QUASI-GASDYNAMICAL DESCRIPTION OF A HOT ELECTRON CLOUD IN A COLD PLASMA

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We study the one-dimensional problem of the expansion of a rarefied cloud of hot electrons through a cold plasma. We show that the expansion process is accompanied by the excitation of Langmuir oscillations; the reaction of those on the hot electron distribution function is calculated in the framework of the quasi-linear approximation. Using a method which is similar to the Chapman-Enskog method we obtain "quasi-gasdynamical" equations which describe the expansion process. We find an analytical solution of these equations for different initial conditions.

1. INTRODUCTION

UNDER astrophysical or laboratory condition a situation is often reached when in some region of space which is filled with a uniform cold plasma for some reason or other a small impurity of "hot" electrons appears with an average energy which appreciably exceeds the temperature of the cold plasma. In the present communication we study the dynamics of the expansion of a cloud of hot electrons, restricting ourselves for the sake of simplicity to the one-dimensional case and assuming that there are no binary collisions.

Let at time $t = 0$ the hot electrons fill the half-space $x < 0$ and let their initial concentration n_0' be small compared to the concentration n of the cold plasma. As under the condition $n_0' \ll n$ we need only an insignificant change in the concentration of the cold electrons to guarantee the quasi-neutrality of the polarized electric field arising when the hot electrons disperse is small and does not turn out to affect their motion significantly. We could therefore expect that the hot electrons disperse completely freely and that their distribution function at any time $t > 0$ is given by the equation

$$f(v,x,t)=\begin{cases}0, & v<x/t\\ f_0(v), & v>x/t\end{cases}, \tag{1}$$

where $f_0(v)$ is the initial distribution function of the fast electrons; we assume that it has a single maximum at $v = 0$.

One sees easily, however, that the distribution function (1) is unstable in the half-space $x > 0$ with respect to the excitation of Langmuir oscillations in the cold plasma (the so-called two-stream instability). The characteristic time τ for the development of this instability is equal to $\omega_{pe}^{-1}n/n'$ (where ω_{pe} is the electron plasma frequency, evaluated using the cold electron concentration) which in the cases of interest is small compared to the observation time. The problem therefore arises about the effect of the instability on the expansion of a hot electron cloud. In the following we shall show that when $t \gg \tau$ the motion of the cloud is described by relatively simple equations which we shall call (for reasons which become clear from what follows) "quasi-gasdynamical." We shall also find an analytical solution of these equations.

For information we mention that the problem of the dispersion of a dense ($n_0' = n$) hot electron cloud was solved in [1], neglecting instability effects.

2. DERIVATION OF THE QUASI-GASDYNAMICAL EQUATIONS

To take into account the interaction of the fast electrons with the Langmuir oscillations excited in the plasma we use a set of quasi-linear equations[2,3] 1)

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}=\frac{\partial}{\partial v}D\frac{\partial f}{\partial v}\equiv \mathrm{St}\,f, \tag{2}$$

$$\frac{\partial D}{\partial t}=D\frac{\pi\omega_{pe}}{n}v^2\frac{\partial f}{\partial v}, \tag{3}$$

where $D \equiv D(v, x, t)$ is the quasi-linear diffusion coefficient which is connected with the energy spectral density of the Langmuir oscillations, $W(v, x, t)$ by the relation $D = (4\pi^2e^2/m^2)W$, while the rest of the notation is the usual one. The group velocity of the Langmuir oscillations v_g is small compared to the characteristic dispersion velocity of the fast electrons which is of the same order of magnitude as their thermal velocity and we drop therefore the term $v_g\,\partial D/\partial x$ on the left-hand side of Eq. (3).

On a time scale $t \gg \tau$ the instability is "fast," i.e., the right-hand sides of Eqs. (2), (3) which can formally be estimated to be Df/v^2 and D/τ are large compared to the left-hand sides. When solving the set (2) and (3), we can thus use an expansion in the parameter τ/t. The position is here in many respects analogous to the one which occurs in the dynamics of a normal gas where when we consider motions on spatial and time scales which are appreciably larger than, respectively, the mean free path and the time between binary collisions the kinetic equation is solved through an expansion in the inverse collision frequency.

In zeroth approximation the function f is simply that source function f_s which is established at each point in space as a result of quasi-linear relaxation and is determined from the condition

1) We limit ourselves to the one-dimensional (in velocity space) quasi-linear equations, bearing in mind that in a plasma there is a rather strong magnetic field in the x-direction.

$$\mathrm{St}\, f_s = 0, \quad \partial f_s / \partial v \leqslant 0.$$

From this it follows that for the initial conditions considered by us

$$f_s(v, x, t) = \begin{cases} p(x,t) & v < u(x,t) \\ f_0(v) & v > u(x,t) \end{cases}, \tag{4}$$

where p and u are some so far undetermined functions of the coordinates and the time. The quasi-linear diffusion coefficient D is also not yet known; we can only state that it differs appreciably from zero in the velocity range $[0, u]$. Outside this range diffusion can be neglected.

By complete analogy with what happens in usual gas-dynamics, the zeroth approximation distribution function is uniquely characterized by a finite number of parameters (in our case there are two such parameters: p and u; in one-dimensional gas-dynamics there are three: density, temperature, and mass velocity). The problem now consists in obtaining equations for these parameters. In gasdynamics this problem is solved by taking the zeroth, first, and second moments of the kinetic equation (taking into account that the collision integral leaves concentration, momentum, and energy of the particles invariant). In our case St f conserves only the particle concentration so that we can obtain only one equation through this method; this equation is analogous to the gas-dynamical continuity equation:

$$\frac{\partial}{\partial t} pu + \frac{\partial}{\partial x}\frac{pu^2}{2} - f_0(u)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 0. \tag{5}$$

We can obtain a second equation for p and u as follows. As $D\,\partial f/\partial v \to 0$ as $v \to 0$, it follows from Eqs. (2) and (3) that

$$\frac{\partial}{\partial t}\int_0^v f\,dv' + \frac{\partial}{\partial x}\int_0^v v' f\,dv' = D\frac{\partial f}{\partial v} = \frac{n}{\pi\omega_{pe}v^2}\frac{\partial D}{\partial t}.$$

In that velocity range $(0 < v < u)$ where the quasi-linear diffusion coefficient is non-vanishing, the distribution function is simply equal to $p(x, t)$ so that

$$\frac{\partial D}{\partial t} = \frac{\pi\omega_{pe}}{n} v^2 \left(v\frac{\partial p}{\partial t} + \frac{v^2}{2}\frac{\partial p}{\partial x}\right)$$

and

$$D = \frac{\pi\omega_{pe}}{n} v^3 \left(p + \frac{v}{2}\frac{\partial}{\partial x}\int_0^t p\,dt'\right). \tag{6}$$

As in the point v = 0 the quasi-linear diffusion coefficient must vanish we get from (6) the required second equation for p and u:

$$p + \frac{u}{2}\frac{\partial}{\partial x}\int_0^t p\,dt' = 0$$

One can easily reduce it to differential form:

$$u\frac{\partial p}{\partial t} - p\frac{\partial u}{\partial t} + \frac{u^2}{2}\frac{\partial p}{\partial x} = 0. \tag{7}$$

It is interesting to note that just as the actual value of the effective collision frequency does not enter into the equations of ideal gasdynamics, similarly the parameters characterizing the beam instability do not enter into the "quasi-gasdynamical" Eqs. (5) and (7).

Knowing p and u we can find all macroscopic characteristics of a hot plasma. For instance, we have for its concentration n′ and average velocity $\bar{v}'$:

$$n' = pu + \int_u^\infty f_0(v)\,dv, \tag{8}$$

$$\bar{v}' = \frac{1}{n'}\left[\frac{pu^2}{2} + \int_u^\infty v f_0(v)\,dv\right]. \tag{9}$$

We can also express the energy density of the Langmuir oscillations in terms of p and u:

$$U = \omega_{pe}\int_0^\infty \frac{W}{v^2}\,dv = \omega_{pe}\frac{m^2}{4\pi^2 e^2}\int_0^u \frac{D}{v}\,dv = \frac{mpu^3}{12}. \tag{10}$$

3. SOLUTION OF THE QUASI-GASDYNAMICAL EQUATIONS

To solve the set of Eqs. (5) and (7) we note that in the quasi-gasdynamical problem there are no parameters of the dimensions of a length. Its solution is therefore self-similar and has the form

$$p = t^\alpha q(\xi), \quad u = u(\xi), \quad \xi = x/t. \tag{11}$$

We determine the parameter α from the condition that the total number of particles in the half-space $x > 0$ linearly increases with time. One sees easily that this condition leads to the value $\alpha = 0$.

By substituting (11) we get from (5) and (7) the ordinary differential equations

$$u(u - 2\xi)\frac{dp}{d\xi} + 2(u - \xi)\left[p - f_0(u)\right]\frac{du}{d\xi} = 0,$$

$$u(u - 2\xi)\frac{dp}{d\xi} + 2p\xi\frac{du}{d\xi} = 0,$$

the solution of which can be found elementarily:

$$p = u^2 \Big/ 2\int_0^u \frac{v\,dv}{f_0(v)}, \quad \xi = u\frac{p - f_0}{2p - f_0} \tag{12}$$

This solution determines implicitly the functions $u(\xi)$ and $p(\xi)$ in the half-space $x > 0$. As regards the half-space $x < 0$ there the solution (1) is valid, as the corresponding distribution function is stable.

We can find from Eqs. (8) to (10) the macroscopic parameters of the hot plasma and the energy density of the Langmuir oscillations. We give in Fig. 1 the results of the appropriate calculations for the case when the initial distribution function of the electrons is Maxwellian:

$$f_0 = n_0'\sqrt{\frac{m}{2\pi T_e'}}\exp\left(-\frac{mv^2}{2T_e'}\right).$$

We study now the problem of the expansion of a hot electron cloud in the case when its initial thickness is equal to zero. In other words, we consider an initial condition of the form

$$f|_{t=0} = N g_0(v)\,\delta(x), \tag{13}$$

where N is the total number of hot electrons while the function $g_0(v)$ is assumed to be normalized to unity

$$\int_{-\infty}^{+\infty} g_0(v)\,dv = 1.$$

Of course, in actual circumstances the initial thickness L of the cloud is always different from zero but

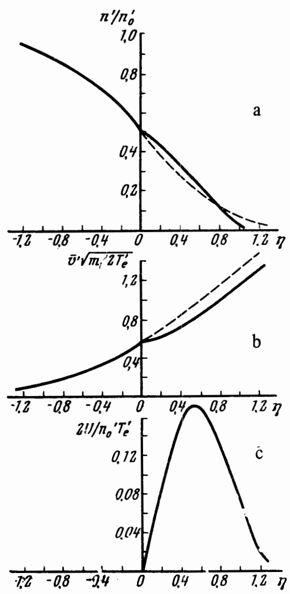

FIG. 1. The general character of the solution in the case where the function $f_0(v)$ is Maxwellian. Along the abscissa axis we plotted the dimensionless parameter $\eta = \xi(m/2T_e')^{1/2}$. The dotted curves correspond to the free dispersion of fast electrons.

if we consider motion at distances $x \gg L$ the approximation (13) is rather good. We note that the problem solved above of the dispersion of an initially half-bounded cloud in fact corresponds to the opposite limiting case, $x \ll L$.

When the initial condition (13) is satisfied, the function $f_s(v, x, t)$ is determined by the equation[2)]

$$f_s(v, x, t) = \begin{cases} p(x, t), & v < u(x, t) \\ 0, & v > u(x, t) \end{cases}, \tag{14}$$

and the quasi-gasdynamical equations take the form

$$\frac{\partial}{\partial t} pu + \frac{\partial}{\partial x}\frac{pu^2}{2} = 0, \tag{15}$$

$$u\frac{\partial p}{\partial t} - p\frac{\partial u}{\partial t} + \frac{u^2}{2}\frac{\partial p}{\partial x} = 0. \tag{16}$$

The solution of these equations is as before self-similar but now the parameter $\alpha = -1$ since in the case considered the total number of hot electrons is conserved, and not their flux through the plane $x = 0$. Substituting $p = t^{-1}q(\xi)$, $u = u(\xi)$ into Eqs. (15) and (16) we get

$$u(u - 2\xi)\frac{dq}{d\xi} + 2(u - \xi)q\frac{du}{d\xi} - 2qu = 0,$$

$$u(u - 2\xi)\frac{dq}{d\xi} + 2\xi q\frac{du}{d\xi} - 2qu = 0,$$

or, which is equivalent,

$$(u - 2\xi)q\frac{du}{d\xi} = 0, \tag{17}$$

$$\frac{d}{d\xi}[u(u - 2\xi)q] = 0. \tag{18}$$

It follows from (17) that

$$u = 2\xi$$

(the solution u = constant, clearly, is a redundant one) and Eq. (18) is satisfied identically for any $q(\xi)$.

To find the function $q(\xi)$ we use the exact equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{n}{\pi\omega_{pe}}\frac{\partial}{\partial v}\frac{1}{v^2}\frac{\partial D}{\partial t}$$

which is valid for any x (and not only when $x \gg L$) and which therefore makes it possible for us to take the initial condition correctly into account. We integrate this equation first over the time from 0 to ∞ and then over the coordinate between the limits (− 0, x) and use the initial condition (13). As a result we find that

$$-Ng_0(v) + v\int_0^\infty f(v, x, t')\,dt' = 0.$$

As the distribution function for $x \gg L$ is given by Eq. (14) in which we must put, as we showed above, $u = 2\xi$, $p = t^{-1}q(\xi)$, the last equations reduces to the form

$$-Ng_0(v) + v\int_{v/2}^\infty \frac{q(\xi)}{\xi}\,d\xi = 0,$$

from which we easily determine $q(\xi)$:

$$q(\xi) = -\frac{N}{2}\xi\frac{d}{d\xi}\frac{g_0(2\xi)}{\xi}$$

Using now Eqs. (8) to (10) we can find n', $\bar{v}'$, and U:

$$n' = -\frac{N}{t}\xi^2\frac{d}{d\xi}\frac{g_0(2\xi)}{\xi},$$
$$\bar{v}' = \xi, \qquad U = -\frac{mN}{3t}\xi^4\frac{d}{d\xi}\frac{g_0(2\xi)}{\xi}. \tag{19}$$

We have shown the form of the solution (19) in Fig. 2 for the case when

$$g_0(v) = \sqrt{\frac{m}{2\pi T_e'}}\exp\left(-\frac{mv^2}{2T_e'}\right).$$

4. DISCUSSION OF THE RESULTS

It is clear from Figs. 1 and 2 that taking the quasi-linear effects into account leads to an appreciable change

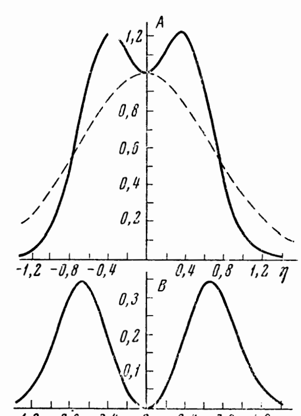

FIG. 2. Solution of the problem with initial condition (13) for a Maxwellian function $g_0(v)$. Along the abscissa axis we plot the non-dimensional parameter $\eta = \xi(m/2T_e')^{1/2}$ and along the ordinate axis the quantities $A = (n't/N)(2\pi T_e'/m)^{1/2}$ and $B = 3(Ut/N)(\pi/2mT_e')^{1/2}$. The dotted curve corresponds to the free dispersion of the fast electrons.

[2)]To be specific, we consider the solution in the half-space $x > 0$.

in the form of the solution as compared to the case of free dispersion: the velocity of the motion is decreased and the leading front of the concentration becomes steeper; moreover, in the region $x \gg L$ there is a clear maximum in the function $n'(x)$. These peculiarities of the motion become even more pronounced if the function $f_0(v)$ decreases at large v faster than the Maxwell distribution. We note that the energy density of the Langmuir oscillations which are excited in the system is of the same order of magnitude as the energy density of the fast electrons.

The quasi-neutrality of the plasma during the dispersion of the hot electrons is guaranteed when we take into account the opposing motion of the cold electrons relative to the ions. It is clear that the solution given above is valid only so long as the velocity of that motion, which is of the order of magnitude of $(n'/n)\sqrt{(T_e'/m)}$, does not exceed a critical velocity corresponding to the threshold for exciting ion-acoustic type oscillations in the cold plasma. The latter depends on the ratio of the temperature of the cold electrons T_e and of the ions T_i and in the particular case when $T_e = T_i$ it is of order of magnitude $\sqrt{(T_e/m)}$. Bearing this in mind we get the following limitation on the allowable value of the hot electron concentration (for $T_e = T_i$):

$$n' < n\sqrt{T_c / T_e'}.$$

If this condition is not satisfied the cold electrons suffer friction from the ions and as a result the polarized electrical field increases its magnitude and the speed of the dispersion of the hot electrons decreases.

Our considerations must also be changed in the case when the external magnetic field is small or is not at all present so that the quasi-linear relaxation is three-dimensional (in velocity space). It is not excluded that there may then occur a solution of the collisionless shock wave type.

In conclusion we note that the results obtained in sections 2 and 3 can be directly transferred to the case when in a strongly non-isothermal ($T_e \gg T_i$) uniform plasma there is a cloud of "warm" ions with a temperature T_i' satisfying the inequality

$$T_i \ll T_i' \ll T_e.$$

The dispersion of such a cloud is accompanied by the excitation of ion-acoustic oscillations with a phase velocity of order $\sqrt{(T_i'/M)}$ (where M is the ion mass) and a frequency close to ω_{pi}. Under those conditions the distribution function of the "warm" ions satisfies as before Eqs. (1) and (2) (in which we must substitute M for m) and hence all derivations given in Secs. 2 and 3 retain their validity.

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