

BREAKDOWN IN SEMICONDUCTORS IN AN ALTERNATING ELECTRIC FIELD

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A consistent theory of behavior of semiconductors in an alternating electric field is developed on basis of the analogy between the adiabatic perturbation theory and the equations describing the behavior of a particle in a crystal in the presence of an electric field which arbitrarily depends on time. A number of problems which can be approximately solved are considered.

INTRODUCTION

THE behavior of a semiconductor in an alternating electric field has been investigated in a large number of papers. Mention must be made above all of the work of Keldysh.^[1, 2] In particular, an attempt was made in^[2] to find the probability of production of an electron-hole pair in a wide range of frequencies of the alternating electric field.

In that paper, however, as well as in all the papers known to us dealing with the electric breakdown of a semiconductor, use is made of perturbation theory, which is not valid in these problems, but is justified by the smallness of the electric field. In this connection, the previously obtained results are only qualitatively correct (they describe only the exponential character of the dependence of the breakdown probability on the field). At the same time, there is undisputed interest in this group of problems, in connection with the question of developing a consistent theory of the behavior of a semiconductor in an alternating electric field. Such an attempt is made in the present paper.

After deriving the fundamental equations, we shall present approximate solutions for a number of concrete problems.

DERIVATION OF FUNDAMENTAL EQUATIONS

In the coordinate representation, the Schrödinger equation in the presence of an electric field is given by ($\hbar = 1$)

$$i \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \left\{ \frac{1}{2m_0} \left(\mathbf{p} + \int_0^t \mathbf{F}(t') dt' \right)^2 + U(\mathbf{r}) \right\} \psi(\mathbf{r}, t), \quad (1)$$

where $U(\mathbf{r})$ is the potential of the crystal, $\mathbf{F} = e\mathbf{E}(t)$, and $\mathbf{E}(t)$ is an alternating but homogeneous electric field. We change over to the momentum representation, putting $\mathbf{p}' = \mathbf{p} + \int_0^t \mathbf{F}(t') dt'$

$$\psi(\mathbf{r}, t) = \sum_n \int a_n(\mathbf{p}, t) \psi_{n\mathbf{p}}(\mathbf{r}) d\mathbf{p}, \quad (2)$$

where $\psi_{n\mathbf{p}}(\mathbf{r})$ are the stationary Bloch functions

$$\psi_{n\mathbf{p}}(\mathbf{r}) = e^{i\mathbf{p}\mathbf{r}} u_{n\mathbf{p}}(\mathbf{r}).$$

Omitting the trivial transformations, we obtain a system of equations for $a_n(\mathbf{p}, t)$:

$$i \frac{\partial a_n(\mathbf{p}, t)}{\partial t} = \epsilon_n(\mathbf{p}') a_n(\mathbf{p}, t) - i \sum_m' \left(\frac{\partial}{\partial t} \right)_{nm} a_m(\mathbf{p}, t). \quad (3)$$

Here

$$\left(\frac{\partial}{\partial t} \right)_{nm} = \int u_{n\mathbf{p}'}^*(\mathbf{r}) \frac{\partial u_{m\mathbf{p}'}(\mathbf{r})}{\partial t} d\mathbf{r}. \quad (4)$$

In (3) there is omitted the term with $(\partial/\partial t)_{nn}$, since it can be included in the phase a_n . It is quite obvious that

$$(\partial/\partial t)_{nm} = \mathbf{F}(t) (\partial/\partial \mathbf{p})_{nm}. \quad (5)$$

We introduce in place of a_n a new quantity c_n by means of the formula

$$a_n(\mathbf{p}, t) = \exp \left(-i \int_0^t \epsilon_n(t') dt' \right) c_n(\mathbf{p}, t),$$

and then we obtain in lieu of (3)

$$\frac{\partial c_n(\mathbf{p}, t)}{\partial t} = - \sum_m' \exp \left\{ -i \int_0^t (\epsilon_m - \epsilon_n) dt' \right\} \left(\frac{\partial}{\partial t} \right)_{nm} c_m(\mathbf{p}, t). \quad (6)$$

This system of equations coincides exactly with the system of equations of the adiabatic perturbation theory (see, for example, [3]). We confine ourselves to the case of weak electric fields, when transitions are significant only between two neighboring bands, and we therefore retain in (6) two equations ($n = 1, 2$) for the valence and conduction bands. Usually, however, the smallness of the field is immediately connected with the possibility of solving the system (6) with the aid of perturbation theory, since the right side of (6) is proportional to E . We shall show that perturbation theory is not applicable to the solution of the system (6). It is well known (see [4]) that the matrix element $(\partial/\partial t)_{12}$ has a pole at the point t_0 where $\epsilon_1 = \epsilon_2$, and that near t_0

$$\epsilon_1 - \epsilon_2 \sim \sqrt{|t - t_0|}.$$

Within the characteristic time of variation of the field T , the exponential in the right side of (6) changes by an amount $T\Delta$ (Δ is the width of the interband gap). In the case of slow variation of F , we have $T\Delta \gg 1$, i.e., the exponential oscillates rapidly within a time T . The saddle point method is therefore employed. However, as we have seen above, the saddle point ($\epsilon_1 = \epsilon_2$) coincides with the singularity of the matrix element, and the smallness of F yields nothing, since in the second order in F the singularity becomes stronger (this can be shown by integrating with respect to the frequencies), and exactly cancels the smallness of F . It follows therefore that slowly varying fields $F(t)$ cannot be considered by perturbation theory. An exact criterion for the slowness of variation of $F(t)$ will be derived below. At the present stage, our problem consists of showing

that besides the smallness of F the method of solving system (6) is determined also by the scale of the temporal variation of the field. We shall therefore consider in this section, by way of a simple illustration, the case of a very short electric pulse, such that the time of its action T is given by

$$\int_0^T (\varepsilon_1 - \varepsilon_2) dt \ll 1 \quad (T\Delta \ll 1).$$

The exponentials in (6) can now be set equal to unity, and we get

$$\begin{aligned} \frac{\partial c_1}{\partial t} &= -\left(\frac{\partial}{\partial t}\right)_{12} c_2 = -\gamma(t) c_2, \\ \frac{\partial c_2}{\partial t} &= \left(\frac{\partial}{\partial t}\right)_{12} c_1 = \gamma(t) c_1. \end{aligned} \quad (7)$$

If prior to turning on of the field we have $c_2 = 0$ and $c_1 = 1$, then after turning off the field

$$c_2(\infty) = \sin \left[\int_{-\infty}^{\infty} \gamma(t) dt \right] = \sin \left[-i \int_{-\infty}^{\infty} r_{12} F(t) dt \right]. \quad (8)$$

Obviously, $r_{12} \sim 1/\sqrt{m\Delta}$, and the argument of the sine function is $\sim T\Delta F/\sqrt{m\Delta^3}$. Thus, a short pulse can produce strong ionization if

$$F \gg F_0 = \sqrt{m\Delta^3},$$

and perturbation theory is valid when $F \lesssim F_0$.

We begin the calculations for an arbitrary $F(t)$ dependence with the simple case of a square-root dispersion law

$$\varepsilon_{1,2} = \pm^{1/2} \Delta \sqrt{1 + p^2/m\Delta}. \quad (9)$$

Then, as is well known, the matrix element x_{12} (the field F is directed along the x axis) is given by

$$x_{12} = \frac{i}{2} \sqrt{\frac{\Delta}{m}} \left(\Delta^2 + \frac{\Delta p_{\perp}^2}{m} \right)^{1/2} \left(\Delta^2 + \frac{\Delta p^2}{m} \right)^{-1}, \quad (10)$$

where p_{\perp} is the momentum perpendicular to the field F . We introduce the following variables:

$$\tau = \Delta_{\perp} t, \quad f(\tau) = \frac{F(t)}{F_0}, \quad p_x' = \sqrt{\frac{m\Delta_{\perp}^2}{\Delta}} q(\tau), \quad (11)$$

where

$$\Delta_{\perp}^2 = \Delta^2 + \frac{\Delta p_{\perp}^2}{m}, \quad F_0 = \Delta_{\perp}^2 \sqrt{\frac{m}{\Delta}}, \quad q(\tau) = q_0 + \int_0^{\tau} f(\tau') d\tau'. \quad (12)$$

In terms of these variables, the system (3) takes the form

$$\frac{\partial a_{1,2}}{\partial \tau} = \pm \frac{i}{2} \sqrt{1 + q^2(\tau)} a_{1,2} \mp \frac{f(\tau) a_{2,1}}{2(1 + q^2(\tau))}, \quad (13)$$

and the system (6) becomes

$$\frac{\partial c_{1,2}}{\partial \tau} = \pm \frac{f(\tau) c_{2,1}}{2(1 + q^2(\tau))} \exp \left\{ \pm i \int \sqrt{1 + q^2(\tau')} d\tau' \right\}. \quad (14)$$

Equations (13) and (14), just as before, are not conveniently solved. We therefore use the transformations used in a paper by one of the authors and Chaplik,^[5] i.e., we introduce in place of $a_{1,2}$ the quantities A and B defined by the formulas

$$a_1 = \frac{1}{2} \left\{ (A + iB) \left(\frac{q-i}{q+i} \right)^{1/4} + (A - iB) \left(\frac{q+i}{q-i} \right)^{1/4} \right\}, \quad (15)$$

$$a_2 = \frac{i}{2} \left\{ (A + iB) \left(\frac{q-i}{q+i} \right)^{1/4} - (A - iB) \left(\frac{q+i}{q-i} \right)^{1/4} \right\} \quad (16)$$

and then we obtain finally for $A(q_0, \tau)$ and $B(q_0, \tau)$

$$A + \frac{iq(\tau)}{2} A + \frac{i}{2} B = 0, \quad B - \frac{iq(\tau)}{2} B + \frac{i}{2} A = 0, \quad (17)$$

where the dot over the letter denotes the partial derivative with respect to τ . For A we get from (17)

$$\ddot{A} + \left[\frac{iq(\tau)}{2} + \frac{1}{4} + \frac{1}{4} q^2(\tau) \right] A = 0, \quad (18)$$

i.e., A as the function of τ satisfies an ordinary second-order differential equation.

It follows from (18) that certain cases of the $F(t)$ dependence admit of an exact solution.^[6] These include the case of a constant electric field, when the function A is expressed in terms of the parabolic-cylinder formula, and obviously the case of a rectangular electric pulse, when it is necessary to join together the region $q = \text{const}$ and the region of action of the field $f = f_0$.

We now proceed to consider different $f(\tau)$ dependences.

2. PERIODIC ELECTRIC FIELD

In the case of a purely periodic field

$$f(\tau) = f_0 \cos \omega \tau, \quad q(\tau) = q_0 + \frac{f_0}{\omega} \sin \omega \tau, \quad (19)$$

where $\omega = \Omega/\Delta_{\perp}$ and Ω is the frequency of the external field. We note first that the quantities $a_{1,2}(\tau)$ at an arbitrary $f(\tau)$ dependence have the following property:

$$|a_1(\tau)|^2 + |a_2(\tau)|^2 = \text{const},$$

i.e., they are independent of the time τ . It follows therefore that the values of $a_{1,2}$ at different times are connected with the aid of the unitary matrix

$$U = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}, \quad (20)$$

with

$$|a|^2 + |\beta|^2 = 1, \quad a = \delta^* = D, \quad \gamma = -\beta^* = R,$$

where D and R have the meaning of the transmission and reflection amplitudes (in our case the latter is the amplitude of the probability of production of an electron-hole pair). By the same token, in a periodic field $f(\tau)$ the problem will be completely solved if we know the quantities α and γ corresponding to the shift of the field by one period. Naturally, in the general case they cannot be determined.

We start with the case $\omega \ll 1$, or more accurately

$$\frac{1}{2} \int_0^{\tau} \sqrt{1 + q^2(\tau')} d\tau' \gg 1. \quad (21)$$

If it is recognized that A satisfies Eq. (18), then the condition (21) coincides with the criterion for the applicability of the quasiclassical approximation for A . In the case when the condition (21) is satisfied and $q(\tau)$ is periodic, we can solve (19) by using a method developed by one of the authors^[7] in the problem of the quasiclassical approximation. In our case the problem obviously corresponds to over-the-barrier passage, since the turning point turned out to be complex. These points τ_0 are determined from the condition

$$1 + q^2(\tau_0) = 0.$$

In determining the turning points in (18) we have discarded the term with $\dot{q} \sim f_0 \ll 1$. However, this term

cannot be discarded completely. Allowance for this term leads to the appearance of a pre-exponential factor in A, analogous to those functions that connect $a_{1,2}$ and A(B). Without stopping to discuss the detailed transformations, we shall indicate the final result. Obviously, the turning points are pairwise complex conjugate.

Let us consider the case of sufficiently weak fields, when the condition

$$\frac{\omega'^{3/2} \cos 2\omega\tau_0}{f_{\omega}^{3/2} \sin'^{3/2} 2\omega\tau_0} \ll 1. \quad (22)$$

is satisfied. This condition is equivalent to the requirement that it be possible to consider each turning point independently (the ionization probability in this case is exponentially small).

When the inequality (22) is satisfied, the phase D on going from the point τ_1 to the point τ_2 is determined by the integral

$$\frac{1}{2} \int_{\tau_1}^{\tau_2} \sqrt{1+q^2(\tau)} d\tau, \quad (23)$$

and

$$|R| = \exp \left\{ -\frac{1}{2} \operatorname{Im} \int_{\tau_{10}}^{\tau_{20}} \sqrt{1+q^2(\tau)} d\tau \right\}, \quad (24)$$

with τ_{10} and τ_{20} the complex turning points, such that

$$\tau_1 < \operatorname{Re} \tau_{10} = \operatorname{Re} \tau_{20} < \tau_2.$$

As will be shown later, the phase R does not enter in the transition probabilities calculated by us.

By the same token we have found that knowledge of the position of the turning point in the period $f(\tau)$ determines completely the matrix U. If we put

$$\tau_0 = \tau_0' + i\tau_0'',$$

then we obtain from the condition $1+q^2(\tau_0) = 0$

$$\sin \omega\tau_0' \operatorname{ch} \omega\tau_0'' = -\frac{\omega}{f_{\omega}} q_0, \quad \cos \omega\tau_0' \operatorname{sh} \omega\tau_0'' = \pm \frac{\omega}{f_{\omega}}. \quad (25)$$

It follows from (25) that if $\tau_0' \omega$ is a solution, then the point $\pi - \tau_0' \omega$ is also a solution. It follows therefore that when $q_0 \neq 0$ and $\omega\tau_0' \neq n\pi$ there are four turning points in the period:

$$\tau_{10} = \tau_{10}' \pm i\tau_{10}'', \quad \tau_{20} = \pi/\omega - \tau_{10}' + i\tau_{10}''.$$

We now readily obtain an expression for the components of the matrix U, corresponding to the passage through the period T of $f(\tau)$.

$$\alpha = e^{i(\varphi_1+\varphi_2)} |D|^2 - e^{i(\varphi_2-\varphi_1)} |R|^2, \quad (26)$$

$$\gamma = 2R|D| e^{-i\varphi_2} \cos \varphi_1,$$

with

$$\varphi_1 = \frac{1}{2} \int_{\tau_{10}'}^{\pi/\omega - \tau_{10}'} \sqrt{1+q^2(\tau)} d\tau, \quad \varphi_2 = \frac{1}{2} \int_{\pi/\omega - \tau_{10}'}^{\tau_{10}'+T} \sqrt{1+q^2(\tau)} d\tau. \quad (27)$$

We note here that we have used the condition

$$|R(\tau_{10}')| = |R(\pi/\omega - \tau_{10}')|.$$

We shall not present the intermediate steps connected with the multiplication of the matrix U (they are trivial), and present only the final formulas.

The probability of production of a pair as the result of passage through m turning points (with different values of τ_0') is

$$|\gamma_{2n}|^2 = 4|R|^2 |D|^2 \cos^2 \varphi_1 \frac{\sin^2 n\varphi}{\sin^2 \varphi}, \quad (28)$$

$$|\gamma_{2n+1}|^2 = |R|^2 \left| \frac{\cos \varphi + e^{-i(\varphi_1-\varphi_2)}}{\sin \varphi} \sin n\varphi + \cos n\varphi \right|^2, \\ m = 2n + 1,$$

with

$$\cos \varphi = \cos(\varphi_1 + \varphi_2) - 2|R|^2 \cos \varphi_1 \cos \varphi_2. \quad (29)$$

We proceed now to analyze the formulas (28). We first rewrite the first formula in (28) in a different form:

$$|\gamma_{2n}|^2 = |R|^2 |D|^2 \cos^2 \varphi_1 \sin^2 n\varphi \left[|D|^2 \cos^2 \frac{\varphi_1 + \varphi_2}{2} + |R|^2 \sin^2 \frac{\varphi_1 - \varphi_2}{2} \right]^{-1} \\ \times \left[|D|^2 \sin^2 \frac{\varphi_1 + \varphi_2}{2} + |R|^2 \cos^2 \frac{\varphi_1 - \varphi_2}{2} \right]. \quad (30)$$

We see therefore that when $\varphi_1 + \varphi_2 = 2k\pi$

$$\varphi \approx 2k\pi + 2 \left| R \cos \frac{\varphi_1 - \varphi_2}{2} \right|,$$

where k is an integer and

$$|\gamma_{2n}|^2 \approx \sin^2 \left[2n \left| R \cos \frac{\varphi_1 - \varphi_2}{2} \right| \right].$$

The result is perfectly understandable. The condition $\varphi_1 + \varphi_2 = 2k\pi$ is obviously none other than the resonance condition ($\Delta = 2k\Omega$), and near the resonance the particle, within a number of periods

$$n = \left(2 \left| R \cos \frac{\varphi_1 - \varphi_2}{2} \right| \right)^{-1}$$

passes, with a probability equal to unity, from one band to the other—a situation well known in quantum mechanics.^[4] (The same occurs also when $\varphi_1 + \varphi_2 = (2k+1)\pi$, when $k = |2R \sin((\varphi_1 - \varphi_2)/2)|^{-1}$). Of course, this effect becomes smeared out by collisions and becomes noticeable when $\tau_0 \approx T/|R|$ (τ_0 —free path time). It is clear that the second term in (29) describes the energy uncertainty connected with these transitions. If we consider times $\tau < T/|R|$, then

$$\varphi = \varphi_1 + \varphi_2. \quad (31)$$

The main contribution to the interband transitions will then be made by terms in (28), where $\varphi = m\pi$ when $|\gamma_{2n}|^2 \sim n^2$. In these cases both formulas in (28) coincide. The usual quantum-mechanical procedure leads to the following formula for the transition probability

$$|\gamma|^2 = 2\Delta\omega |R|^2 \cos^2 \left(\frac{m\pi}{2} + \frac{\varphi_1 - \varphi_2}{2} \right) \delta(\varphi_1 + \varphi_2 - m\pi) \\ = \frac{2\Delta\omega^2}{\pi} |R|^2 \cos^2 \left(\frac{m\pi}{2} + \frac{\varphi_1 - \varphi_2}{2} \right) \delta \left(\frac{1}{T} \int_0^T \sqrt{1+q^2(\tau)} d\tau - m\omega \right). \quad (32)$$

This formula is analogous to corresponding formula in Keldysh's paper.^[2] The difference consists, first, in the fact that Keldysh's formula has an incorrect pre-exponential factor, and second that according to (32) the resonances with absorption of even and odd numbers of quanta have different probabilities.

Leaving out the intermediate calculations (for details see^[2]), we present the final expression for the total interband transition probability

$$w = \frac{2\Omega}{\pi^3} \left(\frac{m\Omega}{\kappa} \right)^{3/2} \exp \left\{ -\pi \left\langle \frac{\tilde{\Delta}}{\Omega} + 1 \right\rangle \frac{K(\kappa) - E(\kappa)}{E(\kappa')} \right\} \\ \times \left[\frac{\pi}{2K(\kappa')} \right]^{1/2} \sum_{n=0}^{\infty} \exp \left\{ -n\pi \frac{K(\kappa) - E(\kappa)}{E(\kappa')} \right\} F_n \left(\kappa, \frac{\tilde{\Delta}}{\Omega} \right). \quad (33)$$

Here

$$\tilde{\Delta} = \frac{2}{\pi} \Delta \frac{E(\kappa')}{\kappa}, \quad \kappa = \frac{\gamma}{\sqrt{1+\gamma^2}}, \quad (34)$$

$$\kappa' = \sqrt{1-\kappa^2}, \quad \gamma = \Omega \sqrt{m\Delta / F};$$

the angle brackets denote the integer part of the quantity, and $K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kind. The function $F_n(\kappa, \mathbf{x})$ in (33) is given by

$$F_n(\kappa, \mathbf{x}) = e^{-v_0} \int_0^{v_0} e^{v^2} \cos^2 \left(\frac{n + \langle x + 1 \rangle}{2} \pi + cy \right) dy. \quad (35)$$

In the last formula

$$y_0 = \left[\frac{\pi^2 (n + \langle x + 1 \rangle - x)}{2E(\kappa')K(\kappa')} \right]^{1/2}, \quad c = \sqrt{\frac{2\Omega K(\kappa')}{\pi \Delta \kappa}} \left(\frac{\gamma}{\text{ch } \gamma} + \arcsin \kappa' \right). \quad (36)$$

The difference between formula (33) and corresponding formula (37) of [2] lies in the fact that (33), first, has a different numerical coefficient (in [2] the pre-exponential factor for $|R|^2$ is equal to $(\pi/3)^2$, and in our case it is equal to unity), and second, in our case F_n contains the square of the cosine.

We note the particular case when $\gamma \gg 1$, i.e., $1 - \kappa \ll 1$. In this case the terms of the sum in (33) behave like $(1/\gamma)^{2n}$, i.e., it is necessary to retain only the term with $n = 0$. We then obtain for the transition probability

$$w = \frac{2\Omega}{\pi^2} (m\Omega)^{1/2} \exp \left(2 \left\langle \frac{\Delta}{\Omega} + 1 \right\rangle \right) \left(\frac{1}{16\gamma^2} \right)^{\langle \Delta/\Omega + 1 \rangle} F_0 \left(1, \frac{\Delta}{\Omega} \right). \quad (37)$$

Here F_0 is given by formula (35) with

$$y_0 = \left[2 \left(\left\langle \frac{\Delta}{\Omega} + 1 \right\rangle - \frac{\Delta}{\Omega} \right) \right]^{1/2}, \quad c \approx 2 \sqrt{\frac{\Omega}{\Delta}} \frac{1}{\gamma}. \quad (38)$$

Obviously, $y_0 \sim 1$ and $c \ll 1$, i.e., we get with good accuracy, putting $c = 0$,

$$F_0 \left(1, \frac{\Delta}{\Omega} \right) = \cos^2 \left(\frac{\pi}{2} \left\langle \frac{\Delta}{\Omega} + 1 \right\rangle \right) e^{-v_0} \int_0^{v_0} e^{v^2} dy. \quad (39)$$

It follows from (39) that in the approximation under consideration the probability of absorption of an even number of quanta ($\Delta = 2m\Omega$) is equal to zero.

We now make a few remarks concerning the case of sufficiently strong fields (confining ourselves as before to the two-band approximation and to low frequencies). From the equations for the turning point (25) we see that if $\omega/f_\omega \ll 1$ we get

$$\omega\tau_0 = n\pi + \varphi, \quad \varphi \ll 1.$$

The latter circumstance makes it possible to expand the expressions in the square brackets in (18) in powers of τ near τ_0 . We assume here that $f_\omega \gtrsim 1$. We then obtain for A near $\omega\tau_0 = n\pi$

$$\ddot{A} + \left\{ (-1)^n \frac{if_\omega}{2} + \frac{1}{4} + \frac{1}{4} [q_0 + (-1)^n f_\omega \tau]^2 \right\} A = 0. \quad (40)$$

The solution of this equation is expressed in terms of the parabolic-cylinder functions, and Eq. (40) is valid up to times τ for which $\tau \gg 1$ but $\omega\tau \ll 1$.

After simple calculations we find that the phase of the quantity D (see (20)), on going from the point τ_1 to the point τ_2 is now given by

$$\frac{1}{2} \int_{\tau_1}^{\tau_2} \sqrt{1+q^2(\tau)} d\tau + \frac{\pi}{4} - \frac{1}{4f_\omega} \ln(4ef_\omega) + \varphi_0, \quad (41)$$

$$e^{2i\varphi_0} = \Gamma \left(1 - \frac{i}{4f_\omega} \right) / \Gamma \left(1 + \frac{i}{4f_\omega} \right)$$

(when $f_\omega \ll 1$ formula (41) goes over into (23)), and the reflection probability is

$$|R| = \exp(-\pi/4f_\omega). \quad (42)$$

The transformations that lead to the probability of the production of a pair per unit time are perfectly analogous to those performed for the case $f_\omega \ll 1$, and will not be presented here.

3. CONSTANT AND PERIODIC ELECTRIC FIELDS

The last case which we shall consider is when the field is a superposition of a constant and periodic electric field. For simplicity we assume both fields to be directed along the x axis, i.e.,

$$f(\tau) = f_0 + f_\omega \cos \omega\tau.$$

The probability of the photonless pair production cannot be calculated by perturbation theory in the case of weak fields at any frequency. The situation is different with the probability of pair production with absorption of a single photon. In first order of perturbation theory in f_ω we have

$$c_2(\infty) = -\frac{1}{4} f_\omega \int_{-\infty}^{\infty} \frac{d\tau}{1+q^2(\tau)} \exp \left\{ i\omega\tau - i \int_{-\infty}^{\tau} \sqrt{1+q^2(\tau')} d\tau' \right\}. \quad (43)$$

Using the saddle-point method, we find that the pre-exponential factor is equal to $\sim f_\omega/\omega^2$. If the condition $f_\omega/\omega^2 \ll 1$ is satisfied, then we can confine ourselves in the calculation of c_2 to the first order of the adiabatic perturbation theory. The saddle points are determined in this case from a system of equations analogous to (25):

$$f_0 \tau_0' + \frac{f_\omega}{\omega} \sin \omega\tau_0' \text{ch } \omega\tau_0'' = -q_0, \quad (44)$$

$$f_0 \tau_0'' + \frac{f_\omega}{\omega} \cos \omega\tau_0' \text{sh } \omega\tau_0'' = \pm \sqrt{1-\omega^2}.$$

The usual method of finding the absorption probability consists in neglecting the terms $\sim f_\omega$ in (44). In this case we obtained formulas for the Franz-Keldysh effect.

In the general case it is of course impossible to solve (44). When $q_0 = 0$ (near the bottom of the band), the first equation of (44) takes the form

$$\frac{\sin \omega\tau_0'}{\omega\tau_0'} = -\frac{f_0}{f_\omega \text{ch } \omega\tau_0''}. \quad (45)$$

It follows from (45) that, besides the turning point with $\tau_0' = 0$, there are other turning points, for which

$$\text{ch } \omega\tau_0'' > f_0/f_\omega.$$

When $1 - \omega \ll 1$ and $f_\omega < f_0$, we get with good accuracy at $\tau_0' = 0$

$$\tau_0'' = \pm \sqrt{1-\omega^2}/f_0.$$

The absorption then decreases at frequencies $1 - \omega \sim f_0^{2/3}$. This means that allowance for the other turning point is immaterial in the case when

$$f_\omega \exp(-f_0^{2/3}) \ll f_0. \quad (46)$$

We now consider the case $\omega \ll 1$, i.e., the question of the influence of the constant field on multiquantum absorption. The derivation of (32) is based on the existence of a large ($n \gg 1$) number of turning points of identical level. If $f_0 \ll f_\omega$, then it follows from (45)

that there are at least k solutions $\omega \tau'_0 = k\pi$, if

$$f_0/f_\omega \operatorname{ch} \omega \tau_0'' < 1/k\pi. \quad (47)$$

In this case τ_0'' is determined from the equation

$$f_0 \tau_0'' \pm \frac{f_\omega}{\omega} \operatorname{sh} \omega \tau_0'' = \pm 1. \quad (48)$$

If the condition (47) is satisfied, then it is possible to neglect in (48) the term with f_0 , and we obtain

$$\operatorname{sh} \omega \tau_0'' = \pm \omega/f_\omega.$$

By the same token the final condition for the existence of the $k \gg 1$ turning points of almost identical level is

$$\frac{f_0}{f_\omega} \left(1 + \frac{\omega^2}{f_\omega^2}\right)^{-1/2} < \frac{1}{k\pi}. \quad (49)$$

To find the probability of the interband transition in this case it is necessary to introduce a set of matrices U_i , describing the transitions between successive turning points. Of course, in the general case they cannot be multiplied.

In conclusion we note one circumstance. All the calculations, starting with formula (14), were carried out for a square root dispersion law. However, if the conditions (21) and (22) are satisfied, then the main contribution to the transition is made only by close vicinities of the turning points. At the same time it is well known (see [1, 4]) that near the point where $\epsilon_1 = \epsilon_2$ the matrix element is of the form

$$\frac{1}{2i(\epsilon_1 - \epsilon_2)} \frac{\partial(\epsilon_1 - \epsilon_2)}{\partial t}.$$

By a simple repetition of the reasoning given in a paper by one of the authors, [8] we arrive at all the results obtained by us (for $\omega \ll 1$), except that the square-root dispersion law is now replaced everywhere by an arbitrary dispersion law.

¹ L. V. Keldysh, *Zh. Eksp. Teor. Fiz.* **33**, 994 (1957) and **34**, 1138 (1958) [*Sov. Phys.-JETP* **6**, 763 (1958) and **7**, 788 (1958)].

² L. V. Keldysh, *ibid.* **47**, 1945 (1964) [**20**, 1307 (1965)].

³ L. Schiff, *Quantum Mechanics*, McGraw-Hill, 1955.

⁴ L. D. Landau and E. M. Lifshitz, *Kvantovaya mekhanika*, Fizmatgiz, 1963 [*Quantum Mechanics*, Addison-Wesley, 1968].

⁵ A. M. Dykhne and A. V. Chaplik, *Zh. Eksp. Teor. Fiz.* **43**, 889 (1962) [*Sov. Phys.-JETP* **16**, 631 (1963)].

⁶ Yu. A. Bychkov and A. M. Dykhne, *ibid.* **48**, 1174 (1965) [**21**, 783 (1965)].

⁷ A. M. Dykhne, *ibid.* **40**, 1423 (1961) [**13**, 999 (1961)].

⁸ A. M. Dykhne, *ibid.* **41**, 1324 (1961) [**14**, 941 (1962)].