EVOLUTION OF THE WAVE SPECTRUM IN A PLASMA AS A RESULT OF STIMULATED SCATTERING

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It is shown that nonlinear transfer of sufficiently narrow wave packets to the long-wave spectral region as a result of scattering by particles does not occur via diffusion "creeping" but via a satellite system. For Langmuir oscillations, the process may represent either a discontinuous variation of wavelength or nonlinear reflection involving a change of the wave vector direction. Conditions are found under which evolution of broad wave packets leads ultimately to the same type of two-level transfer.

WE consider in this paper the evolution of the spectrum of oscillations of a plasma as a result of induced scattering by particles. In a collisionless plasma, in the spectral region where the Landau damping is small, the evolution of the spectrum is determined by nonlinear effects, principal among which are (with respect to the parameter W/nT, where W is the wave energy density) decay processes and scattering of waves by plasma particles. We shall henceforth consider only the case when the decay processes are forbidden by conservation laws, or have a much lower probability than nonlinear scattering. Examples of "non-decay spectra" can be ion-acoustic waves, and in an isothermal plasma also Langmuir waves and electromagnetic waves at low frequencies ($\omega < 2\omega_{0e}$).

1. PRINCIPAL LAWS OF THE PROCESS

To establish the principal laws of the spectral redistribution due to nonlinear scattering, we turn to the following simple model.

Assume that there exists in the plasma a narrow one-dimensional wave packet (the criterion for the packet width will be given below). As a characteristic of the spectrum, we choose the "number of waves" $N_k = W_k/\omega_k$, where W_k is the spectral density of the oscillation energy. Using the second-quantization formulas and going over to the limit of large quantum-state occupation numbers, we can obtain a kinetic equation for the waves, describing the evolution of the spectrum due to nonlinear scattering[1]:

$$\partial N_k/\partial t = \gamma_k N_k, \qquad \gamma_k = \sum_{k'} w_k^{k'} N_{k'}, \tag{1}$$

where $w_k^{k'}$ is the transition probability. Inasmuch as the number of waves in each act of scattering is conserved, we can write

$$\sum_k \frac{\partial N_k}{\partial t} = \sum_{k',k} w_k^{k'} N_{k'} N_k = 0,$$

whence $w_k^{k'} = -w_{k'}^{k}$, and in particular $w_k^k = 0$.

Further, inasmuch as the particles can only become heated during the scattering process, and consequently acquire energy from the waves, we obtain $w_k^{k'} > 0$ when $k' > k$. Finally, the probability of the transition into a state with zero wave number is $w_0^k = 0$.

We can now reconstruct approximately the form of the entire nonlinear increment γ_k as a function of the wave number (Fig. 1). The increment γ_k vanishes at $k = 0$, and $k = k_0$ at the "center of gravity" of the packet. Obviously, when $0 < k < k_0$ the function γ_k has at least one maximum. Let the principal maximum be located at the point k_1, and assume that in the entire interval $0 < k < k_0$ there is a certain initial background of oscillations n_k. If the initial width of the spectrum δ_0 is small compared with $k_0 - k_1$, then at the point k_1 the oscillations that will build up predominantly will be those due to the nonlinear redistribution of the initial spectrum N_k. Until an appreciable fraction of the noise of the initial spectrum is transferred to the region $k \sim k_1$, the number of waves at the point k_1 will increase exponentially:

$$N(k_1) = n(k_1)\exp[\gamma(k_1)t]. \tag{2}$$

If, as is usually the case in an equilibrium plasma, n_k is a power-law function of k, then the n(k) dependence turns out to be negligible compared with the γ(k) dependence, which enters in the argument of the exponential in (2). Thus, within the time

$$\tau \sim \gamma^{-1}(k_1)\Lambda, \qquad \Lambda = \ln[N(k_0)/n(k_1)] \tag{3}$$

the number of waves in the region $k \sim k_1$ increases in order of magnitude to the number of waves in the initial spectrum. The position of the maximum γ(k) remains unchanged so long as $N(k_1) \ll N(k_0)$. The width of the new line at $k = k_1$ can be obtained from the relation

$$\frac{N(k_1 \pm \delta k)}{N(k_1)} \sim \exp\left[-(\delta k)^2 \frac{\partial^2\gamma}{\partial k^2}\tau\right]$$

and turns out to equal

$$\delta_1 \sim \left[\gamma(k_1) \Big/ \left(\frac{\partial^2\gamma}{\partial k^2}\right)_{k=k_1}\Lambda\right]^{1/2}. \tag{4}$$

The new spectrum also is a narrow line if $\Lambda \gg 1$.

As soon as the number of waves in the new line

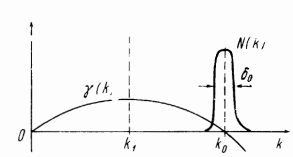

FIG. 1

reaches an order of $N(k_0)$, the position of the zero and of the maximum of $\gamma(k)$ begins to shift towards smaller k. Here, however, the new line is located all the time in the region $\gamma > 0$, and the time required for all the waves of the initial spectrum to become transferred to it is of the order of $\gamma^{-1}(k_1)$. Simultaneously, in order for the number of waves at some other point of phase space to increase to a value of the order of $N(k_0)$, the required time is $\tau \sim \Lambda\gamma^{-1}(k_1)$. Thus, practically the entire initial spectrum becomes transferred to a new narrow line $\langle k\rangle \approx k_1$, and the entire process is repeated. It should be noted that at each instant of time there can exist no more than two lines with an energy density greatly exceeding the energy density of the background, this also being connected with the large value of the time τ.

Let us find the time variation of the transformation of the spectra, assuming that the transfer is only between two narrow lines. The total increment, accurate to terms $\sim\delta_0/k_0$ is written in the form

$$\gamma(k) = w_k^{k_1}N(k_1) + w_k^{k_0}(N_0 - N(k_1)). \tag{5}$$

(We have taken it into account that the total number of waves N_0 remains constant during the transfer process.) Substituting (5) in (1), we obtain the following equation describing the evolution of the spectrum:

$$\frac{\partial}{\partial t}\ln\frac{N(k_1)}{N_0} = \gamma_0\left(1 - \frac{N(k_1)}{N_0}\right), \tag{6}$$

$$\gamma_0 \equiv N_0 w_{k_1}^{k_0} = \gamma(k_1)\,|_{t=0}.$$

Solving (6) under the condition $N(k_1)|_{t=0} = n(k_1)$, we arrive at the following result:

$$N(k_1, t) \approx N_0 \Big/ \left[1 + \frac{N_0}{n}\exp(-\gamma_0 t)\right], \tag{7}$$

$$N(k_0, t) = N_0 - N(k_1, t).$$

The total transfer time τ_0 is determined from the condition $N(k_0, \tau_0) = n$, from which we get

$$\tau_0 \approx \frac{2}{\gamma_0}\ln\frac{N_0}{n} = \frac{2\Lambda}{\gamma_0}. \tag{8}$$

The "two-level" transfer process considered above may experience competition on the part of the process of "diffusion" transfer of the wave from the center of the line to the "left wing" (Fig. 1). In a time on the order of $\gamma^{-1}(k_0 - \delta_0)$, the spectrum shifts by an amount equal to the line width δ_0 into the region of smaller wave numbers. Accordingly, the shift of the spectrum within the time τ_0 can be estimated at $\delta_0\tau_0\gamma(k_0 - \delta_0)$. In order for two-level transfer to predominate, we must have

$$\delta_0\tau_0\gamma(k_0 - \delta_0) \ll k_0 - k_1.$$

When $k_0 - k_1 \sim k_0$ we get from this a limitation on the width of the packet:

$$\frac{\delta_0}{k_0}\frac{\gamma(k_0 - \delta_0)}{\gamma_0} \ll \frac{1}{\Lambda}. \tag{9}$$

2. TWO-LEVEL TRANSFER OF A NARROW PACKET OF PLASMA WAVES

Let us illustrate the effect considered above using as an example the evolution of a narrow spectrum of Langmuir waves ($N(\mathbf{k}) \approx N_0\delta(k - k_0)$) as a result of nonlinear scattering of waves by ions and electrons of a plasma, which at $T_e = T_i$ are described by the following kinetic equations[2]:

$$\left(\frac{\partial N_{\mathbf{k}}}{\partial t}\right)_{Li} = \frac{\sqrt{2\pi}}{16mnD_e^2}N_{\mathbf{k}}\int N_{\mathbf{k}'}\cos^2\widehat{\mathbf{k}\mathbf{k}'}\,\frac{\omega_{k'} - \omega_k}{|\mathbf{k}' - \mathbf{k}|v_{Ti}} \times\exp\left[-\left(\frac{\omega_{k'} - \omega_k}{|\mathbf{k}' - \mathbf{k}|v_{Ti}}\right)^2\right]d\mathbf{k}', \tag{10}$$

$$\left(\frac{\partial N_{\mathbf{k}}}{\partial t}\right)_{Le} = \frac{3\sqrt{\pi}D_e}{2mn}N_{\mathbf{k}}\int N_{\mathbf{k}'}\cos^2\widehat{\mathbf{k}\mathbf{k}'}\,\frac{[\mathbf{k}\mathbf{k}']^2}{|\mathbf{k}' - \mathbf{k}|^3}(k'^2 - k^2)\,d\mathbf{k}'; \tag{11}$$

$$v_{T\alpha}^2 = 2T_\alpha/m_\alpha, \qquad D_\alpha = v_{T\alpha}/\omega_{0\alpha}.$$

It is seen from (10) and (11) that the maximum frequency of scattering of the oscillations by the ions is $\omega_{0e}N_0\omega/nT_e$, and that for scattering by electrons is $\omega_{0e}(N_0\omega/nT_e)(kD_e)^2$. Since two-level transfer has the maximum increment, it will be realized for plasma waves as the result of scattering by ions. Solving the equation $\partial\gamma_i/\partial\mathbf{k} = 0$, we obtain the position of the new line $\mathbf{k}_1$ in wave-vector space:

$$k_0D_e < \frac{2}{3}\sqrt{\frac{m}{M}}, \quad \widehat{\mathbf{k}_1\mathbf{k}_0} = 0, \quad \frac{k_1}{k_0} = \frac{2}{3}\sqrt{\frac{m}{M}}\frac{1}{k_0D_e} - 1;$$

$$k_0D_e > \frac{2}{3}\sqrt{\frac{m}{M}}, \quad \widehat{\mathbf{k}_1\mathbf{k}_0} = \pi, \quad \frac{k_1}{k_0} = 1 - \frac{2}{3}\sqrt{\frac{m}{M}}\frac{1}{k_0D_e}, \tag{12}$$

i.e., for sufficiently short Langmuir waves, the two-level transfer leads to a linear reflection effect. The effective range of plasma waves

$$L \approx \frac{1}{\gamma}\frac{\partial\omega}{\partial k}\Lambda \sim kD_e\frac{nT_e}{N_0\omega}D_e\Lambda \tag{13}$$

decreases upon each succeeding reflection. This leads to "entanglement" of the packet of waves, which continues so long as the condition $kD_e > (2/3)\,m/M$ is satisfied.

3. GATHERING OF BROAD SPECTRA INTO NARROW ONES

Let us return to the one-dimensional model and let us consider a spectrum of final width, bounded by certain values of wave numbers k_1 and k_2 (Fig. 2). Outside the region $k_1 < k < k_2$, the noise density vanishes more rapidly than exponentially. Then, as can be readily seen, the nonlinear increment describing the scattering of the waves by the particles vanishes at a certain point k_0 and is positive when $k < k_0$. Thus, as a result of scattering of waves by particles, the "center of gravity" of the packet will shift towards the point k_1 (Fig. 2), but owing to the rapid decrease of the number of waves on the boundary, the point k_1 itself will shift towards longer wavelengths much more slowly. As a result, the initially broad spectrum contracts into a narrow line near $k = k_1$. The time of transfer is of the order of

$$\tau_{eff} \sim \frac{1}{\gamma_0}\ln\frac{k_2 - k_1}{\delta}, \tag{14}$$

where δ is the final line width

$$\gamma_0 \equiv \gamma(k_1)\,|_{t=0} = \int_{k_1}^{k_2} w_{k_1}^{k'}N(k')\,|_{t=0}\,dk'.$$

As the spectrum contracts, the maximum of the increment shifts gradually to the left, and is located outside the packet after a certain time. Then, at $\tau_{eff} \gtrsim \tau_0$, where τ_0 is given by (8), a new line can form, i.e., the already considered time of two-level transfer begins.

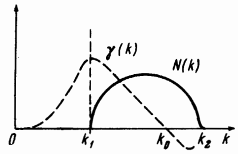

FIG. 2

All the effects of nonlinear transfer considered above, take place, strictly speaking, only in the case when we can neglect the processes of higher orders, which broaden the spectrum, for example four-plasmon interaction.

Let us consider, by way of an example, a one-dimensional packet of Langmuir waves with $kD_e < \sqrt{m/M}$. The process that contracts the spectrum is the scattering of waves by ions, and its frequency can be determined from (10):

$$\gamma_{pi} \sim \omega_{0e}\sqrt{\frac{M}{m}}kD_e\frac{N\omega}{nT_e}.$$

The competing process, which broadens the packet, is plasmon-plasmon scattering, the maximum frequency of which in the indicated spectral region does not exceed[3]

$$\gamma_{pp} \leqslant \omega_{0e}(N\omega / nT_e)^2.$$

We can see that at sufficiently high wave energy density

$$N\omega / nT_e > kD_e\sqrt{M/m}$$

the four-plasmon interaction is capable of effectively hindering the gathering of the spectrum.

[1] A. A. Vedenov, in: Voprosy teorii plazmy (Problems of Plasma Theory), No. 3, Gosatomizdat, 1963, p. 203.

[2] V. N. Tsytovich, Nelineinye effecty v plasme (Nonlinear Effects in a Plasma), Nauka, 1967.

[3] A. A. Vedenov, A. V. Gordeev, and L. I. Rudakov, Plasma physics 9, 719 (1967).

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