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THE ANOMALOUS SKIN EFFECT IN METALS IN A MAGNETIC FIELD

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We obtain an asymptotically exact solution of the problem of the anomalous skin-effect in a magnetic field parallel to the metal surface for the case of specular reflection at the metal-vacuum boundary. Surface electrons "slipping" along the boundary play the main role in the production of the screening current. As a result the magnitude of the impedance is changed and also its dependence on frequency, magnetic field, and other parameters when compared to the case of diffuse scattering, and the surface impedance also becomes anisotropic.

1. INTRODUCTION

Reuter and Sondheimer[1] solved already in 1948 the problem of the penetration of an electromagnetic field into a metal under the conditions for the anomalous skin effect. They obtained exact formulae for the field distribution and for the surface impedance in two cases: specular and diffuse reflection of the electrons from the metal surface. In the case of specular reflection the problem reduces to an integro-differential equation with a difference kernel, and can easily be solved by Fourier transformation. The case of diffuse scattering of the electrons is more complicated as one needs to use the Wiener–Hopf method. In a paper by Azbel' and one of the authors,[2] an integral equation was obtained for the Fourier components of the electromagnetic field in the metal for an arbitrary law for the reflection of the electrons from the boundary in the case of the limiting anomalous skin effect. Hartmann and Luttinger[3] solved this integral equation with a non-difference kernel, subsequently using an ingenious application of Mellin transforms. In their paper they also found an exact solution of the problem of the anomalous skin effect in a parallel magnetic field for the case of diffuse scattering of the electrons by the boundary. The corresponding integral equation was obtained in [4]. The influence of the electron scattering on the cyclotron resonance and the anomalous skin effect in a parallel magnetic field was studied in [5]. This study showed that in the range of strong magnetic fields and also near the cyclotron resonance the scattering of the electrons by the surface does not play an important role unless the reflection coefficient ρ is close to unity. The case of specular reflection ($\rho = 1$) is a peculiar one. The asymptotic behavior of the surface current and of the surface impedance is then qualitatively changed as compared to the non-specular reflection ($\rho \neq 1$). The physical reason for such an abrupt dependence of the impedance and the current on the reflection coefficient lies in the fact that when $\rho = 1$ the main contribution to the high-frequency current is given by the electrons which slip along the metal surface thanks to multiple collisions with the surface of the sample. For specular reflection therefore the main term of the asymptotic expansion of the impedance in terms of the parameter $(\delta/R)^{1/2}$ has no resonance character and cyclotron resonance occurs only in the next approximation in $(\delta/R)^{1/2}$. Here δ is the effective thickness of the skin-layer and R the radius of the electron orbit in the magnetic field. However, up to now an exact solution of the problem of the field distribution in the metal in that case has not been found. The main difficulty lies in the fact that the skin-layer is formed by the glancing electrons whose conductivity has an essentially non-local character. One must therefore solve a complicated integral equation on the semi-axis with a non-difference kernel. In the present paper we use a method developed in [3] to obtain an asymptotically exact solution of the problem of the anomalous skin-effect in a parallel magnetic field when the electrons are reflected specularly from the metal surface.

2. STATEMENT OF THE PROBLEM

We consider an alkali metal with a spherical Fermi surface. The constant and uniform magnetic field **H** is oriented parallel to the metal-vacuum interface. The x-axis lies along the inward normal to the surface and the z-axis along the vector **H** (see Fig. 1). Let there be incident upon the boundary an electromagnetic wave of frequency ω, the electric vector of which, **E**, is linearly polarized at right angles to the magnetic field (**E** $\parallel$ y).

We write down the Maxwell equation for the spatial Fourier component of the electric field:

$$\mathscr{E}(k) \equiv 2\int_0^\infty E(x)\cos(kx)\,dx. \tag{2.1}$$

Neglecting the displacement current we have

$$k^2\mathscr{E}(k) + 2E'(0) = 4\pi i\omega c^{-2} j(k), \tag{2.2}$$

where j(k) is the Fourier component of the y-component of the current density, c is the velocity of light, and a prime indicates differentiation with respect to z.

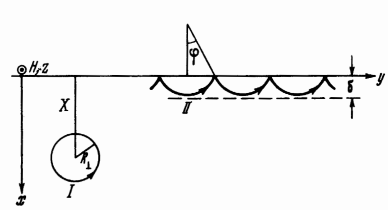

FIG. 1. Trajectories of "volume" (I) and "glancing" (II) electrons.

When evaluating the y-component of the current density we can neglect the field component $\mathscr{E}_x(k)$. This function must be found from the quasi-neutrality condition $j_x(k) = 0$. If the term in $\mathscr{E}_x$ in the current j(k) is small, the electrical neutrality condition $j_x = 0$ can completely be neglected when we find $\mathscr{E}(k)$.

We must use the kinetic equation for the electron distribution function to obtain a connection between j(k) and $\mathscr{E}(k)$. Such calculations were made in [5] for arbitrary reflection coefficients ($0 \le \rho \le 1$). Using these calculations we get the final result for the current:

$$j(k) = K(k)\mathscr{E}(k) - \frac{1}{\pi}\int_0^\infty dk' Q(k, k')\mathscr{E}(k'). \tag{2.3}$$

Here

$$K(k) = \frac{3\sigma\gamma}{\pi}\int_0^{\pi/2} d\theta \sin^3\theta \int_0^\infty dx\, e^{-\gamma x}\int_0^\pi d\tau \cos\tau \cos(\tau - x) \times \cos\{kR_\perp(\cos\tau - \cos(\tau - x))\}$$

is the Fourier component of the conductivity of the unbounded metal

$$\sigma = \frac{Ne^2}{m(\nu - i\omega)}, \quad \gamma = \frac{\nu - i\omega}{\Omega}, \quad R_\perp = \frac{v\sin\theta}{\Omega}, \tag{2.5}$$

N is the concentration, m the effective mass, v the Fermi velocity, e the absolute value of the electron charge, $\Omega = eH/mc$ the cyclotron frequency, $R_\perp$ the radius of the electron orbit in the magnetic field, θ the polar angle with the polar z-axis, τ the azimuthal angle ($v_y = v_\perp \cos\tau$, $v_x = v_\perp \sin\tau$), ν the frequency of collisions of electrons and scatterers, and $R = v/\Omega$.

According to [5]

$$Q(k, k') = \frac{6\sigma\gamma R}{\pi}\int_0^{\pi/2} d\theta \sin^4\theta \int_0^\infty dx\, e^{-\gamma x}\int_0^\pi \frac{d\tau \sin\tau}{e^{\gamma\pi} - \rho e^{-\gamma\pi}}\{\cos(\tau + x)\cos\{kR_\perp(\cos\tau - \cos(\tau + x))\} - \rho\cos(\tau - x)\cos\{kR_\perp(\cos\tau - \cos(\tau - x))\}\} \times \int_0^\tau d\lambda \cos\lambda\, \mathrm{ch}(\gamma\lambda)\cos[k'R_\perp(\cos\tau - \cos\lambda)]. \tag{2.6}$$

The kernel of the conductivity operator Q(k, k') takes into account the contribution of all electrons, both those which collide with the metal surface and those which move far from it and are not scattered at the boundary. Equations (2.3), (2.4), and (2.6) are exact and valid for the normal ($k \to 0$) and the anomalous ($k \to \infty$) skin effects.

By solving Eqs. (2.2) and (2.3) we can find the distribution of the field inside the metal and the surface impedance

$$Z = \frac{4\pi i\omega}{c^2}\frac{E(0)}{E'(0)}, \quad E(0) = \frac{1}{\pi}\int_0^\infty dk\,\mathscr{E}(k). \tag{2.7}$$

It is clear that in the general case it is impossible to obtain an analytical solution of Eqs. (2.2) and (2.3). We shall therefore consider in what follows the region of the limiting anomalous skin effect

$$kR \gg 1 + |\gamma|^2, \tag{2.8}$$

in which the kernels K(k) and Q(k, k') can be replaced by their asymptotic expressions for large values of their arguments. After this it is then possible to find an exact solution of the approximate equation.

3. ASYMPTOTIC BEHAVIOR OF THE CURRENT DENSITY

To obtain the asymptotic expression of K(k) it is convenient to transform Eq. (2.4) somewhat by integrating over x and τ. As a result we obtain

$$K(k) = \frac{3\sigma\gamma}{4}\int_0^{\pi/2} d\theta \sin^3\theta \sum_{n=-\infty}^{\infty}\frac{1}{\gamma + in}[J_{n+1}(kR_\perp) - J_{n-1}(kR_\perp)]^2. \tag{3.1}$$

We replace the Bessel functions by their asymptotic values for large values of the argument and neglect fast oscillating terms. After simple calculations we find

$$K(k) \approx \frac{3\pi}{4}\frac{\sigma\gamma}{kR}\mathrm{cth}(\pi\gamma) = \frac{3\pi}{4}\frac{Ne^2}{mvk}\mathrm{cth}(\pi\gamma). \tag{3.2}$$

This formula is valid when $kR \gg 1 + |n|$; as in the sum over n terms with $|n| \sim |\gamma|$ play an essential role role, the asymptotic behavior (3.2) occurs when

$$kR \gg 1 + |\gamma|. \tag{3.3}$$

It is very complicated to obtain the asymptotic behavior of the function Q(k, k'). The calculational difficulties are to a large extent caused by the fact that expression (2.6) takes into account contributions to the conductivity from two different groups of electrons. One of them is formed by the electrons whose orbit centers are well inside the metal at distances larger than the orbit radius $R_\perp$. We shall call such electrons in what follows "volume" electrons. They are not scattered at the metal boundary. Into the other group enter the so-called "surface" electrons. The centers of their orbits lie outside the metal or at a depth inside it which is less than $R_\perp$. The surface electrons collide with the boundary in each loop (see Fig. 1). Only the electrons of the second group satisfy the boundary condition for the distribution function. The distribution function has a singularity at $x = x_0 \equiv R_\perp(1 - \cos\tau)$ where the transition from surface ($x < x_0$) to volume ($x > x_0$) electrons takes place. Expression (2.6) which takes into account the contribution from both electron groups can therefore conveniently be written as a sum of two terms

$$Q(k, k') = Q_{\mathrm{vol}} + Q_{\mathrm{surf}} \tag{3.4}$$

The additivity of the contributions from the surface and volume electrons to the current density can be proved by a direct evaluation of their distribution functions. The volume electrons give

$$Q_{\mathrm{vol}} = \frac{3\sigma\gamma}{4}\int_0^{\pi/2} d\theta \sin^3\theta \sum_{n=-\infty}^{\infty}\frac{1}{\gamma + in}\left\{\frac{\sin[(k - k')R_\perp]}{k - k'} - (-1)^n\frac{\sin[(k + k')R_\perp]}{k + k'}\right\}[J_{n-1}(kR_\perp) - J_{n+1}(kR_\perp)] \times [J_{n-1}(k'R_\perp) - J_{n+1}(k'R_\perp)]. \tag{3.5}$$

The contribution from the surface electrons can for the case of specular reflection ($\rho = 1$) be written in the form

$$Q_{\mathrm{surf}} = -\frac{6\sigma\gamma R}{\pi}\int_0^{\pi/2} d\theta \sin^4\theta \int_0^\pi d\varphi \sin\varphi \sum_{n=-\infty}^{\infty}\frac{1}{\gamma\varphi + i\pi n} \times \int_0^\varphi d\lambda \cos\lambda \cos\frac{\pi n\lambda}{\varphi}\cos[kR_\perp(\cos\varphi - \cos\lambda)] \times \int_0^\varphi d\mu \cos\mu \cos\frac{\pi n\mu}{\varphi}\cos[k'R_\perp(\cos\varphi - \cos\mu)]. \tag{3.6}$$

The variable φ is defined by the equation $\cos\varphi = -X/R_\perp$ where X is the x-component of the coordinate of the center, i.e., φ is the angle along which the electron is slipping at the moment it collides with the metal surface.

A direct calculation shows that the sum of Eqs. (3.5) and (3.6) is the same as (2.6) with $\rho = 1$.

We can split off from Q_{surf} the term which exactly compensates Q_{vol}. To do this we must integrate by parts in the integral over φ in Eq. (3.6). It is necessary first to write the cosines in whose arguments occur $kR_\perp$ and $k'R_\perp$ as sums of oscillating exponentials. The integrated term then turns out to be equal to and of the opposite sign of Q_{vol}. The asymptotic form of the kernel Q is determined by the terms which remain after the integration by parts. In the limiting case (2.8) only the term in the sum with n = 0 gives the main contribution to the asymptotic form of the kernel Q. We shall not give here all these transformations which are simple but rather cumbersome and we only write down the result of the calculations:

$$Q(k,k') \approx -\frac{9\pi^2\sigma}{5\sqrt{2}\Gamma^2(1/4)\,(kk'R)^{1/2}}[|k-k'|^{-1/2}-(k+k')^{-1/2}], \quad (3.7)$$

where $\Gamma(x)$ is the Euler function. The asymptotic form (3.7) is, strictly speaking, valid when $|k-k'|R \gg 1 + |\gamma|^2$. However, as the singularity in $Q(k, k')$ at $k = k'$ is integrable we can use the asymptotic expression (3.7) for all k' satisfying inequality (2.8).

The main contribution to the asymptotic form of the kernel Q is given by electrons which drift along the surface of the metal with their orbital centers outside the metal at a distance $|X| \approx R_\perp$ (trajectory II in Fig. 1). A characteristic slipping angle φ is of the order

$$\varphi \sim (kR_\perp)^{-1/2} \sim (\delta/R)^{1/2}. \quad (3.8)$$

This estimate follows directly from Eq. (3.6) in which the point $\varphi = 0$ is the point of phase stationarity. The width of the integration interval is determined by Eq. (3.8). Physically this result is evident as the maximum contribution to the skin current is made only by those surface electrons whose trajectories are completely situated within the skin-layer ("glancing" electrons). The estimate (3.8) follows at once from these considerations.

We compare the volume conductivity K(k) and the surface conductivity kQ. As to order of magnitude, the ratio

$$|K/kQ| \lesssim |\gamma|/(kR)^{1/2} \ll 1 \quad (3.9)$$

is small compared to unity by virtue of inequality (2.8). We must note that in the opposite limiting case of a weak magnetic field, when $|\gamma| \gg (kR)^{1/2} \gg 1$, the surface conductivity kQ is small compared with the volume conductivity and one can neglect it. The physical meaning of condition (3.9) consists in that the section of the trajectory of "glancing" electrons between two consecutive collisions with the surface $(R\delta)^{1/2}$ must be small compared with the effective mean free path $|l^*| = v/|\nu - i\omega|$. In that case the skin layer is primarily formed by the "glancing" electrons and one can neglect the volume conductivity K(k). We can thus write Eq. (2.2) for the y-component of the electric field in the form

$$k^2\mathscr{E}(k) - i\frac{(3\pi)2^{2^{1/2}}}{5\Gamma^2(1/4)}\frac{\omega\sigma}{c^2R^{1/2}}\int_0^\infty \frac{dk'\mathscr{E}(k')}{(kk')^{1/2}}[|k-k'|^{-1/2}-(k+k')^{-1/2}] = -2E'(0). \quad (3.10)$$

The equation for the orthogonal polarization of the electric field $\mathscr{E}_z(k)$ has also a similar form with that difference that the numerical factor in the integral term turns out to be 1.5 times smaller than in (3.10).[1)] In other words, the Fourier components of the surface current along the vector H is 1.5 times smaller than the current in the perpendicular direction. This difference is caused by the fact that

$$Q_{yy}(k,k') \sim \int_0^{\pi/2} d\theta \sin^{5/2}\theta, \quad Q_{zz}(k,k') \sim \int_0^{\pi/2} d\theta \sin^{1/2}\theta\cos^2\theta.$$

The ratio of these two integrals is equal to 3/2. Owing to this difference in the magnitude of the surface currents even in an isotropic metal there must occur an anisotropy in the surface impedance ($Z_{yy} \neq Z_{zz}$).

4. SOLUTION OF THE INTEGRAL EQUATION

We now solve Eq. (3.10). We write it in dimensionless variables

$$\xi^2F(\xi) - i\int_0^\infty \frac{d\xi'F(\xi')}{(\xi\xi')^{1/2}}[|\xi-\xi'|^{-1/2}-(\xi+\xi')^{-1/2}] = 1. \quad (4.1)$$

Here ξ is the dimensionless wave number

$$\xi = \frac{k}{k_0}, \quad k_0 = \left(\frac{18\pi^2\sqrt{2}}{5\Gamma^2(1/4)}\frac{\omega\sigma}{c^2R^{1/2}}\right)^{2/5} \quad (4.2)$$

while the function $F(\xi)$ is connected to $\mathscr{E}(k)$ through the relation

$$F(\xi) = -k_0^2\mathscr{E}(k_0\xi)/2E'(0). \quad (4.3)$$

The surface impedance Z can be expressed in terms of k_0 and $F(\xi)$ through the following formula:

$$Z = -\frac{8i\omega}{c^2k_0}\int_0^\infty F(\xi)\,d\xi. \quad (4.4)$$

We change variables in Eq. (4.1): $\xi = e^t$, $\xi' = e^\tau$ and change the sought function: $F(e^t) = g(t)$. We can then rewrite (4.1) in the form

$$g(t)e^{5t/2} - i\int_{-\infty}^{\infty} g(\tau)\Phi(t-\tau)\,d\tau = e^{t/2}, \quad (4.5)$$

where

$$\Phi(x) = |e^x-1|^{-1/2} - (e^x+1)^{-1/2}. \quad (4.6)$$

We shall solve Eq. (4.5) using a two-sided Laplace transform

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(z)e^{zt}dz, \quad M(z) = \int_{-\infty}^{\infty} g(t)e^{-zt}dt, \quad c = \operatorname{Re} z. \quad (4.7)$$

The real number c is arbitrary and is chosen inside the region where M(z) is a regular function. If the function we are looking for, g(t), is a function which increases exponentially ($g(t) \sim \exp(at)$ as $t \to \infty$ and

1) We use this opportunity to correct an error made in [5] when calculating the asymptotic form of $Q_{zz}(k,k')$ for the case $\rho = 1$. Equation (3.6) of [5] although it gives the qualitatively correct asymptotic behavior in k and k' differs from Eq. (3.7). It is necessary, though, to note that this error does not change those qualitative conclusions which were drawn in [5].

$g(t) \sim \exp(bt)$ as $t \to -\infty$), its Laplace transform M(z) is regular in the region $a < \operatorname{Re} z < b$. It follows from Eq. (4.5) that as $t \to \infty$ the function $g(t) \sim \exp(-2t)$, i.e., $a = -2$. As $t \to -\infty$ we can neglect the first term on the left-hand side of (4.5) and by comparing the integral term with exp (t/2) it follows that $g(t) \approx \exp(t/2)$, i.e., $b \equiv 1/2$. The function M(z) is thus regular in the region

$$-2 < \operatorname{Re} z < 1/2. \quad (4.8)$$

We shall look for such a solution of Eq. (4.5) whose Laplace transform is regular in a somewhat wider region, e.g.,

$$-2\tfrac{1}{4} < \operatorname{Re} z < 1/2, \quad (4.9)$$

excluding the singularity $z = -2$. As will become clear in what follows the singularity of M(z) at $z = -2$ must be a simple pole with a residue equal to unity in order that we obtain for the function M a homogeneous functional equation. This is why it is necessary to widen the region of regularity of M(z).

We now substitute (4.7) into (4.5):

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz\, M(z)\exp\left[\left(z+\frac{5}{2}\right)t\right] - \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} dz\, M(z)\varphi(z)e^{zt} = e^{t/2}, \quad (4.10)$$

where

$$\varphi(z) = \int_{-\infty}^{\infty}\Phi(x)e^{-zx}dx = B\left(z+\frac{1}{2},\frac{1}{2}\right)\left[1+\operatorname{tg}\left(\frac{\pi z}{2}\right)\right], \quad (4.11)$$

where B(x, y) is the Euler function.

We choose c inside the interval (1/4, 1/2). In the first integral in (4.10) we shift the integration contour to the left by 2.5 (Fig. 2). As a result we obtain the integral over the straight line $\operatorname{Re} z = c - 2.5$ and the residue in the point $z = -2$, which is equal to exp (t/2). This last contribution cancels the right-hand side of (4.10) and after a change in variable we get from (4.10) the relation

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[M\left(z-\frac{5}{2}\right) - i\varphi(z)M(z)\right]e^{zt}dz = 0.$$

From this it follows that the function M(z) must satisfy the difference equation

$$M(z-5/2) = i\varphi(z)M(z). \quad (4.12)$$

The general solution of this equation has as factor an arbitrary periodic function with period 5/2. The condition that the solution must be regular in the region (4.9) enables us to determine that arbitrary periodic function unambiguously.

We shall look for the solution of (4.12) in the form

$$M(z) = \frac{2\pi}{5}\frac{\exp[\pi i(z+2)/5]}{\sin[2\pi(z+2)/5]}\exp[V(z)-V(-2)+W(z)-W(-2)]. \quad (4.13)$$

The unknown functions V(z) and W(z) are now regular in the region (4.9) and satisfy the equations

$$\exp[V(z-5/2)-V(z)] = B(z+1/2, 1/2), \quad (4.14)$$

$$\exp[W(z-5/2)-W(z)] = 1+\operatorname{tg}(\pi z/2). \quad (4.15)$$

We first of all find the function V(z). To do this we differentiate (4.14) twice. For the first derivatives we have

$$V'(z-5/2) - V'(z) = \psi(z+1/2) - \psi(z+1),$$

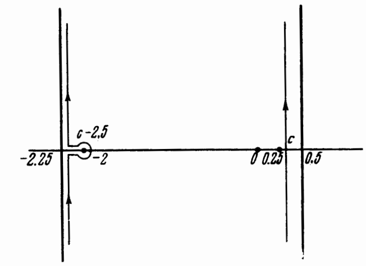

FIG. 2

and for the second derivatives

$$V''(z-5/2) - V''(z) = \zeta(2, z+1/2) - \zeta(2, z+1) \quad (4.16)$$

Here $\psi(x) = d\ln\Gamma(x)/dx$ is the logarithmic derivative of the gamma-function, $\zeta(x, y)$ is the Riemann zeta-function. If we use the well-known integral representation of $\zeta(2, x)$ we can easily find a solution of (4.16) in the form

$$V''(z) = \int_0^\infty du \frac{ue^{-zu}}{(e^{u/2}+1)(e^{5u/2}-1)} + \text{const.} \quad (4.17)$$

We added a constant to (4.17) and not an arbitrary periodic function with period 5/2 as this periodic function must be regular in the band of width 5/2, that is in the whole complex z-plane. Integrating Eq. (4.17) over z from 0 to z and substituting the result into the equation for the first derivatives we see easily that const = 0. Hence,

$$V'(z) = \int_0^\infty du \frac{1-e^{-zu}}{(e^{u/2}+1)(e^{5u/2}-1)} + A. \quad (4.18)$$

We can determine the constant A as follows. The difference

$$V\left(z-\frac{5}{2}\right) - V(z) = \int_z^{z-5/2} dx\, V'(x)$$

we substitute into (4.14). Using Eq. (4.18) and the well-known integral representation for $\ln B(z+\frac{1}{2}, \frac{1}{2})$ we get from (4.14) A in the form

$$A = -\frac{1}{5}\ln\pi + \int_0^\infty\frac{du}{u}\left[\frac{1}{5}e^{-u} - \frac{u}{(e^{u/2}+1)(e^{5u/2}-1)}\right].$$

Finally,

$$V(z) - V(-2) = \int_{-2}^{z} V'(x)dx = -\frac{z+2}{5}\ln\pi + \int_0^\infty\frac{du}{u}\left\{\frac{z+2}{5}e^{-u} + \frac{e^{-zu}-e^{2u}}{(e^{u/2}+1)(e^{5u/2}-1)}\right\}. \quad (4.19)$$

It is clear from the solution which we have obtained that the function $V(z) - V(-2)$ is regular in the half-plane $\operatorname{Re} z > -3$ and certainly analytic in the region (4.9).

We now find the function W(z). In order not to have to deal with logarithms of trigonometric functions we write down the equation for the derivatives

$$W'(z-5/2) - W'(z) = -1/2\pi\lambda(z). \quad (4.20)$$

The function

$$\lambda(z) = -\frac{2}{\pi}\left[\ln\left(1+\operatorname{tg}\frac{\pi z}{2}\right)\right]' \equiv -\left\{\cos\left(\frac{\pi z}{2}\right)\left[\cos\left(\frac{\pi z}{2}\right)+\sin\left(\frac{\pi z}{2}\right)\right]\right\}^{-1}$$

is periodic with period 2. We can thus write the general solution of (4.20) in the form

$$W'(z) = {}^1/_{20}\pi[z\lambda(z) + (z - {}^5/_2)\lambda(z - {}^1/_2) + (z-5)\lambda(z-1) + (z - {}^{15}/_2)\lambda(z - {}^3/_2)] + {}^1/_2\pi\Psi(z) + C.$$

Here C is a constant and $\Psi(z)$ an arbitrary periodic function with period $^5/_2$, the average value of which is zero. After simple transformations we find

$$W'(z) = \frac{\pi}{2}\left[\frac{1-2\cos(\pi z)}{\sin(2\pi z)} + \Psi(z)\right] + C. \qquad (4.21)$$

The first term in the square brackets has simple poles for all odd half-integer integral and all integer values of z. We must choose the arbitrary function $\Psi(z)$ in such a way that $W'(z)$ will be regular in the region (4.9). For instance, to cancel the singularity at z = 0 we must add $+{}^1\!/_5 \cot 2\pi z/5$. In the table we have indicated the position of the poles of the first term, their residues, and those periodic functions which cancel the singularity of $W'(z)$ in the region considered. The function $\Psi(z)$ is the sum of all terms written down in the last column of the table. We easily find the constant C if we substitute the difference

$$W\left(z-\frac{5}{2}\right) - W(z) = \int_z^{z-5/2} dx W'(x)$$

in the initial Eq. (4.15). Then

$$C = +\frac{2}{5\pi}\int_0^{\pi}\frac{x dx}{\cos x(\cos x + \sin x)} = -\frac{1}{5}\ln 2. \qquad (4.22)$$

The difference we are looking for is thus

$$W(z) - W(-2) = -\frac{z+2}{5}\ln 2 + \frac{\pi}{2}\int_{-2}^{z} d\zeta\left\{\frac{1-2\cos(\pi\zeta)}{\sin(2\pi\zeta)} - \frac{3}{5}\mathrm{ctg}\left[\frac{2\pi}{5}(\zeta+1)\right] + \frac{1}{5}\left(\mathrm{ctg}\left[\frac{2\pi\zeta}{5}\right] + \mathrm{ctg}\left[\frac{2\pi}{5}\left(\zeta+\frac{1}{2}\right)\right] + \mathrm{ctg}\left[\frac{2\pi}{5}\left(\zeta+\frac{3}{2}\right)\right] + \mathrm{ctg}\left[\frac{2\pi}{5}(\zeta+2)\right]\right)\right\}. \qquad (4.23)$$

Equations (4.13), (4.19), and (4.23) give us the final solution of Eq. (4.12) for M(z). The expression for g(t), i.e., for $F(\xi)$ is obtained in the form of a contour integral. We shall not evaluate the electromagnetic field distribution inside the metal but give the result for the surface impedance. It follows from Eqs. (4.4) and (4.7) that

$$Z = -\frac{8i\omega}{c^2 k_0}M(-1). \qquad (4.24)$$

According to (4.13)

$$M(-1) = \frac{(2\pi)^{4/5}}{10}e^{-I}e^{\pi i/5}, \qquad (4.25)$$

where

$$I = -\int_0^{\infty}\frac{dt}{t}\left[\frac{1}{5}e^{-t} + \frac{e^t - e^{3t/2}}{e^{5t/2}-1}\right] \approx 0{,}215.$$

Thus

$$Z = \frac{8(2\pi)^{4/5}}{10}\frac{\omega}{k_0 c^2}\exp\left(-0{,}215 - \frac{3\pi i}{10}\right) \approx 1{,}75\left(\frac{\omega^3 v}{\Omega c^6 \sigma^2}\right)^{1/5}\exp\left(-\frac{3\pi i}{10}\right) \qquad (4.26)$$

Pole in the band (-2.25; 0.5)	Residue at the pole	Periodic function cancelling the singularity
0	$-1/2\pi$	$+{}^1/_5\,\mathrm{ctg}\,(2\pi z/5)$
$-{}^1/_2$	$-1/2\pi$	$+{}^1/_5\,\mathrm{ctg}\,[2\pi(z+{}^1/_2)/5]$
-1	$3/2\pi$	$-{}^3/_5\,\mathrm{ctg}\,[2\pi(z+1)/5]$
$-{}^3/_2$	$-1/2\pi$	$+{}^1/_5\,\mathrm{ctg}\,[2\pi(z+{}^3/_2)/5]$
-2	$-1/2\pi$	$+{}^1/_5\,\mathrm{ctg}\,[2\pi(z+2)/5]$

The impedance Z_{zz} of an electromagnetic wave which is polarized along the magnetic field is larger than (4.26) by a factor $(^3\!/_2)^{2/5} \approx 1.18$. The surface impedance is proportional to $H^{-1/5}$ and decreases when the magnetic field increases. The dependence on the frequency of the external field, ω, is determined by the equation

$$Z \infty \omega^{3/5}(\nu^2+\omega^2)^{1/5}\exp\left(-\frac{2i}{5}\mathrm{arctg}\frac{\omega}{\nu} - \frac{3\pi i}{10}\right); \qquad (4.27)$$

at low frequencies ($\omega \ll \nu$) the real and the imaginary part of Z are proportional to $\omega^{3/5}$ while when $\omega \gg \nu$

$$Z \infty -i\omega + {}^2/_5\nu. \qquad (4.28)$$

It is of interest to compare the impedance of the metal in a parallel magnetic field with specular and diffuse reflection of the electrons from the boundary. According to [3] the impedance of a metal in a strong magnetic field, when $|\gamma| \ll 1$, is equal to

$$Z_0 = \frac{8}{\sqrt{3}}\left(\frac{2\pi^2}{3}\right)^{1/3}\left(\frac{\omega^2 v}{c^4\sigma\Omega}\right)^{1/3}\exp\left(-\frac{\pi i}{3}\right). \qquad (4.29)$$

By comparing this expression with (4.26) it is clear that the frequency and magnetic field dependence of the impedance are different for specular reflection from what they are for diffuse scattering. The impedance turns out to be smaller in the case $\rho = 1$ than for $\rho = 0$. The difference is caused by the fact that for specular reflection the skin-layer is formed by the surface electrons which "glance" along the surface of the metal while in the case of diffuse scattering the screening current is basically produced by the "volume" electrons.

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