STABILITY OF A MONOCHROMATIC WAVE IN A PLASMA

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It is shown that the equilibrium state corresponding to propagation of a monochromatic wave in a plasma is unstable. Two instability mechanisms are investigated. These are the instability due to stimulated scattering of waves on thermal particles in the plasma and the instability due to particles trapped in the potential well produced by the wave. The trapped-particle instability leads to the excitation of satellite lines in the spectrum, these satellite lines being shifted in frequency with respect to the main wave by an amount $k_0\sqrt{e\varphi_0/m}$ (k_0 is the wave number and φ_0 is the peak potential in the wave).

1. It is well-known that the Landau damping of a plasma wave, which is predicted by the linear theory and is associated with the absorption of wave energy by resonant particles in the plasma, operates only for times shorter than the period of particle oscillations in the potential well produced by the wave $\tau_0 \approx (k_0\sqrt{e\varphi_0/m})^{-1}$ (k_0 is the wave number and φ_0 is the peak potential). The nonlinear solutions obtained by Mazitov[1] and O'Neil[2] describe the transition, in a time τ_0, to a wave of constant amplitude $-\varphi_0(x - v_\varphi t)$. The vanishing of the damping decrement in the nonlinear solution is due to the smearing of resonance particles in phase space, a process which occurs because of the dependence of the oscillation period of the particle in the potential well on energy.[3,4] Because of the smearing effect the distribution function for the resonant particles is found to be a rapidly oscillating function of velocity. The wavelength of the oscillations diminishes without limit in the course of time so that at long times one is dealing with a distribution function which is essentially averaged over the rapid oscillations. In the reference system fixed in the wave this distribution function, as in any stationary state, depends only on the particle energy $f = f_0(\epsilon)$, $\epsilon = (1/2)mv^2 - e\varphi_0(x)$.

In the present work we investigate the equilibrium state found in[1,2] which corresponds to the propagation of a monochromatic wave of fixed amplitude in the plasma; we are interested in stability with respect to the excitation of plasma waves with close values of phase velocity. In Sec. 2 we find the perturbation of the distribution function for the resonant particles due to a test wave and obtain the equation that describes the time variation of the amplitude of this wave. In Sec. 3 we investigate this equation and determine the conditions for the appearance of the instability of the monochromatic wave. If there is a large discrepancy between the phase velocities of the primary wave and the test wave $|v_\varphi - v'_\varphi| \gg \sqrt{e\varphi_0/m}$ only that instability is possible which is associated with stimulated scattering of the wave on thermal particles of the plasma. This instability has been investigated earlier for the case of waves with random phases.[5-7] This instability leads to the excitation of oscillations in the longwave region of the spectrum. For small values of the discrepancy $|v_\varphi - v'_\varphi| \gtrsim \sqrt{e\varphi_0/m}$ an instability arises which is associated with particles that are trapped in the potential well produced by the primary wave. As the result of the instability the oscillation spectrum exhibits satellite lines that are shifted in frequency by an amount proportional to $\sqrt{\varphi_0}$, in agreement with the results of the experimental investigations reported by Wharton et al.[8]

2. The analysis is carried out in the reference system of the primary wave, in which the equilibrium state is characterized by a stationary potential distribution $\varphi_0(x)$ and a stationary distribution function $f_0(\epsilon)$.[1)] As in[1,2] we will assume that the following restriction holds on the amplitude φ_0:

$$e\varphi_0 \ll mv_{ph}^2. \tag{1}$$

The perturbation of the equilibrium state will be sought in the form of a plane wave of charge density, in which the electric field is given by[2)]

$$E(t,x) = E_k(t)\exp[i(kx - \omega_k t)], \tag{2}$$

where $\omega_k = \omega_p(1 + {}^3/_2k^2\lambda_D^2) - kv_{ph}$ is the frequency of the plasma oscillations in the reference system fixed in the wave, $\omega_p = (4\pi e^2n_0/m)^{1/2}$ is the plasma frequency and $\lambda_D = (T/4\pi e^2n_0)^{1/2}$ is the Debye radius; we shall limit ourselves to the longwave case $k\lambda_D \ll 1$.

We shall first find the perturbation in the electron distribution function in the field given by (2). The distribution function for the initial time is written in the form

$$f|_{t=0} = f_0(\varepsilon) + f_1(v)e^{ikx}, \tag{3}$$

where the second term $\sim e^{ikx}$ is the initial perturbation of the equilibrium state that leads to the plasma oscillations.

Since the distribution function is an integral of the motion in the absence of collisions, at any arbitrary time, limiting ourselves to terms that are linear in the amplitude of the perturbation, we can write

[1)]In the present problem the averaging of the distribution function over the fast oscillations occurs by virtue of the fact that resonances between the test wave and particles have a finite width according to the velocity $\sim\gamma/k$ (γ is the damping rate).

[2)]If the condition in (1) is satisfied, the field harmonics due to the presence of the potential φ_0, $\sim\exp[i(k + nk_0)x]$, $n \neq 0$, are weak.

$$f(t, x, v) = f_0(\varepsilon) - (\varepsilon - \varepsilon_0) df_0 / d\varepsilon + f_1(v_0) e^{ikx_0}. \quad (4)$$

In this expression $\epsilon_0(t, x, v)$, $v_0(t, x, v)$, and $x_0(t, x, v)$ are initial coordinates of the particles in phase space along trajectories which, at time t, go through the point x and v of this space. In the last term, which is proportional to the initial perturbation of the distribution function, the unperturbed particle trajectory can be used to find the relation between x_0 and v_0 and t, x and v.

Thus, the problem of determining the distribution function reduces to the search for the variation of the particle energy in the field given by (2). It is then necessary to integrate the following equation:

$$d\varepsilon / dt = -eE_k v(t) \exp[i(kx(t) - \omega_k t)], \quad (5)$$

where v(t) and x(t) on the right side of Eq. (5) are the velocity and coordinates of the particle along the unperturbed trajectory which are due to the effect of the potential $\varphi_0(x)$. If the amplitude φ_0 is not too large, so that the condition in (1) is satisfied, it can be assumed that $\varphi_0(x)$ is a harmonic function of x: $\varphi_0(x) = \varphi_0 \cos k_0 x$. Then the unperturbed trajectories of transiting particles with energy $\epsilon > e\varphi_0$ are described by the equation (cf.[2])

$$F\left(\xi, \frac{1}{\varkappa}\right) = F\left(\xi_0, \frac{1}{\varkappa}\right) + \frac{\varkappa}{\tau_0} t \equiv \Theta(t, \xi_0). \quad (6)$$

In this equation

$$\xi = \frac{k_0 x}{2}, \quad \xi_0 = \frac{k_0 x_0}{2}, \quad \tau_0 = \frac{1}{k_0 \sqrt{e\varphi_0 / m}}, \quad \varkappa^2 = \frac{\varepsilon + e\varphi_0}{2e\varphi_0}$$

where $F(\xi, 1/\kappa)$ is the elliptic function of the first kind.

The relation in (6) holds for the case $v_0 > 0$, that is, for particles moving in the positive x direction. When $v_0 < 0$ it is necessary to change the sign of the term proportional to t. Using the well-known expansion for the elliptic function amΘ in terms of the Fourier series (cf., for example[9],) we can write the function x(t) that appears in the solution of Eq. (6) in the following form:

$$x = x_0 + \frac{\pi\varkappa}{K(1/\varkappa)} \frac{t}{k_0 \tau_0} + \frac{4}{k_0} \sum_{n=1}^{\infty} \frac{g^{-n}}{n(1 + g^{-2n})} \times \left[\sin \frac{\pi n \Theta}{K(1/\varkappa)} - \sin \frac{\pi n \Theta_0}{K(1/\varkappa)}\right] = x_0 + ut + \tilde{x}(t). \quad (7)$$

Here $g = \exp[\pi K'(1/\kappa)/K(1/\kappa)]$, $K(1/\kappa)$ is the complete elliptic function of the first kind, $K'(1/\kappa) = K(\sqrt{1 - 1/\kappa^2})$, and $\Theta_c = \Theta(t = 0)$.

The term $\tilde{x}(t)$ in Eq. (7), which describes the deviation from the equilibrium motion of the transiting particles, is a periodic function of time with period

$$\tau(\varepsilon) = \int_0^a \frac{dx}{\sqrt{2m^{-1}(\varepsilon + e\varphi_0(x))}} = 2\frac{K(1/\varkappa)}{\varkappa}\tau_0, \quad a = \frac{2\pi}{k_0} \quad (8)$$

In accordance with Eq. (7), the velocity of the uniform motion u is

$$u = \frac{a}{\tau} = \frac{\pi\varkappa}{K(1/\varkappa)} \frac{1}{k_0 \tau_0}. \quad (8')$$

In the present work we limit ourselves to the case in which the phase velocities of the primary and test waves are rather close $|\omega_k|/k \ll v_\varphi$. In this case the detuning between the wave numbers of these waves is also small:

$$\frac{\delta k}{k} = \frac{|k - k_0|}{k} \approx \frac{|\omega_k|}{k v_{ph}} \ll 1.$$

Then, neglecting small terms proportional to $\delta k/k$, we can write the following relation:

$$\exp[ikx(t)] = \exp[i(k - k_0)x_0 + i(k - k_0)ut + ik_0 x(t)] = \exp[i(k - k_0)x_0] \sum_{n=-\infty} \psi_n(x_0) \exp\left[i\left((k - k_0)u + \frac{2\pi n}{\tau}\right)t\right] \quad (9)$$

The coefficients for the Fourier expansion of the function $\exp[ik_0 x(t)]$ are computed in the Appendix:

$$\psi_n = -\frac{4\pi^2\varkappa^2}{K^2(1/\varkappa)} \frac{ng^{3n}}{1 - g^{4n}} \exp\left[\frac{\pi n i}{K(1/\varkappa)} F\left(\xi_0, \frac{1}{\varkappa}\right)\right] \quad (n \neq 0),$$
$$\psi_0 = -2\varkappa^2\left[\frac{E(1/\varkappa)}{K(1/\varkappa)} - 1 + \frac{1}{2\varkappa^2}\right]. \quad (9')$$

To this same accuracy we have the expansion

$$v(t)\exp[ikx(t)] = \exp[i(k - k_0)x_0] \sum_{n=-\infty}^{\infty} \Phi_n(x_0) \times \exp\left[i\left((k - k_0)u + \frac{2\pi n}{\tau}\right)t\right], \quad (10)$$

where

$$\Phi_n = \psi_n \frac{(k - k_0)u + 2\pi n/\tau}{k}. \quad (10')$$

As expected, the coefficients $\psi_n(x_0)$ and $\Phi_n(x_0)$ are periodic functions of x_0 with period a. Equations (9) and (10) for these coefficients have been obtained for the case $v_0 > 0$. When $v_0 < 0$ the coefficients $\Phi_n(x_0)$ and $\psi_n(x_0)$ are determined from the relations

$$\Phi_n(x_0, v_0) = -\Phi_{-n}^*(-x_0, -v_0); \quad \psi_n(x_0, v_0) = \psi_{-n}^*(-x_0, -v_0). \quad (11)$$

Substituting Eq. (10) in Eq. (6), integrating over time and neglecting the term that depends on the initial conditions, which is unimportant when $t \gg 1/kv_T$ (v_T is the thermal velocity), we have

$$\delta\varepsilon = \varepsilon - \varepsilon_0 = -eE_k \exp[i(k - k_0)x_0] \times \sum_{n=-\infty}^{\infty} \Phi_n(x_0) \frac{\exp[i((k - k_0)u - \omega_k + 2\pi n/\tau)t]}{i[(k - k_0)u - \omega_k + 2\pi n/\tau]}. \quad (12)$$

In accordance with[2] the unperturbed trajectories of the trapped particles, characterized by energy $\epsilon \ll \epsilon\varphi_0$, can be obtained from the equation

$$F(\zeta, \varkappa) = F(\zeta_0, \varkappa) + t / \tau_0 = \Theta^{tr}(t, \zeta_0). \quad (13)$$

In this equation $\zeta(t)$ is associated with $\xi = (1/2)k_0 x$ through the relation

$$\sin\zeta = \varkappa^{-1}\sin\xi, \quad \zeta_0 = \zeta(t = 0).$$

From Eq. (13) we have

$$x = x_0 + \frac{8}{k_0} \sum_{n=1}^{\infty} \frac{g_{tr}^{-n-1/2}}{1 + g_{tr}^{-2n-1}} \left[\sin(2n - 1)\frac{\pi\Theta^{tr}}{2K(\varkappa)} - \sin(2n - 1)\frac{\pi\Theta_0^{tr}}{2K(\varkappa)}\right] = x_0 + \tilde{x}^{tr}(t).$$
$$g_{tr} = \exp(\pi K'(\varkappa) / K(\varkappa)). \quad (14)$$

Here, $\tilde{x}^{tr}(t)$ is a periodic function of time with period

$$\tau^{tr}(\varepsilon) = 4K(\varkappa)\tau_0. \quad (15)$$

Neglecting terms of order $\delta k/k$, for the trapped particles we find the following expansions in place of

Eqs. (9) and (10):

$$\exp[ikx(t)] = \exp[i(k-k_0)x_0] \sum_{n=-\infty}^{\infty} \psi_n^{tr}(x_0)\exp\left[2\pi i n \frac{t}{\tau^{tr}}\right], \tag{16}$$

$$v(t)\exp[ikx(t)] = \exp[i(k-k_0)x_0] \sum_{n=-\infty}^{\infty} \Phi_n^{tr}(x_0)\exp\left[2\pi i n \frac{t}{\tau^{tr}}\right],$$

where

$$\psi_n^{tr} = -\frac{\pi^2}{K^2(\varkappa)} \frac{(-1)^n n g_{tr}^{n/2}}{1-(-1)^n g_{tr}^n} \exp\left[\frac{\pi i n}{2K(\varkappa)} F(\zeta_0, \varkappa)\right], \quad n \neq 0,$$
$$\psi_0^{tr} = \frac{2E(\varkappa) - K(\varkappa)}{K(\varkappa)}, \quad \Phi_n^{tr} = \frac{2\pi n}{k\tau^{tr}} \psi_n^{tr}. \tag{17}$$

Equation (17) has been obtained for the case $v_0 > 0$. When $v_0 < 0$, the coefficients Φ_n^{tr} and ψ_n^{tr} are expressed in terms of Φ_n^{tr} and ψ_n^{tr} for $v_0 > 0$ through the use of (11).

The change in the energy of the trapped particles in the oscillatory field (2) can be obtained by integration of Eq. (5):

$$\delta\varepsilon = \varepsilon - \varepsilon_0 = -eE_k \exp[i(k-k_0)x_0] \sum_{n=-\infty}^{\infty} \Phi_n^{tr}(x_0) \frac{\exp[i(2\pi n/\tau^{tr} - \omega_k)t]}{i(2\pi n/\tau^{tr} - \omega_k)}. \tag{18}$$

We now consider the derivation of the equation for the amplitude of the test wave. We start from the conservation of energy which, in the case being considered (plasma waves), is written in the form

$$\frac{1}{4\pi}\frac{d|E_k|^2}{dt} - \frac{e}{2a}\left[\int_{-a/2}^{a/2} dx\, E_k^* \exp[-i(kx - \omega_k t)] \cdot \cdot \int dv (v + v_{ph})\delta f + \text{c.c.}\right] = 0. \tag{19}$$

In this equation the first term corresponds to the variation of wave energy (the potential energy $|E_k|^2/8\pi$ and the kinetic energy associated with the oscillatory particle motion in the wave $(1/2) n_0 m |v_k|^2 = |E_k|^2/8\pi$ while the second term describes the change in the energy of particles that interact with the wave; δf is the perturbation of the distribution function for these particles. Averaging over the period a in Eq. (19) we can determine the current

$$j = -e\int (v + v_{ph})\delta f\, dv$$

and also the component e^{ikx}, which corresponds to the change in time of the amplitude E_k. For convenience in the further calculations in Eq. (19) we shall integrate with respect to the initial coordinate of the particles in phase space, making use of the conservation of phase volume:

$$dx\, dv = dx_0\, dv_0 = d\varepsilon \frac{dx_0}{mv_0(x_0, \varepsilon)}$$

Then, substituting δf from Eq. (4), we can write Eq. (19) in the form

$$\frac{1}{4\pi}\frac{d|E_k|^2}{dt} = -\frac{eE_k^*}{2a}\left[\int_{e\varphi_0}^{\infty} d\varepsilon \int_{-a/2}^{a/2} \frac{dx_0}{mv_0(x_0,\varepsilon)} \exp[-i(kx - \omega_k t)] \right.$$
$$\times (v + v_{ph})\left(\frac{df_0}{d\varepsilon}\delta\varepsilon - f_1(v_0)e^{ikx_0}\right) + \int_{-e\varphi_0}^{e\varphi_0} d\varepsilon \int_{-\bar{x}}^{\bar{x}} \frac{dx_0}{mv_0(x_0, \varepsilon)} \tag{20}$$
$$\left.\times \exp[-i(kx - \omega_k t)](v + v_{ph})\left(\frac{df_0}{d\varepsilon}\delta\varepsilon - f_1(v_0)e^{ikx_0}\right)\right] + \text{c.c.},$$

where, in the integral with respect to ϵ, we separate the contributions due to the trapped particles and the transiting particles; $\bar{x}(\epsilon)$ and $-\bar{x}(\epsilon)$ are the turning points for the trapped particles:

In the integral with respect to ϵ on the right side of Eq. (20) at long times the most important term is $\sim(df_0/d\epsilon)\delta\epsilon$. It is easy to show that the term that depends on the initial correction to the distribution function f_1 makes a contribution in (20) which decays in a time $t \sim 1/kv_T$. Now, in Eq. (20) we substitute $\exp[-ikx(t)]$, $v(t)\exp[-ikx(t)]$ and $\delta\epsilon$ in the form of the Fourier expansion Eqs. (9), (10), (12), (16), and (18) and then carry out the integration with respect to x_0 (cf. Appendix). In this way, after a number of elementary transformations we obtain the following equation for $|E_k|^2$, which holds at times $t \gg 1/kv_T$:

$$\frac{d|E_k|^2}{dt} = \frac{64\pi^5 e^2}{mk}(\omega_k + kv_{ph})\omega_k\tau_0|E_k|^2\left\{\int_{e\varphi_0}^{\infty} d\varepsilon \frac{\varkappa^3}{K^3(1/\varkappa)}\right.$$
$$\times \sum_{n\neq 0}{}' \frac{n^2 g^{2n}}{(g^{2n} - g^{-2n})^2}\delta[(k-k_0)u + nk_0 u - \omega_k]\frac{df_0}{d\varepsilon} \tag{21}$$
$$\left. + \frac{1}{8}\int_{-e\varphi_0}^{e\varphi_0} \frac{d\varepsilon}{K^3(\varkappa)} \sum_{n\neq 0} \frac{n^2}{(g_{tr}^{-n/2} - (-1)^n g_{tr}^{n/2})^2}\delta\left[\frac{2\pi n}{\tau^{tr}} - \omega_k\right]\frac{df_0^{tr}}{d\varepsilon}\right\}.$$

In this equation $f_0(\epsilon)$ is the equilibrium distribution function for the transiting particles while $f_0^{tr}(\epsilon)$ is the distribution function for the trapped particles; the integral with respect to the transiting particles includes particles that move in the positive and negative sense along the x axis ($u > 0$, $u < 0$).

Equation (21) determines the change in the amplitude of the test wave due to the resonance particles, the velocities of these particles being approximately equal to the phase velocity of the wave ($u \approx \omega_k/nk_0 \sim \sqrt{e\varphi_0/m}$, n = 1, 2, ... In the summation over the transiting particles in Eq. (21) we omit the term characterized by n = 0, which corresponds to the resonance $(k - k_0)u = \omega_k$. In the laboratory reference system this resonance condition can be written

$$(k - k_0)(u + v_{ph}) = \omega_k^{lab} - \omega_{k_0}^{lab}$$

and coincides with the condition for stimulated scattering of oscillations on thermal particles of the plasma, the velocities of these particles being equal to $u + v_\varphi \simeq 3k\lambda_D^2\omega_p \ll v_T$ in the laboratory system. For a monochromatic wave the stimulated scattering occurs in the same way as in the case of wave which has been considered earlier, and leads to an instability of the original wave with respect to longwave perturbations characterized by $k < k_0$. The coefficient for stimulated scattering γ can be expressed in terms of the scattering cross section for waves with random phases $\gamma^{ll}(k, k_0)$ which had been obtained earlier:[3)]

$$\gamma_s = \frac{1}{4}k_0^2\varphi_0^2\gamma^{ll}(k, k_0). \tag{22}$$

The quantity $\gamma^{ll}(k, k_0)$ for various cases (scattering by thermal electrons or ions) is given, for example, in[11].

3. We now consider Eq. (21). In the argument of the

3)We note that in the analysis of stimulated scattering on thermal particles in addition to considering the radiation of particles associated with oscillations in the electric field of the wave E_k, which is computed in the present work, it is also necessary to take account of the transition radiation due to density inhomogeneities produced by the wave (cf. [10]).

δ-function we neglect terms of order $(k - k_0)/k$ and carry out the integration over ϵ, obtaining the following expression for the growth rate γ_R associated with the interaction of the wave with resonance particles:

$$\gamma_R = \frac{32\pi^5 e^2}{mk^2}(\omega_k + kv_{ph})\,\omega_k\tau_0$$
$$\cdot\left\{\sum_{n\neq 0}{}' \left\langle \frac{\varkappa^8}{K^3(1/\varkappa)}\frac{|n|g^{2n}}{(g^{2n}-g^{-2n})^2}\left|\frac{du}{d\varepsilon}\right|^{-1}\frac{df_0}{d\varepsilon}\right\rangle_{nk_0u(\varepsilon)=\omega_k}\right.$$
$$\times\frac{1}{8}\sum_{n\neq 0}{}'\left\langle\frac{1}{K^3(\varkappa)}\frac{|n|}{(g_{tr}^{-n\,2}-(-1)^n g_{tr}^{n\,2})^2}\frac{k}{|d\Omega^{tr}/d\varepsilon|}\frac{df_0^{tr}}{d\varepsilon}\right\rangle\Bigg|_{n\Omega^{tr}(\varepsilon)=\omega_k}$$
$$(\Omega^{tr} = 2\pi/\tau^{tr}). \quad (23)$$

We first consider large detuning between the phase velocities of the main wave and the test wave, in which case $|\omega_k|/k \gg \sqrt{e\varphi_0/m}$. In this case the resonance values of the energy of the transiting particles lies somewhat higher than the limit of the potential well $\epsilon_R \sim m\omega_k^2/k^2 \gg e\varphi_0$. Thus

$$\varkappa^2 = \frac{\varepsilon}{2e\varphi_0} \gg 1, \quad K(1/\varkappa) \approx \frac{\pi}{2}, \quad K'(1/\varkappa) \approx \ln 4\varkappa, \quad g \approx 16\varkappa^2,$$

and in the summation corresponding to the transiting particles in Eq. (23) we are left only with the term $n = 1$. Obviously the contribution of the trapped particles disappears in this case. As a result, we find from Eq. (23) that when $|\omega_k|/k \gg \sqrt{e\varphi_0/m}$, because of the interaction with the resonance particles the test wave damps with the Landau damping factor

$$\gamma_L = \frac{2\pi^2 e^2}{mk^2}(\omega_k + kv_{ph})\frac{df_0}{du}, \quad \left(u(\varepsilon) = \sqrt{\frac{2\varepsilon}{m}} = \frac{\omega_k}{k}\right). \quad (24)$$

Under these conditions, when $df_0/du < 0$ only the instability which is associated with stimulated scattering of oscillations is possible. The instability arises when φ_0 is large enough, in which case the increment for stimulated scattering exceeds the damping factor due to the resonance particles. Assuming that the scattering is due only to the thermal electrons, we find that the loss of stability is given by the condition

$$\frac{e\varphi_0}{mv_{ph}^2} > \frac{2}{3}\left(\frac{8}{9\pi}\right)^{1/4}\left(\frac{\gamma_L}{\omega_p k^3\lambda_D^3}\right)^{1/2} \approx \frac{2}{(3\sqrt{e})^{3/2}}\frac{\exp(-1/4k^2\lambda_D^2)}{k^3\lambda_D^3} \quad (25)$$

In obtaining the last relation we have assumed that the distribution function of the resonance particles is a Maxwellian. The instability condition in (25) is first satisfied for longwave perturbations.

We now investigate the possibility of an instability due to resonance particles when $|\omega_k|/k \lesssim \sqrt{e\varphi_0/m}$. It follows from Eq. (23) that when $\omega_k > 0$ the excitation of test waves is associated with particles whose distribution function increases with energy in the reference system fixed in the primary wave $df_0/d\epsilon > 0$. As a result of the interaction with the test wave these particles diffuse toward lower energies and a plateau appears on the distribution function; the particle energy is thus converted into wave energy. When $\omega_k < 0$ the energy of the test waves in the reference system of the primary wave

$$W = \frac{\omega_k}{8\pi}\frac{\partial\varepsilon(\omega_k, k)}{\partial\omega_k}|E_k|^2$$

is negative $[\epsilon(\omega_k, k) = 1 - \omega_p^2/(\omega_k^2 + kv_{ph})^2$ is the dielectric constant in this reference system$]$. In this case the excitation of test waves is associated with particles for which $df_0/d\epsilon < 0$; these particles acquire the energy of the wave in the diffusion process.

The distribution function of the resonance particles $f_0(\epsilon)$, which is established under the effect of the wave φ_0, can be expressed in terms of the original distribution function $f_0^0(v)$, which exists in the plasma at the time φ_0 is switched on; for this purpose we use the relation (cf.[2])

$$f_0(\varepsilon) = \left(\int\frac{dx}{|v(x,\varepsilon)|}\right)^{-1}\int\frac{dx}{|v(x,\varepsilon)|}f_0^0[v(x,\varepsilon)]. \quad (26)$$

Here, for the transiting particles $(\epsilon > e\varphi_0)$ the integration over x is carried out between the points $-a/2$ and $a/2$; for the trapped particles $(\epsilon < e\varphi_0)$ the integration is carried out over a closed trajectory between the turning points: $-\bar{x}(\epsilon)$ and $\bar{x}(\epsilon)$. Equation (26) derives from the fact that under the action of the wave there is a uniform smearing of particles in phase along the lines ϵ = const.

In what follows we shall assume that the initial distribution function $f_0^0(v)$ is a Maxwellian in the laboratory reference system. It follows from Eq. (26) that for a Maxwellian or any other stable distribution function $f_0(v)$ when $df_0^0/dv < 0$ the condition $df_0/d\epsilon > 0$ is satisfied for the transiting particles when $u < 0$ while the derivative $df_0/d\epsilon < 0$ when $u > 0$. From the resonance condition $u(\epsilon) = \omega_k/nk_0$ (n is an integer); it is then easy to show that the transiting particles lead to excitation of the test wave by virtue of resonances characterized by $n < 0$, which correspond to the anomalous Doppler effect; the resonances characterized by $n > 0$ correspond to the Cerenkov effect and the normal Doppler effect and are stabilizing.[4)]

When $\sqrt{e\varphi_0/m} \ll v_T^2/v_\varphi$ the distribution function $f_0^0(v)$ under the integral sign in Eq. (26) can be expanded in powers of v. Then, taking the transiting particles and limiting ourselves to the first two terms in the expansion, we have

$$f_0(\varepsilon) = \frac{n_0}{\sqrt{2\pi}\,v_T}\exp\left(-\frac{v_{ph}^2}{2v_T^2}\right)\left(1 - \frac{v_{ph}}{v_T^2}u(\varepsilon)\right). \quad (27)$$

Substituting $f_0(\epsilon)$ in Eq. (23) we obtain the following expression for γ_R associated with the transiting particles:

$$\gamma_R' = \frac{32\pi^6 e^2}{mk^2}(\omega_k + kv_{ph})\frac{\partial f_0^0}{\partial v}(0)\cdot \quad (28)$$
$$\cdot\alpha^4\sum_{n=1}^{\infty}\left(\frac{1}{g^{2n}(\varkappa) - g^{-2n}(\varkappa)}\right)\Bigg|_{\varkappa' K(\varkappa^{-1})=\alpha/n},$$

where we have used the notation

$$\alpha = \frac{1}{\pi}|\omega_k|\tau_0 = \frac{|\omega_k|}{\pi k_0}\frac{1}{\sqrt{e\varphi_0/m}}.$$

Since

$$g = \exp\left[\pi K'\left(\frac{1}{\varkappa}\right)\Big/K\left(\frac{1}{\varkappa}\right)\right] > 1,$$

4)The presence of the wave φ_0 converts particles which are resonance particles for the test wave into oscillators with characteristic frequency $\omega_0 = k_0 u(\epsilon)$. The resonance condition for the transiting particles is of the form $\omega_k - ku = (n-1)\omega_0$; since $k \approx k_0$, when $n \neq 0$ this condition can be written $\omega_k = nk_0u$.

in the summation that corresponds to the transiting particles in Eq. (23) the contribution of the stabilizing resonances is found to be larger in this case so that $\gamma'_R < 0$.[5)]

We now consider the trapped particles. In computing the distribution function for these particles in Eq. (26) we need not retain the linear terms in small v in the expansion of $f_0^0(v)$, but take account of the following terms, which are quadratic in v. In this way we obtain the following expression for $f_0^{tr}(\epsilon)$ from Eq. (26):

$$f_0^{tr}(\varepsilon) = f_0^0(0) + \frac{\partial^2 f_0^0}{\partial v^2}(0)\frac{\Omega^{tr}(\varepsilon)}{2\pi}\int_{-\bar{x}}^{\bar{x}}\sqrt{\frac{2}{m}(\varepsilon + e\varphi_0\cos k_0 x)}\,dx \tag{29}$$
$$= \frac{n_0}{\sqrt{2\pi}\,v_T}\exp\left(-\frac{v_{ph}^2}{2v_T^2}\right)\left[1 + \frac{4}{\pi}\sqrt{\frac{e\varphi_0}{m}}\frac{v_{ph}^2\Omega^{tr}(\varepsilon)}{v_T^4 k_0}(K(\varkappa)(\varkappa^2 - 1) + E(\varkappa))\right].$$

In computing the equilibrium distribution function for the resonance particles $f_0(\epsilon)$ we have neglected binary collisions. In this connection it should be recalled that we are considering a time $t \sim 1/\gamma_L \ll 1/\nu$ (ν is the binary collision frequency). Under these conditions collisions are important only in a narrow energy range close to the separatrix ($\epsilon = e\varphi_0$) in which the gradients of the distribution function $f_0(\epsilon)$ given by Eqs. (27) and (29) grow without limit:

$$\frac{df_0}{d\varepsilon} = \frac{\partial f_0^0}{\partial v}(0)\frac{du}{d\varepsilon} = \frac{\pi}{4m}\frac{\partial f_0^0}{\partial v}(0)\frac{\operatorname{sign} u}{\sqrt{e\varphi_0/m}} \Big/ (\varkappa^2 - 1)\ln^2\frac{4}{\sqrt{\varkappa^2 - 1}}$$

for the transiting particles for $\kappa \to 1$ ($\epsilon \to e\varphi_0$) and

$$\frac{df_0}{d\varepsilon} = \frac{4}{\pi}\frac{\partial^2 f_0^0}{\partial v^2}(0)\sqrt{\frac{e\varphi_0}{m}}\frac{1}{k_0}\frac{d\Omega^{tr}}{d\varepsilon}$$
$$= -\frac{1}{2m}\frac{\partial^2 f_0^0}{\partial v^2}(0)/(1 - \varkappa^2)\ln^2(4/\sqrt{1 - \varkappa^2})$$

for the trapped particles for $\kappa \to 1$.

Collisions lead to a smoothing of the distribution function $f_0(\epsilon)$ in this energy region (cf.[12]) so that in actuality the quantity $df_0/d\epsilon$ remains finite as $\kappa \to 1$.

It should be noted that the distribution function for the trapped particles, which is given by Eq. (29), is similar to that for a beam: at low particle energies it increases with ϵ, reaching a maximum at $\kappa^2 = (\epsilon + e\varphi_0)/2e\varphi_0 = 0.875$; thereafter it falls off as ϵ increases. By virtue of this distribution function it is possible to have excitation, by the trapped particles, of both red

$$\left(\Delta\omega^{lab} = \frac{d\omega^{lab}}{dk}(k - k_0) = -\frac{v_g}{v_{ph}}\omega_k < 0\right),$$

and violet ($\Delta\omega^{lab} > 0$) satellites in the oscillation spectrum.

The growth rate for the instability due to the trapped particles is determined by the second summation in Eq. (23). The resonance values of the particle energies in this summation are related to the parameter $\alpha = \pi^{-1}|\omega_k|\tau_0$ by the expression $K(\kappa_R) = n/2\alpha$ ($n = 1, 2, \ldots$). When α varies in the range $0, 1/\pi$ the resonance energy of the trapped particles at the fundamental resonance ($n = 1$) varies from the separatrix ($\kappa = 1$) to the bottom of the potential well ($\kappa = 0$) while the other resonance energies of the trapped particles remain close to the separatrix. The resonance values of the energy of the transiting particles given by the relation $K(1/\kappa_R)\kappa_R^{-1} = n/\alpha$ ($n = 1, 2, \ldots$) are also close to the separatrix in this range of α (cf. table).

As we have noted above, near the separatrix collisions lead to a smoothing of the distribution function so that the quantity $df_0/d\epsilon$ remains finite as $\kappa \to 1$ while $|du/d\epsilon|$, $|d\Omega^{tr}/d\epsilon| \to \infty$. For this reason, as follows from Eq. 23, there is a reduction in the contribution of the corresponding resonances to the growth rate. Hence, in computing the growth rate due to the trapped particles, limiting ourselves to the range of variation of α from 0 to $1/\pi$, we take account only of the fundamental resonance $n = 1$, using the distribution function in Eq. (29) for $f_0^{tr}(\epsilon)$. We also limit ourselves to the $n = 1$ resonance for the transiting particles (the higher the value of n the closer κ_R is to unity and the smaller the contribution from the corresponding resonance in the growth rate) using as $f_0(\epsilon)$ the distribution function for the transiting particles close to the separatrix as established by the effect of collisions:[12]

$$f_0(\varepsilon) = \frac{n_0}{\sqrt{2\pi}v_T}\exp\left(-\frac{v_{ph}^2}{2v_T^2}\right)\left[1 - \frac{\pi}{2^{1/2}}\frac{\varepsilon - e\varphi_0}{\sqrt{em\varphi_0}}\frac{v_{ph}}{v_T^2}\operatorname{sign} u\right]$$

(here we assume that $\sqrt{e\varphi_0/m} \ll v_T^2/v_{ph}$).

As a result, we have from Eq. (23)

$$\gamma_n = \frac{2\pi^2 e^2}{mk^2}(\omega_k + kv_\Phi)\frac{\partial f_0^0}{\partial v}(0)\left[\Gamma_1(\alpha) - \sqrt{\frac{e\varphi_0}{m}}\frac{v_{ph}}{v_T^2}\operatorname{sign}\omega_k\,\Gamma_2(\alpha)\right], \tag{30}$$

where the term $\sim\Gamma_1(\alpha)$ determines the contribution from the transiting particles

$$\Gamma_1(\alpha) = 16\sqrt{2}\pi^4\alpha^2\left\langle\frac{\varkappa(\varkappa^2 - 1)}{E\left(\frac{1}{\varkappa}\right)(g^2(\varkappa) - g^{-2}(\varkappa))}\right\rangle\Bigg|_{K(\varkappa^{-1})\varkappa^{-1} = 1/\alpha};$$

while the term $\sim\Gamma_2(\alpha)$ determines the contribution from the trapped particles

$$\Gamma_2(\alpha) = 32\pi^3\alpha^3$$
$$\left\langle\frac{2(\varkappa^2 + E(\varkappa)/K(\varkappa)) - 1 - E(\varkappa)/K^2(\varkappa)(1 - \varkappa^2)}{(g_{tr}^{1/2}(\varkappa) + g_{tr}^{-1/2}(\varkappa))|E(\varkappa)/K(\varkappa)(1 - \varkappa^2) - 1|}\right\rangle\Bigg|_{K(\varkappa) = 1/2\alpha}.$$

Curves of the functions $\Gamma_1(\alpha)$ and $\Gamma_2(\alpha)$ are shown in the figure. It is evident that even when $\sqrt{e\varphi/m}\,v_\varphi/v_T^2 \sim 0.1$ an instability of the test waves due to trapped particles arises. The transiting particles are stabilizing. The instability arises when $|\omega_k|/k_0\sqrt{e\varphi_0/m} < 1$ ($\alpha < 1/\pi$). For large values of α the resonance condition for the trapped particles is satisfied when $n \geq 2$. Under these conditions the stabilizing contribution in the growth rate due to the transiting particles when $\sqrt{e\varphi_0/m}\,v_\varphi/v_T^2 \ll 1$ is found to be large and the instability is quenched when $\alpha > 1/\pi$.

When α varies from 0 to 0.202 the function $\Gamma_2(\alpha)$ is negative. In this range of α, corresponding to trapped particle energies for which $df^{tr}/d\epsilon < 0$, we find that the violet satellites are excited ($\Delta\omega^{lab} = -\omega_k v_g/v_\varphi > 0$). In the range of α from 0.202 to $1/\pi$, which corresponds to values of ϵ for which $df_0^{tr}/d\epsilon > 0$, the function $\Gamma_2(\alpha)$ is positive and the red

5)The instability due to the transiting particles arises when $\sqrt{e\varphi_0/m} \gg v_T^2/v_{ph}$. In this case it is important to extend the distribution function $f_0(\epsilon)$ defined by Eq. (26) to particles characterized by $u < 0$. The contribution of these particles is basic in the growth rate; as a result there arises an instability for test waves characterized by $\omega_k > 0$.

$\alpha = \frac{\lvert\omega_k\rvert}{\pi k_0}\frac{1}{\sqrt{e\varphi_0/m}}$	Trapped particles		Transiting particles	$\varkappa = \frac{\lvert\omega_k\rvert}{\pi k_0}\frac{1}{\sqrt{e\varphi_0/m}}$	Trapped particles		Transiting particles
	$\varkappa_R^2$ $(n=1)$	$\varkappa_R^2$ $(n=2)$	$1/\varkappa_R^2$ $(n=1)$		$\varkappa_R^2$ $(n=1)$	$\varkappa_R^2$ $(n=2)$	$1/\varkappa_R^2$ $(n=1)$
0.10	0.999	1.000	1.000	0.24	0.703	0.996	0.996
0.16	0.968	1.000	1.000	0.28	0.413	0.987	0.988
0.20	0.883	0.999	0.999	0.30	0.216	0.979	0.981

satellites are excited $\Delta\omega^{\text{lab}} < 0$. The maximum value of the growth rate for the red satellite is approximately an order of magnitude higher than for the violet satellite. As the amplitude of the potential in the primary wave φ_0 increases the frequencies of the excited waves $\sim\omega_k$ vary in proportion to $\sqrt{\varphi_0}$. This result has been observed experimentally.[8] Under instability conditions, in which the second term in Eq. (30) is much larger than the first, the growth rate γ_R is proportional to $\sqrt{\varphi_0}$ when $\omega_k/\sqrt{\varphi_0} = \text{const}$.

In this case the growth length $l = v_g/\gamma_R$ is inversely proportional to $\sqrt{\varphi_0}$, as has also been observed in the measurements reported in[8].

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APPENDIX

We wish to compute the coefficients in the expansion of the function $\exp[ik_0x(t)]$ in a Fourier series. For the transiting particles we have

$$\exp[ik_0x(t)] = \sum_{n=-\infty} \psi_n \exp\left(2\pi i n \frac{t}{\tau(\varepsilon)}\right), \quad \tau = \frac{2\pi}{k_0|u|}. \quad \text{(A.1)}$$

The coefficients ψ_n are determined from the relation

$$\psi_n = \frac{1}{4K(1/\varkappa)} \int_{-2K(1/\varkappa)}^{2K(1/\varkappa)} \exp\left[i\left(k_0x(z) - \frac{\pi n}{2K(1/\varkappa)}z\right)\right] dz, \quad \text{(A.2)}$$

where we have used the notation $z = \kappa t/\tau_0$. Substituting $x(t)$ from Eq. (6) we have

$$\exp[ik_0x(z)] = \text{cn}^2\left[F\left(\xi_0, \frac{1}{\varkappa}\right) + z, \frac{1}{\varkappa}\right] - \text{sn}^2\left[F\left(\xi_0, \frac{1}{\varkappa}\right) + z, \frac{1}{\varkappa}\right]$$
$$\pm 2i\,\text{cn}\left[F\left(\xi_0, \frac{1}{\varkappa}\right) + z, \frac{1}{\varkappa}\right] \text{sn}\left[F\left(\xi_0, \frac{1}{\varkappa}\right) + z, \frac{1}{\varkappa}\right], \quad \text{(A.3)}$$

where the upper sign refers to particles for which $u > 0$ and the lower sign to particles for which $u < 0$.

Using the well-known properties of the elliptic function the integral in Eq. (A.2) can be transformed in the following manner when $n \neq 0$ (cf. also[2]):[6)]

$$\psi_n = \frac{1}{4K(1/\varkappa)} \int_C \exp\left[-i\frac{\pi n z}{2K(1/\varkappa)}\right]$$
$$\times \left\{\frac{\text{cn}^2[F(\xi_0, 1/\varkappa) + z, 1/\varkappa] - \text{sn}^2[F(\xi_0, 1/\varkappa) + z, 1/\varkappa]}{1 - (-1)^n g^n}\right. \quad \text{(A.4)}$$
$$\left.\pm \frac{2i\,\text{cn}\{F(\xi_0, 1/\varkappa) + z, 1/\varkappa]\,\text{sn}[F(\xi_0, 1/\varkappa) + z, 1/\varkappa]}{1 + (-1)^n g^n}\right\} dz.$$

The contour of integration C is a parallelogram with vertices at the points

$$-2K\left(\frac{1}{\varkappa}\right), \quad 2K\left(\frac{1}{\varkappa}\right), \quad 2iK'\left(\frac{1}{\varkappa}\right), \quad -4K\left(\frac{1}{\varkappa}\right) + 2iK'\left(\frac{1}{\varkappa}\right).$$

Computing the integral in Eq. (A.4) by means of residues we obtain Eq. (9) and (11) for ψ_n. The expansion in (10) for $v(t)\exp[ikx(t)]$ can be obtained without difficulty by means of the relation

$$v(t)\exp[ikx(t)] = \frac{1}{ik}\frac{d}{dt}\exp[ikx(t)]$$

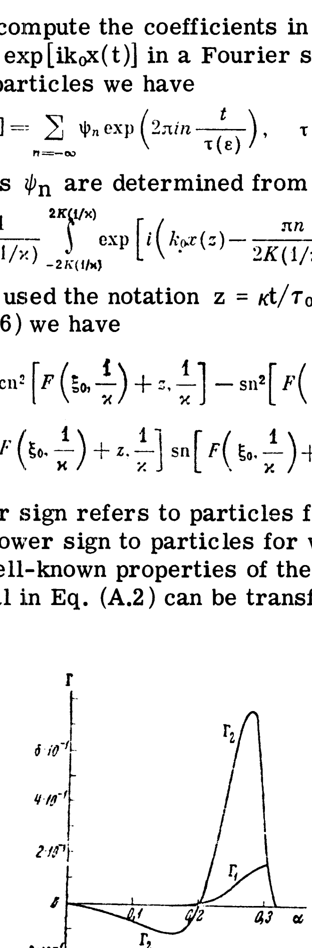
and substitution of $\exp[ikx(t)]$ for Eq. (9). The coefficients ψ_n^{tr} and Φ_n^{tr} for the trapped particles can be obtained in similar fashion.

We now wish to compute the integrals with respect to x_0 on the right side of Eq. (20). For transiting particles with $u > 0$ these integrals are of the form

$$\frac{1}{a}\int_{-a/2}^{a/2} \frac{dx_0}{v_0(x_0, \varepsilon)} \psi_{n'}^*(x_0, \varepsilon)\psi_n(x_0, \varepsilon) = \frac{16\pi^4\varkappa^3}{K^4(1/\varkappa)} \frac{nn' g^{3(n+n')}}{(1 - g^{4n})(1 - g^{4n'})} \frac{\tau_0}{a}$$
$$\times \int_{-\pi/2}^{\pi/2} \frac{d\xi_0 \exp[\pi i(n - n')F(\xi_0, 1/\varkappa)/K(1/\varkappa)]}{\sqrt{1 - \varkappa^{-2}\sin^2\xi_0}} \quad \text{(A.5)}$$
$$= \frac{16\pi^3\varkappa^3}{K^3(1/\varkappa)} k_0\tau_0 \frac{n^2g^{2n}}{(g^{2n} - g^{-2n})^2}\delta_{nn'} \quad (n \neq 0),$$

where we have used the following substitution in computing the integral with respect to ξ_0:

$$\eta = F\left(\xi_0, \frac{1}{\varkappa}\right) = \int_0^{\xi_0} \frac{d\varphi}{\sqrt{1 - \varkappa^{-2}\sin^2\varphi}}.$$



Considering the trapped particles and making use of the symmetry of the distribution function in velocity, we can combine the contributions in the integral over x_0 for particles with $v_0 > 0$ and $v_0 < 0$. Calculating the integral over ζ_0 in the same way as in Eq. (A.5) we obtain the following orthogonality relations for the coefficients ψ_n^{tr}:

$$\frac{1}{a}\int_{-\bar{x}}^{\bar{x}} \frac{dx}{|v_0(x_0\varepsilon)|} [\psi_{n'}^{tr*}(x_0, \varepsilon)\psi_n^{tr}(x_0, \varepsilon)|_{v_0>0} + \psi_{n'}^{tr*}(x_0, \varepsilon)\psi_n^{tr}(x_0, \varepsilon)|_{v_0<0}]$$
$$= \frac{\pi^3 k_0\tau_0}{2K^4(\varkappa)} \frac{(-1)^{n+n'} nn' g_{tr}^{(n+n')/2}}{[1 - (-1)^n g_{tr}^n][1 - (-1)^{n'} g_{tr}^{n'}]} \int_{-\pi/2}^{\pi/2} \frac{d\zeta_0}{\sqrt{1 - \varkappa^2\sin^2\zeta_0}}$$

6)The integral in Eq. (A.2) is tabulated for the case n = 0.

$$\times \left\{\exp\left[\pi i \frac{F(\zeta_0, \varkappa)}{2K(\varkappa)}(n-n')\right] + (-1)^{n-n'} \exp\left[-\pi i \frac{F(\zeta_0, \varkappa)}{2K(\varkappa)}(n-n')\right]\right\}$$
$$= \frac{2\pi^3 k_0 \tau_0}{K^3(\varkappa)} \frac{n^2}{[g_{tr}^{-n/2} - (-1)^n g_{tr}^{n/2}]^2} \delta_{nn'} (n \neq 0). \qquad \text{(A.6)}$$

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