CONTRIBUTION TO THE PHENOMENOLOGICAL THEORY OF SUPERFLUIDITY NEAR THE LAMDA-POINT

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The equations describing the behavior of the order parameter Φ of superfluid helium near the λ-point have been investigated by means of scaling methods of the theory of phase transitions. Restrictions on the form of these equations are derived. It is shown that the differential equations for Φ exist only for definite values of the critical parameters.

GINZBURG and Pitaevskiĭ[1] have proposed a semi-phenomenological theory of superfluidity of helium near the λ-point. This theory is based on Landau's concepts about phase transitions of the second kind. In[2,3] this theory has been modified in such a manner as to make it compatible with the results of measurements of ρ_5/ρ and of the heat capacity[4,5] near the phase transition temperature.

Recently the problem of phase transitions of the second kind has been attacked by means of scaling methods[6,7]. We shall consider below the problem of the phase transition in helium from this point of view. In the spirit of the Ginzburg-Pitaevskiĭ theory we shall assume that the free energy is a functional of the order parameter. We first find out the restrictions imposed on this functional dependence by the requirements of scaling. Scaling theory involves two parameters, the values of which are determined by the law of decrease of the pair-correlation function at the phase-transition point, and also by the thermodynamics of the system.

Following the method of[7], we assume that near the phase transition point

$$\langle\Psi^+(0)\Psi(\mathbf{r})\rangle\sim r^{-\beta},\quad l_0\ll r\ll r_c,\qquad r_c\sim|\tau|^{-\alpha},\quad \tau=(T-T_c)/T_c, \tag{1}$$

where $\Psi(\mathbf{r})$ and $\Psi^+(\mathbf{r}')$ are the creation and annihilation operators of particles at the appropriate points, l_0 is a distance of the order of atomic distances, r_c is the correlation radius of the system, T_c is the phase transition temperature.

We separate from the field operators a classical part describing the macroscopically occupied state

$$\Psi=\Phi+\varphi,\quad\langle\Psi\rangle=\Phi,\quad\langle\varphi\rangle=0,$$
$$\Psi^+=\Phi^*+\varphi^+,\quad\langle\Psi^+\rangle=\Phi^*,\quad\langle\varphi^+\rangle=0.$$

In computing the partition function it becomes now necessary to take the trace over the quantum variables, and to minimize it with respect to the classical part. We introduce the notation

$$F(T,\Phi)=-kT\ln \mathrm{Sp}_{\mathrm{qu}}\, e^{-H/kT}. \tag{2}$$

Then the order parameter satisfies the equations

$$\delta F/\delta\Phi=0,\quad \delta F/\delta\Phi^*=0.$$

In the sequel it will be convenient to consider a more general problem with macroscopic sources, which we introduce in the following manner

$$F\to F-I^*\Phi-I\Phi^*. \tag{3}$$

The sources of condensate particles play a role analogous to that of the external magnetic field in a ferromagnet[8], which is usually used to illustrate scaling laws; in this case the order parameter Φ is to be put in correspondence with the magnetization of the ferromagnet. Repeating the reasoning of[7] it is easy to see that after a minimization with respect to the order parameter, the free energy has the following expression in terms of the particle source I:

$$F\sim Vr_c^{-a}f(Ir_c^{a-\beta/2}), \tag{4}$$

where V is the volume of the system, a is the dimensionality of the space. The function f must have the properties

$$f(0)=\mathrm{const},\qquad f(x)\underset{x\to\infty}{\sim}x^p,\qquad p=(1-\beta/2a)^{-1},$$

which are consequences of the physical considerations discussed in[7].

It follows from (4) that the free energy depends on the order parameter and the correlation radius in the following manner:

$$F\sim Vr_c^{-a}g(|\Phi|^2r_c^\beta)-I^*\Phi-I\Phi^*. \tag{5}$$

One can make the following assertions about the function g:

$$g(0)=\mathrm{const},\qquad g(x)\underset{x\to\infty}{\sim}x^{a/b}.$$

In the derivation of (5) it was assumed that the order parameter does not depend on the coordinates; in spatially inhomogeneous cases the form of the terms in the expression of the free energy which are proportional to the gradients of Φ can be derived from dimensional considerations. Retaining only quadratic terms in the expansion with respect to grad Φ we obtain

$$F\sim\int d\mathbf{r}\{g(|\varphi|^2r_c^\beta)r_c^{-a}-I^*\Phi-I\Phi^*+r_c^{2+\beta-a}|\mathrm{grad}\,\Phi|^2g_1(|\Phi|^2r_c^\beta)\}. \tag{6}$$

It follows directly from (6) that $\rho_s\sim r_c^{2-a}$ (a result first derived by Josephson[9]).

The value of the critical parameter α is uniquely determined by the behavior of the heat capacity or of the superfluid density, and according to the measurements[4,5] one should assume $\alpha = 2/3$. Unfortunately, the value of the critical parameter does not determine any directly measurable quantities. A number of authors[10,11] have advanced considerations according to which the behavior of the thermodynamic functions of He II near the phase transition temperature is analogous to the behavior of the functions of a Heisenberg ferromagnet with a light plane of magnetization. The results of measurements and numerical calculations seem to indicate that independently of the type of anisotropy $\beta = 1$. The degree of anisotropy seems to affect only the value of α, which varies from 2/3 in the anisotropic case to 5/8 for the case of the Ising model[12].

We note that for $\beta \neq 1$ an expansion of the free energy containing terms up to the second degree in the gradients of the order parameter leads to an incorrect behavior of the binary correlation functions. This means, essentially, that for $\beta \neq 1$ the Ginzburg-Pitaevskiĭ equation to exist.[1)]

If one assumes in the spirit of the Ginzburg-Pitaevskiĭ theory that $r_c^{-a}g(|\Phi|^2 r_c^{\beta})$ is a polynomial in the square of the order parameter, then this implies necessarily that the expansion coefficients of the free energy in the powers of Φ^2 will not be analytic functions of the temperature. The apparently simplest form of functional relation satisfying the abovementioned requirements is

$$r_c^{-a}g(|\Phi|^2 r_c^{\beta}) = A_{\pm}|\tau|^{4/3}|\Phi|^2 + {}^1/_3 B|\Phi|^6 + Cr_c^{-a}. \qquad (7)$$

Since above the transition temperature the equilibrium value of the order parameter is zero, the coefficient of $|\Phi|^2$ must be positive for $\tau < 0$ $(A > 0)$; below the transition point $A_+ < 0$. As we already remarked, $g(x) \sim x^3$ for $x \to \infty$, and $g(x) - g(0) \sim x$ for $x \to 0$. Therefore one might expect that the expression (7) is a good interpolation formula.

The simplest expression for g_1 satisfying the requirements of scaling for $\beta = 1$ is

$$r_c^{2+\beta-a} g_1(|\Phi|^2 r_c^{\beta}) = \text{const}.$$

In[2,3], a two-term formula was proposed for g:

$$g = A'\tau^{4/3}|\Phi|^2 + B'\tau^{2/3}|\Phi|^4.$$

This expression does not guarantee the existence of a nonzero limit for the free energy at I = const, $\tau \to 0$, although it satisfies all other necessary requirements. It might seem that this circumstance is not essential, since in realistic problems there are always particle sources; one should however keep in mind that the boundary conditions imposed on the order parameter can be replaced by delta-function sources on the boundary. We therefore assume that for problems in which the presence of the boundary and nonlinearities play a decisive role, our expression might give a more accurate description.

Taking into account the fact that the binary correlation functions are determined by the relations

$$\mathcal{G} = \langle\langle \Psi^+(\mathbf{x})\Psi(\mathbf{x}')\rangle\rangle \equiv \langle \Psi^+(\mathbf{x})\Psi(\mathbf{x}')\rangle - \langle \Psi^+(\mathbf{x})\rangle\langle \Psi(\mathbf{x}')\rangle$$

$$= \delta^2 F/\delta I(\mathbf{x})\delta I^*(\mathbf{x}')\,|_{I\to 0,\ \Phi\to\Phi_{\text{equil}}},$$

$$\mathcal{F} = \langle\langle \Psi(\mathbf{x})\Psi(\mathbf{x}')\rangle\rangle \equiv \langle \Psi(\mathbf{x})\Psi(\mathbf{x}')\rangle - \langle \Psi(\mathbf{x})\rangle\langle \Psi(\mathbf{x}')\rangle$$

$$= \delta^2 F/\delta I^*(\mathbf{x})\delta I^*(\mathbf{x}')\,|_{I\to 0,\ \Phi\to\Phi_{\text{equil}}},$$

a variation of (6) with the use of the expansion (7) leads for $\tau > 0$ to the following relation for the Fourier components $\tilde{\mathcal{G}}(k)$, $\tilde{\mathcal{F}}(k)$

$$\tilde{\mathcal{G}}(k) + \tilde{\mathcal{F}}(k) = \left[\frac{\hbar^2 k^2}{2m} + 4B\Phi^4_{\text{equil}}\right]^{-1},$$

$$\tilde{\mathcal{G}}(k) - \tilde{\mathcal{F}}(k) = \left(\frac{\hbar^2 k^2}{2m}\right)^{-1}, \quad \Phi^4_{\text{equil}} = -\frac{A_+\tau^{4/3}}{B}, \quad A_+ < 0.$$

Similarly, for $\tau < 0$

$$\tilde{\mathcal{G}}(k) = \left[\frac{\hbar^2 k^2}{2m} + |\tau|^{4/3} A_-\right]^{-1}, \quad \tilde{\mathcal{F}}(k) = 0,$$

$$\Phi_{\text{equil}} = 0, \quad A_- < 0.$$

It follows that A_- must be different from zero (in contradistinction from the assumption made in[14]) since the correlation functions must decrease exponentially for $r \gtrsim r_c$. In the sequel we shall assume for simplicity that $|A_-| = |A_+|$[2)].

If one assumes, in analogy with[2,3], that the regular part of the discontinuity in heat capacity is due to the appearance of a nonvanishing order parameter, it becomes possible to estimate the phenomenological constants A and B. Setting

$$A = -akT_c, \quad B = bkT_c/n^2, \quad a, b > 0,$$

where n is the atomic density of helium, and making use of the results of[4,5] we obtain a = 0.78, b = 0.13.

We consider several problems regarding the behavior of the order parameter on the basis of the equation

$$-\frac{\hbar^2}{2m}\Delta\Phi - akT_c|\tau|^{4/3}\Phi\,\text{sign}\,\tau + \frac{bkT_c}{n^2}|\Phi|^4\Phi = 0. \qquad (8)$$

A. The Half-space z > 0 Bounded by a Rigid Wall

Assuming, as is usual, that the boundary condition satisfied by Φ on the rigid wall is $\Phi = 0$, we obtain a solution of Eq. (8) under the condition that at infinity $\Phi \to \Phi_{\text{equil}}$

$$\Phi = \Phi_{\text{equil}}\psi, \quad |\psi| \equiv f = \sqrt{\frac{2}{3 - \text{th}^2\,\zeta/\sqrt{3}}}\,\text{th}\frac{\zeta}{\sqrt{3}},$$

$$\zeta = \frac{x}{\xi(T)}, \quad \xi^2(T) = \frac{\hbar^2}{2makT_c\tau^{4/3}}. \qquad (9)$$

If the boundary condition at infinity is $\Phi \to 0$ $(x \to \infty)$ the solution can be written in the form

$$f \sim \text{ch}^{-1/2}\frac{\zeta - \zeta_0}{\sqrt{3}}. \qquad (9')$$

1) On the basis of independent considerations, Patashinskiĭ [13] has brought forward some arguments in favor of the fact that for a system of strongly interacting Bosons one should expect $\beta = 1$.

2) In the case $\beta \neq 1$ a similar computation of correlation functions based on an expansion of the expression (6) leads to the result

$$\mathcal{G}(r) \sim r_c^{-\beta}(r_c/r)e^{-r/r_c},$$

which contradicts the original assumption (1). Consequently, as was remarked above, for $\beta \neq 1$ we cannot limit ourselves to a finite number of terms in the expansion of the free energy in terms of the gradients of the order parameter.

B. The Critical Thickness of a Parallel Capillary

We consider a plane-parallel capillary of thickness d. The maximal value of the order parameter which we denote by $\Phi_{equil} f_m$, is attained in the middle of the capillary. In this case Eq. (8) has a first integral

$$(df/d\zeta)^2 = f_m^2 - f^2 - {}^1/_3(f_m^6 - f^6).$$

The quantity f_m is determined by the relation

$$\int_0^{f_m} \frac{df}{\sqrt{f_m^2 - f^2 - {}^1/_3(f_m^6 - f^6)}} = \frac{d}{2\xi(T)}.$$

Letting $d \to d_c$, $f_m \to 0$, we obtain

$$d_c = \pi\xi(T). \tag{10}$$

It can be seen from this that the critical thickness of a capillary is determined only by the dependence of the correlation radius on the temperature, and its magnitude agrees naturally with the result derived by Mamaladze[2].

C. The Hydrostatic Effect

We consider a column of liquid helium placed in the field of gravity at a temperature $T < T_c$. Since the pressure displaces T_c towards lower temperatures, at a sufficiently large depth the helium should undergo a transition into the normal state. Analyzing Eq. (8) one can derive an estimate for the thickness of the transition layer. A similar problem was solved in[14] on the basis of the Ginzburg-Pitaevskiĭ equation, but that solution refers to a very simplified model, and as indicated above, the value of the phenomenological coefficients selected there exhibit an inadmissible temperature and pressure dependence. We give the solution both for the Ginzburg-Pitaevskiĭ equation and the modified equation (8).

The critical temperature depends linearly on the pressure ($dT_c/dp \approx 0.015$ degree/atm) and the critical height h_c is determined by the relation

$$h_c = \tau\varepsilon l_0, \quad T_c = \frac{\hbar^2}{2ml_0^2}, \quad \varepsilon = \frac{1}{T_c}\frac{dT_c}{dp}\rho g l_0 \approx 3\cdot 10^{14}$$

(here we have introduced the natural unit of length l_0).

In the equation (8) τ should be considered a linear function of the coordinate x. Measuring coordinates from the height h_c, we have $\tau(x) = x/\epsilon l_0$. For sufficiently large positive x one may neglect the first term in Eq. (8). Then

$$\Phi = n(a/b)^{1/2}\tau^{2/3}(x) = \Phi_{equil}(p(x)).$$

Estimating the correction we obtain that the local approximation is valid for $x/l_0 \gg \epsilon^{2/5}$. For sufficiently large (in absolute value) negative x, the nonlinear term in Eq. (8) can be neglected. In this case

$$\Phi \sim \exp(-{}^3/_5|z|^{5/3}), \quad z = x/\lambda, \quad \lambda = l_0 a^{-3/10}\varepsilon^{2/5}.$$

In the region $|z| \lesssim 1$ the equation (8) can be solved only numerically, since all terms become of the same order. Introducing the dimensionless variables

$$\Phi = D\psi, \quad x = \lambda z, \quad D^4 = \frac{a}{b}n^2\left(\frac{\lambda}{l_0}\right)^{4/3}\varepsilon^{-4/3},$$

we rewrite (8) in the form

$$-\Delta\psi - |z|^{4/3}\psi\,\mathrm{sign}\,z + |\psi|^4\psi = 0.$$

The quantity λ determines the thickness of the transition layer; estimates show that $\lambda \approx l_0\epsilon^{2/5} \sim 5\times10^{-3}$ cm.

The same magnitude is found for λ in the analysis of the equation proposed by Mamaladze[2]. A similar calculation based on the Ginzburg-Pitaevskiĭ equation leads to the form $\lambda \sim l_0\epsilon^{1/2}$.

D. The Josephson Effect

The problems considered so far were essentially linear, so that the results obtained, (10) and (11), coincided apart from inessential numerical coefficients with the results derived on the basis of the equations of[2]. The value of the critical Josephson current in liquid helium is under certain circumstances determined by the nonlinear term in Ginzburg-Pitaevskiĭ equations. Unfortunately all conceivable experiments for liquid helium seem to be hard to realize.

We consider the following model problem. Liquid helium in a capillary is compressed in the interval (−d/2, d/2) by means of an electric field, in such a manner that the critical temperature in this section turns out to be higher than the temperature of the system[3)]. For simplicity we shall assume that the quantity τ suffers a jump at the boundary. (If the gradient of the field is small, it is necessary to treat the transition region similarly to the preceding section.) Inside the transition region (−d/2, d/2) the equation for the order parameter takes the form

$$\Delta\psi - \varkappa^2\psi - |\psi|^4\psi = 0, \tag{11}$$

where ψ is defined by the relation (9), $\kappa^2 = (\tau_1/\tau)^{4/3}$, $\tau_1 = (T - T_c^*)/T_c \approx (T - T_c^*)/T_c^*$, where T_c^* is the critical temperature in the transition region.

For the case of thick barriers ($d/\xi \gg 1$, $\kappa d/\xi \gg 1$) the equation (11) linearizes within the thickness of the transition layer (from now on we consider the problem as one-dimensional:

$$d^2\psi/dx^2 - \varkappa^2\psi = 0,$$
$$\psi = Ae^{-\varkappa x} + Be^{\varkappa x}, \quad A = Ce^{i\alpha}, \quad B = Ce^{i\beta}. \tag{12}$$

Making use of the following relation:

$$j = \frac{1}{2i}\left(\Phi^*\frac{\partial\Phi}{\partial x} - \Phi\frac{\partial\Phi^*}{\partial x}\right) = j_0 2\varkappa C^2\sin(\alpha-\beta), \quad j_0 = \frac{\hbar\Phi^2_{equil}}{\xi(T)},$$

we obtain the value of the critical current (in dimensionless units)

$$j_c = 2\varkappa C^2. \tag{13}$$

For the determination of C the solution of the equation (11) inside the barrier has to be joined, on one side (at $x = d/\xi$) with the expression (9) and on the other with the solution (12). The calculations yield

$$j_c = 4\sqrt{3}D(\varkappa)\exp(-\varkappa d/\xi),$$
$$D(\varkappa) = 1, \quad \varkappa \ll 1, \quad D(\varkappa) = 1/3\sqrt{3}\varkappa^3, \quad \varkappa \gg 1. \tag{14}$$

Similar computations for the equation in which the nonlinear part is proportional to Φ^4 lead to the result

3)The idea of a similar experiment was proposed by B. N. Esel'son and V. I. Solov'ev.

$$j_c = 16\varkappa^3 D(\varkappa)\exp(-\varkappa d/\xi), \quad D(\varkappa) = (\varkappa + \sqrt{1+\varkappa^2})^{-4},$$
$$j_c = \varkappa^{-1}\exp(-\varkappa d/\xi), \quad \varkappa \gg 1, \quad j_c = 16\varkappa^3\exp(-\varkappa d/\xi), \quad \varkappa \ll 1. \tag{14'}$$

In the general case the value of the critical current can be obtained by solving the equation for the absolute value f of the order parameter[4)]

$$f'' - \varkappa^2 f - j^2/f^3 - f^n = 0, \tag{15}$$

$$n = \begin{cases} 3, \text{ Mamaladze, Ginzburg-Pitaevskiĭ} \\ 5, \text{ Equation (8)} \end{cases}$$

Equation (15) can be solved by direct integrations:

$$K(f_m, j) = \int_{f_m}^{f_0}\left[\varkappa^2(f^2 - f_m^2) + j^2(f_m^{-2} - f^{-2}) + \frac{2}{n+1}(f^{n+1} - f_m^{n+1})\right]^{-1/2} df = \frac{d}{2\xi}, \tag{16}$$

where f_m is the minimal value of the function at the point x = 0, and $f_0 = f(d/\xi)$. The solution ceases to be real starting at some $j = j_c$, where the left-hand side of (16) reaches an extremal point as a function of f_m. Thus, the condition $\partial K/\partial f_m$ determines the value of the critical current.

In particular, if $\kappa d/\xi \ll 1$, $d/\xi \gg 1$, one can neglect the linear term in Eq. (15). Setting $\kappa = 0$ in (16), we obtain up to a coefficient, the following estimate:

$$j_c \sim \left(\frac{\xi}{d}\right)^{(n+3)/(n-1)} = \begin{cases} (\xi/d)^3, & n = 3 \\ (\xi/d)^2, & n = 5 \end{cases}. \tag{17}$$

We note that the results of solving spatially inhomogeneous problems on the basis of equations derived for a different selection of the nonlinear term can be different in the case when an essential role is played by regions in which $\Phi \gg \Phi_{equil}$ or $\Phi \ll \Phi_{equil}$. In the first case one should retain the asymptotic form of the function g in (5) for large values of its argument. In the second case the selection of the nonlinear term proposed by Mamaladze[2] is justified.

4) A similar equation was solved by Jacobson [15] with the purpose of determining the Josephson effect in superconductors. However this author only calculated the value of the critical current for the case $d/\xi \gg 1$, $\kappa \gg 1$, and the method of solution was unduly complicated.

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