proportional to the effective piezo coupling constant $\eta_{\alpha\alpha}^2 J_l^2(\xi)$, which depends on the number of the wave, the amplitude and frequency of the external field. The contribution of such a wave to the damping or amplification of the sound wave is determined by the phase shift between these waves.[1] The sign of the shift is identical with the sign of Re $P_l(\omega_\alpha) \sim (\omega_\alpha - \mathbf{q}\cdot\mathbf{v}_0 + l\Omega)$. Therefore, the electron density waves, for which Re $P_l > 0$, produce attenuation while the electron-density waves, for which Re $P_l < 0$, produce amplification of the acoustic wave. Its decrement (increment) is determined by the total effect. In the limiting case, in which the amplitude of the varying field tends to zero, only the term with $l = 0$ remains in (2.1). Here the expression for γ transforms into the well-known formula (see[1]) for the decrement of acoustic waves propagating in a piezosemiconductor to which is a applied a constant electric field.

The character of the amplification (attenuation) of the sound depends materially on the relation between the parameters Ω and a. In the case in which the period of oscillation of the external field is much less than the characteristic time of dissipation of the electron-density excitations, i.e., $a \ll \Omega$, the width of the resonance curve for an induced electron-density wave is much less than the difference between the frequencies of these waves. Therefore, under conditions for which the frequency of the l-th electron wave is close to the frequency of the plasma-drift branch of the oscillations, which is equal to $\mathbf{q}\cdot\mathbf{v}_0$:

$$\left|\frac{\omega_\alpha - \mathbf{q}\mathbf{v}_0 + l\Omega}{\Omega}\right| \ll 1, \quad l = 0, \pm 1, \ldots, \tag{2.2}$$

the amplitude of this wave is $\Omega/a \gg 1$ greater than the amplitude of the remaining electron waves and the decrement of the sound wave is determined by the contribution of just this resonating wave. From now on, we

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2)We shall call the direct wave the wave that propagates along the direction of the vector q, and the reverse wave the one traveling in the opposite direction.

shall call this case the case of l-th plasma resonance. Far from the conditions of plasma resonance, it is necessary to take into account a large number of waves with approximately the same amplitude in order to find γ.

It is easy to obtain an expression for the decrement from (2.1). It is convenient here to replace the sum over l by an integral, in accord with the formula (A.1), in which one must set n = 0. Expanding γ in powers of the small parameter a/Ω, and using the relation (A.3), we can obtain a simple form of the decrement. Near the n-th plasma resonance we have

$$\gamma = -\frac{\eta^2_{\alpha\alpha}\omega_\alpha\omega_M}{2}\left[\frac{\Delta_n}{\Delta_n^2+a^2}J_n^2(\xi) - A_n(\xi)\right], \qquad (2.3)$$

$$\Delta_n = \omega_\alpha - \mathbf{q}\mathbf{v}_0 + n\Omega, \quad A_n(\xi) = \frac{2}{\pi\Omega}\int_0^\pi dy\, y \sin(2ny) J_0(2\xi\sin y).$$

In this expression, the first term corresponds to the resonating wave; the small correction $A_n(\xi)$ takes into account the contribution of the remaining electron-density waves. It becomes important first when $\Delta_n \ll a$ and, second, near the zeroes of the function $J_n(\xi)$.

When the relation (2.2) is not satisfied for any l, the expression for the decrement has the form

$$\gamma = -\frac{\pi\eta^2_{\alpha\alpha}\omega_\alpha\omega_M}{2\Omega\sin\pi\lambda}J_\lambda(\xi)J_{-\lambda}(\xi), \quad \lambda = \frac{\omega - \mathbf{q}\mathbf{v}_0}{\Omega}. \qquad (2.4)$$

The amplification (attenuation) of the sound in this case is much less than the effect which is generated under the conditions of plasma resonance. This is connected with the fact that, far from resonance, γ is determined by the electron-density waves, the amplitudes of which are much smaller than the amplitudes of the resonating waves under resonance conditions. Furthermore, the terms that correspond to these waves enter in with different signs. In the simplest case, when $\omega_\alpha = \mathbf{q}\cdot\mathbf{v}_0 + \Omega/2$, we have

$$\gamma = -\frac{\eta^2_{\alpha\alpha}\omega_\alpha\omega_M\sin 2\xi}{8\Omega\xi}. \qquad (2.4a)$$

The character of the dependence of γ on the quantities v_0 and ξ, which determine the amplitudes of the external waves, is shown in Figs. 1 and 2. The dependence of the decrement on the drift velocity v_0 (Fig. 1) has a resonance character for sufficiently large values of ξ ($\xi > 1$). γ vanishes in the regions of plasma resonance, the width of which is of the order of a, and there is a transition from attenuation to amplification of the ultrasound. For $\Delta_n = \pm a$, maxima in attenuation and amplification occur. In the region between neighboring resonances, γ changes sign (i.e., the reverse transition takes place from amplification of the sound waves to attenuation). For values of γ close to the zeroes of the function $J_n(\xi)$, the extremal values of γ in the region of the n-th plasma resonance are comparable with its values in the nonresonance region. Here the character of the dependence of the decrement on v_0 is maintained, except that a correction $A_n(\xi)$ that does not depend on v_0 appears.

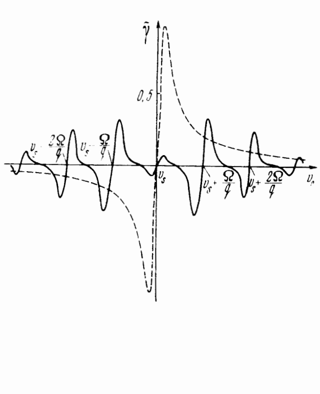

FIG. 1 Dependence of the decrement γ of the sound wave on the dc component v_0 of the drift velocity in the nonresonance case. The dashed curve indicates this same dependence in the absence of an external alternating field ($\tilde{\gamma} = 4a\gamma/\eta_{\alpha\alpha}^2\omega_\alpha\omega_M$, $\xi = 2.4$).

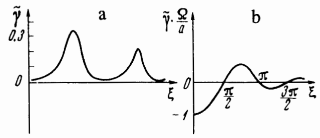

FIG. 2. Dependence of the decrement γ on the amplitude of the alternating electric field ξ (nonresonance case): a—in the region of plasma resonance; b—outside this region ($\tilde{\gamma} = 4a\gamma/\eta_{\alpha\alpha}^2\omega_\alpha\omega_M$, $\xi = \mu q E_1/\Omega$, v_0 = const).

For fixed values of the remaining parameters the decrement is an oscillating function of ξ and, consequently, of the amplitude of the external field.[3)] The character of these oscillations depends materially on the value of the drift velocity v_0. Under conditions of plasma resonance, only the absolute value of γ oscillates (Fig. 2a), the sign of the decrement as a whole being determined by the sign of Δ_n (see (2.3)). Far from resonance, both the absolute value and the sign of γ change with change in ξ (Fig. 2b). Therefore, amplification of ultrasound is possible in the presence of an alternating external field only.

Equation (2.3) is certainly not suitable for small values of ξ, since the function $J_n(\xi) = 0$ close to $\xi = 0$ (n = 1, 2, ...). The value of the decrement for such ξ can be obtained if we expand the function J_0 $J_0(\xi \sin y)$ in Eq. (A.2) in powers of its argument. With account of the first two terms of the expansion, γ takes the form

$$\gamma = \gamma_0\left[1 - \frac{\xi^2\Omega^2}{2}\frac{\Omega^2 - (\omega_\alpha - \mathbf{q}\mathbf{v}_0)^2 + 3a^2}{[\Omega^2 - (\omega_\alpha - \mathbf{q}\mathbf{v}_0)^2 + a^2]^2 + 4a^2(\omega - \mathbf{q}\mathbf{v}_0)^2}\right],$$

$$\gamma_0 = -\frac{1}{2}\eta^2_{\alpha\alpha}\omega_\alpha\omega_M\frac{\omega - \mathbf{q}\mathbf{v}_0}{(\omega - \mathbf{q}\mathbf{v}_0)^2 + a^2}, \qquad (2.5)$$

which is valid for any relation between the parameters Ω and a. When $a \ll \Omega$, the correction to (2.4), which is proportional to ξ^2, becomes important close to the first plasma resonance, i.e., for satisfaction of the condition $\omega - \mathbf{q}\cdot\mathbf{v}_0 \approx \pm\Omega$. Similarly, account in (2.5) of the succeeding terms of the expansion in ξ makes it possible to trace the appearance of the plasma resonances with higher numbers.

If $a \gg \Omega$, i.e., if the time of resolution of the electron density fluctuations is smaller than the period of oscillation of the external field, the effects associated with plasma resonances do not appear, and the dependence of γ on v_0 and ξ has a smooth character. This is due to the fact that the width of the resonance curve for the electron-density wave is much greater than Ω and a large number of waves with combination frequen-

3)The oscillating character of the decrement as a function of the amplitude of an external alternating field was pointed out in [4] for the case in which the dc component of the field is lacking and $\Omega \gg \omega_s$ (ω_s is the characteristic sound frequency). This corresponds to taking into account (in the present work) only of the term in Eq. (2.1) with l = 0, in which v_0 = 0.)

cies $\omega + l\Omega$ falls in the region of its maximum. The approximate expression for the frequency of the sound wave is obtained from (2.1) if we use the formula (A.4):

$$\omega = \omega_\alpha - \frac{i}{2}\eta^2_{\alpha\alpha}\omega_\alpha\omega_M \frac{1}{\sqrt{(\omega - \mathbf{q}\mathbf{v}_0 + ia)^2 - \xi^2}} \qquad (2.6)$$

As is seen from (2.6) the alternating field does not change the threshold value $v_0 = v_s$ at which the transition takes place from attenuation to amplification of the sound.

3. THE CASE OF ACOUSTIC RESONANCE

In this section, we shall consider a resonance in which two characteristic modes are coupled. In accord with (1.12), a coupling of two direct waves of different polarization appears for $\Omega = (\omega_\alpha - \omega_{\alpha'})/r$ (here $\omega_\alpha > \omega_{\alpha'}$); for $\Omega = (\omega_\alpha + \omega_{\alpha'})/r$, direct and reverse waves are coupled (in particular, one can have $\alpha' = \alpha$)[4)] Here, that part of the displacement vector $\mathbf{u}(\mathbf{x}, t)$ which describes one of the resultant coupled oscillations, has the form

$$u_\alpha(0)e^{iqx}[\mathbf{b}_\alpha e^{-i\omega t} + \varkappa\mathbf{b}_{\alpha'}e^{-i(\omega - r\Omega)t}] + \text{c.c.} \qquad (3.1)$$

here, $\mathbf{b}_\alpha$ and $\mathbf{b}_{\alpha'}$ are the polarization vectors, $\kappa = u_{\alpha'}(r)/u_\alpha(0)$ is the ratio of the amplitudes of the coupled waves.

By assuming that ω is close to ω_α, we get from (1.13) an expression for ω and κ in terms of the parameter and the functions

$$A_0^\alpha(\omega),\quad A_r^{\alpha'}(\omega),\quad B_r(\omega):$$

$$\omega_{1,2} = \frac{\omega_\alpha + (\pm\,\omega_{\alpha'}) + r\Omega}{2} - i\varphi(\omega) \pm \sqrt{[\Delta - i\chi(\omega)]^2 + D(\omega)}, \qquad (3.2)$$

$$\varkappa_{1,2} = \frac{\Delta - i\chi(\omega) \mp \sqrt{[\Delta - i\chi(\omega)]^2 + D(\omega)}}{i\omega_{\alpha'}B_r/2}, \qquad (3.3)$$

where

$$\varphi(\omega) = {}^1\!/_4[\omega_\alpha A_0^\alpha(\omega) + (\pm\omega_{\alpha'})A_r^{\alpha'}(\omega)],$$

$$\chi(\omega) = {}^1\!/_4[\omega_\alpha A_0^\alpha(\omega) - (\pm\omega_{\alpha'})A_r^{\alpha'}(\omega)],$$

$$D(\omega) = -\frac{(\pm\,\omega_{\alpha'})\omega_\alpha}{4}B_r^2(\omega), \quad \Delta = \frac{\omega_\alpha - (\pm\,\omega_{\alpha'}) - r\Omega}{2}.$$

In these expressions, the upper sign in front of $\omega_{\alpha'}$ corresponds to the intersection of the direct wave α with the direct wave α', the lower, with reverse α'. The sign $\pm$ in front of the square root corresponds to the existence of two branches of the solution of Eq. (1.13), which are encountered for corresponding values of the wave vector close to $\omega = \omega_\alpha$.

Equation (3.2) is transcendental, but it is easy to obtain its solution by expanding the argument of the expressions $A_0^\alpha(\omega)$, $A_r^{\alpha'}(\omega)$ and $B_r(\omega)$ in powers of η^2. In the zeroth approximation, (3.2) has two solutions ω_α and $r\Omega \pm \omega_{\alpha'}$. The second of these corresponds to the characteristic wave with frequency $\pm\omega_{\alpha'}$, the amplitude of which has the number r in the expansion (1.8). The curve corresponding to this branch in the plane (ω', q) is shifted above the ω' axis by $r\Omega$ (see Figs. 3 and 4). The width of the region of parametric

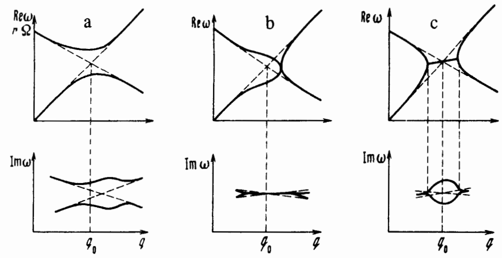

FIG. 3. Modification of the spectrum of sound oscillations under conditions of parametric resonance ($\chi'(\omega_0) = 0$). The dashed lines show the spectra of sound vibrations without account of parametric coupling.

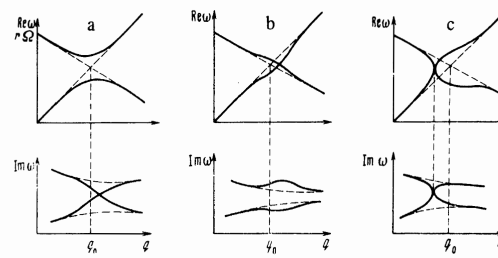

FIG. 4. Modification of the spectrum of oscillations for $\chi'(\omega_0) \neq 0$.

resonance is determined by η^2: $\Delta_{res} \lesssim \eta^2\omega_\alpha$. In the limit of large detuning, when $|\Delta| \gg \eta^2\omega_\alpha$, the solutions ω_1 and ω_2 transform into the solutions obtained in the nonresonance approximation for the branches α and α'. However, the solution into which ω_1 (or ω_2) goes over is determined, first, by the character of the modification of the spectrum and, second, by the sign of Δ. In accord with this, the expression for $\kappa_{1,2}$ at large detunings tends to 0 or ∞, depending on the branch (α or α') of the selected solution.

Inside the resonance region, the expressions (3.2) and (3.3) for ω and κ are valid; on the right sides of these, $\omega_0 = v_\alpha q_0$ enters in place of ω, where $q_0 = \Omega/(v_\alpha \mp v_{\alpha'})$ is the value of the wave vector for which Eq. (1.12) becomes exact:

$$\omega'_{1,2} \equiv \operatorname{Re}\omega_{1,2} = \frac{\omega_\alpha + (\pm\,\omega_{\alpha'}) + r\Omega}{2} + \varphi''(\omega_0) \pm \left[\frac{\sqrt{K'^2 + K''^2} + K'}{2}\right]^{1/2}, \qquad (3.4)$$

$$\gamma_{1,2} \equiv \operatorname{Im}\omega_{1,2} = -\varphi'(\omega_2) \pm \left[\frac{\sqrt{K'^2 + K''^2} - K'}{2}\right]^{1/2}, \qquad (3.5)$$

where K denotes the radicand in (3.1)

$$K'(\Delta) = [\Delta + \chi''(\omega_0)]^2 - \chi'^2(\omega_0) + D'(\omega_0),$$

$$K''(\Delta) = -2\chi'(\omega_0)[\Delta + \chi''(\omega_0)] + D''(\omega_0),$$

the prime denotes the real and the double prime the imaginary parts of a complex number. The sign in front of the root in (3.4) and (3.5) should be taken with account of the circuiting around the branch point by the curve $K(\Delta)$ in the complex K plane. The dependence of K on the wave vector is expressed only in terms of the detuning Δ, so that K' is a *quadratic function of* q and K'' is a linear function. The features of the spectra of parametrically coupled waves are determined by the signs and zeroes of these functions.

Let $\chi'(\omega_0) = 0$. This means that without account of the parametric coupling between the waves, their

[4)]The problem of the parametric resonance of sound waves in piezosemiconductors was first put forward in [2,3]. However, the method of consideration of the resonance in [2] appears unsuitable to us, inasmuch as the redistribution of the spectrum of parametrically coupled waves in the region of resonance has not been taken into account.

decrements also intersect in the region of resonance. If $D''(\omega_0) \neq 0$, then $K'' = \text{const}$, and, regardless of the behavior of the function $K'(\Delta)$ in the region of resonance, the usual redistribution of the spectra takes place: the frequencies and the decrements become further separated, and as q changes continuouly inside the resonance region each solution $\omega_{1,2}$ changes over from one branch of the characteristic waves to the other (Fig. 3a). If $D''(\omega_0) = 0$, then $K'' \equiv 0$, and the spectra of the oscillations will be different, depending on the number of zeroes of the function $K'(\Delta)$.

For $D'(\omega_0) > 0$, the quantity K' has no zeroes, so that the frequencies become further separated, while the decrements are identical in the resonance region.

For $D'(\omega_0) = 0$, the quantity K' has a zero at the point

$$q_1 = \frac{r\Omega - 2\chi''(\omega_0)}{v_\alpha \mp v_{\alpha'}}.$$

The decrements of the two characteristic oscillations are identical as before; however, in the spectrum there appears a root singular point associated with the fact that $K(\Delta) = 0$ for $q = q_1$ (Fig. 3b). At this point, $\partial\omega/\partial q = \pm\infty$ and the concept of a group velocity loses meaning: close to such a value of the wave vector, it is not possible to form a wave packet because of the very strong dispersion. Inasmuch as the values of the frequencies and the decrements of both characteristic oscillations are identical for $q = q_1$, the choice of the solution in the region $q > q_1$, into which the solution $\omega_1(q)$ (or $\omega_2(q)$) transforms in the region of wave numbers $q < q_1$, can be arbitrary. The presence of such an arbitrariness and the fact that the derivatives $\partial\omega_{1,2}/\partial q$ become infinite are features of the spectra of parametrically coupled waves containing root singular points.

When $D'(\omega_0) < 0$ the function $K'(\Delta)$ has two roots, while $\omega(q)$ has a root singularity already at two points. The form of the spectrum for this case is shown in Fig. 3c. In the region between the roots

$$|\Delta + \chi''(\omega_0)| < |D'(\omega_0)|^{1/2},$$

where $K'(\Delta) < 0$, the resultant solutions describe two oscillations with identical group velocities $\partial\omega_{1,2}/\partial q = (v_\alpha \pm v_{\alpha'})/2$, but with different decrements. Each oscillation, according to (3.1), consists of two waves whose relative-amplitude ratio has a modulus $|\kappa| = \sqrt{v_\alpha/v_{\alpha'}}$. The phase of κ changes in the range from 0 to π with change in Δ. In the special case $\alpha' = \alpha$, the group velocities vanish and two standing waves appear with different decrements.

The condition $\chi'(\omega_0) \neq 0$ means that without account of coupling between the waves their decrements are separated in the resonance region (see Fig. 4). The spectral features are determined here by the mutual location of the zeroes of $K'(\Delta)$ and $K''(\Delta)$. If the zero of the function K'' lies in the region where $K' > 0$, then redistribution of the spectra takes place: the frequencies become more separated and the decrements intersect at a point at which $K''(\Delta) = 0$ (Fig. 4a). Further, if the zero of K'' lies in the region in which $K' < 0$, the redistribution of the spectrum does not occur, and separation of the decrements and a small distortion of the dispersion curves result (Fig. 4b). Finally, when the zero of K'' is identical with the zero of K', a singular point appears in the spectrum (Fig. 4c).

Evidently, all the different spectra above can be realized by the choice of the parameters of the crystal and the corresponding external conditions. As an example, let us consider the parametric resonance of direct and reverse waves of identical polarization under conditions in which the external electric field contains only an alternating component with frequency $\Omega \approx \partial\omega_0/r$ (r an integer). In this case, $\chi'(\omega_0) = 0$, $D''(\omega_0) = 0$, so that $K'' \equiv 0$. The expression for $D'(\omega_0)$ has the form

$$D'(\omega_0) = (-1)^{r+1}\left|\frac{\omega_0\omega_M}{2}\eta_{\alpha\alpha}^2 \sum_{l=-\infty}^{\infty} \frac{J_l(\xi)J_{l+r}(\xi)}{\omega_0 + l\Omega + ia}\right|^2.$$

For odd resonance, $r = 2p - 1$ ($p = 1, 2, \ldots$), the function K' has no zeroes and the spectrum has the shape of separating modes with identical decrements.

For even resonance, $r = 2p$, two singular points appear in the spectrum, of which we have spoken above (see Fig. 3c). We shall give the analytic expression for $\omega(q)$ in the simplest case, in which $a/\Omega \ll 1$:

$$\omega_{1,2}(q) = p\Omega \pm \sqrt{\Delta(\Delta - \Phi)}, \quad \Phi = \frac{\omega_0\eta_{\alpha\alpha}^2 J_p^2(\xi)}{2(1 + q_0^2 r_0^2)}. \tag{3.6}$$

In the derivation of (3.6), we used the approximate formulas (A.2) and (A.3). It is seen from the expression (3.6) that two standing waves appear in the range $0 < \Delta < \Phi$ with $\omega'_{1,2} = p\Omega$ and decrements

$$\gamma_{1,2} = \pm\sqrt{\Delta(\Phi - \Delta)}. \tag{3.7}$$

The plus sign in (3.7) corresponds to a standing wave with increasing amplitude, the minus sign to an attenuating standing wave. The condition $a/\Omega \ll 1$ imposes definite requirements on the parameters of the crystal and the frequency of the external field:

$$\frac{v_\alpha}{v_{Te}}\sqrt{\frac{\nu}{\omega_M}} \gg p, \quad \omega_M \ll \Omega \ll \nu\left(\frac{v_\alpha}{p v_{Te}}\right)^2.$$

Estimates show that the ratio of the maximum increment of the standing wave (3.7) to the maximum increment of the traveling wave in a constant electric field is equal to $J_p^2(\xi)$. For parameters that are usual for crystals of the CdS type: $\mu = 90\ \text{cm}^2\text{-V}^{-1}\text{-sec}^{-1}$, $n_0 = 10^{13}\ \text{cm}^{-3}$, $T = 300°\text{K}$, $\eta^2 = 0.04$, $v_\alpha = 1.75 \times 10^5\ \text{cm-sec}^{-1}$, $\Omega = \omega_\alpha = 1.6 \times 10^9\ \text{sec}^{-1}$, and $E_1 = 3.6 \times 10^3\ \text{V-cm}^{-1}$, the maximum increment is equal to $\gamma_{max} = 2.6 \times 10^6\ \text{sec}^{-1}$. The region of wave vectors inside which modification of the spectrum occurs, $\Delta q \approx 2\gamma_{max}/v_\alpha = 30\ \text{cm}^{-1}$ is much greater than the spectral width of the sound pulses used in the experiment. This assures the observation of the modification of the spectrum in the experiment. For the same condition $a/\Omega \ll 1$ the decrement (increment) of the oscillations for conditions of odd resonance is much less (by Ω/a) than the decrement (3.6). The expression of it is the same as Eq. (2.4).

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APPENDIX

We write down the integral representation of the sum

$$S_n = \sum_{l=-\infty}^{\infty} P_l(\omega) J_l(\xi) J_{l+n}(\xi),$$

and also give the approximate expressions for S_n in different limiting cases. Representing S_n in the form

$$S_n = -\frac{i}{\Omega}\int_0^\infty dy\, e^{i(\omega - \mathbf{q}\mathbf{v}_0 + ia)y} \sum_{l=-\infty}^{\infty} J_l(\xi) J_{l+n}(\xi) e^{il\Omega y}$$

and using Eq. (8.530) from[7], we get

$$S_n = \frac{2i^{n-1}}{\Omega}\int_0^\infty dy e^{2iQ_n y} J_n(2\xi \sin y), \qquad \text{(A.1)}$$

where

$$Q_n = (\omega - \mathbf{q}\mathbf{v}_0 - n\Omega/2 + ia)/\Omega.$$

Dividing the region of integration in (A.1) into a set of intervals of length 2π, it is easy to obtain the result that

$$S_n = \frac{2i^{n-1}}{\Omega}(1 - e^{2\pi i Q_0})^{-1} Z_n(\xi), \qquad \text{(A.2)}$$

where

$$Z_n(\xi) = \int_0^\infty e^{2iQ_n y} J_n(2\xi \sin y)\, dy.$$

Under conditions for which $\pi a/\Omega \ll 1$, the real part of the exponent of exponential in the radicand in (A.2) can be neglected, and the integral $Z_n(\xi)$ takes the form (formula (6.681) in[7]:

$$Z_n(\xi) = \pi(-i)^n e^{i\pi\lambda} J_\lambda(\xi) J_{n-\lambda}(\xi), \quad \lambda = (\omega - \mathbf{q}\mathbf{v}_c)/\Omega. \qquad \text{(A.3)}$$

In the other limiting case, in which $a/\Omega \gg 1$, in the argument of the function J_n in (A.1) sin y can be replaced approximately by y. Using the corresponding formula (6.611) in[7], we get

$$S_n = \frac{1}{\Omega} i^{n-1}\xi^n \frac{[\sqrt{\xi^2 - Q_n^2} - iQ_n]^{-n}}{\sqrt{\xi^2 - Q_n^2}}. \qquad \text{(A.4)}$$

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