ASYMPTOTIC PHOTON AND ELECTRON GREEN'S FUNCTIONS IN THE $\alpha(\alpha L)^n$ APPROXIMATION

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Submitted April 3, 1969

Zh. Eksp. Teor. Fiz. 57, 1280–1289 (October, 1969)

Asymptotic values of the renormalized (after Dyson) photon and electron Green's functions are obtained for large momenta in the so-called $\alpha(\alpha L)^n$ approximation (the definition of which is given in the text). The gauge transformation for the renormalized electron Green's function is considered.

1. INTRODUCTION

THE investigation of the asymptotic renormalized electrodynamic functions $D_{\mu\nu}$, G, and Γ_ν in the region of large momenta has been the subject of an appreciable number of papers[1-5], but this problem is far from completely solved and continues to remain of interest in many respects.

In the asymptotic region, the series of perturbation theory for the renormalized functions become double series in the parameters α and αL, where α is the renormalized coupling constant and L is a generalized symbol for the logarithms of the corresponding large momenta: $L \sim \ln(k^2/m^2)$, $\ln(p^2/m^2)$, etc. The character of these series can be presented schematically in the following form[1)]:

$$f(L,\alpha) = f_0(\alpha L) + \alpha f_1(\alpha L) + \alpha^2 f_2(\alpha L) + \ldots, \quad (1)$$

$$f_m(\alpha L) = \sum_{n=0}^{\infty} f_{mn}(\alpha L)^n. \quad (2)$$

In view of the smallness of α, we can confine ourselves, generally speaking, to several terms in the expansions of the type (1) with respect to α, but in the expansions (2) it suffices to confine oneself to a finite number of terms only in the momentum region $\alpha L \ll 1$, whereas in the region $\alpha L \sim 1$[2)] it is necessary to know the exact form of at least several of the first "coefficient functions" $f_m(\alpha L)$. The approximation in which only $f_0(\alpha L)$ is taken into account will be called the $(\alpha L)^n$ approximation, while allowance for the terms $\alpha f_1(\alpha L)$ will be called the $\alpha(\alpha L)^n$ approximation, etc.[8].

The need for considering the region $\alpha L \sim 1$ arises, for example, in such problems as the determination of the values of the renormalization constant[9], the problem of self-consistency of the renormalized quantum electrodynamics[10], the problem of the existence and properties of solutions of the "superconducting type" in quantum electrodynamics[8], and others. Studies of these problems have been limited so far to the $(\alpha L)^n$ approximation, but, as shown in[8], it is also important to take into account the $\alpha(\alpha L)^n$ approximation, and possibly also higher approximations.

In this paper we calculate in the $\alpha(\alpha L)^n$ approximation the Green's function $D_{\mu\nu}(k)$ and $G(p)$ renormalized after Dyson. This problem can be solved either on the basis of the Dyson equation[1] or with the aid of asymptotic functional relations of Gell-Mann and Low[2]. We use the second method, since it reaches our target in a shorter path.

We note that asymptotic forms of the Green's functions in a similar approximation were considered by Bogolyubov and Shirkov within the framework of the method of renormalization group[5] [3)]. The results obtained in[5], however, are insufficient for our purposes in two respects. First, in[5] are considered not ordinary Green's functions, renormalized after Dyson and dependent on the physical interaction constant α, but functions normalized at certain arbitrary points $k^2 = \lambda^2$ and $p^2 = \lambda'^2$ and dependent on the charge e^2; the correspondence between the asymptotic forms of both types of functions occurs only with logarithmic accuracy, i.e., when terms $\sim\alpha$ are neglected, and these terms must be taken into account in the $\alpha(\alpha L)^n$ approximation. Second, not all the coefficients of the logarithmic terms have been determined in[5]; in addition, the coefficient taken from[4] seems to us to be incorrect (cf. our formula (44)).

In view of the foregoing, we shall calculate here the asymptotic forms of the Dyson Green's functions in the $\alpha(\alpha L)^n$ approximation in an independent manner, with the aid of the Gell-Mann and Low functional relations. We note that this method, just like the renormalization-group method, is not applicable to the vertex function $\Gamma_\nu(p, q)$. The calculation of the $\alpha(\alpha L)^n$ asymptotic form of $\Gamma_\nu(p, q)$ will be carried out in the succeeding article with the aid of Dyson's equation.

2. SOLUTION OF THE ASYMPTOTIC FUNCTIONAL RELATIONS IN THE $\alpha(\alpha L)^n$ APPROXIMATION

The asymptotic functional relations of Gell-Mann and Low hold for the scalar functions d(k), A(p), and

1) We note that for the vertex function there exists also a "double logarithmic" region [6], which is not considered here; for details see article II, which is devoted to the vertex function (Zh. Eksp. Teor. Fiz. 57, 2198 (1969) [Sov. Phys. JETP 30, No. 12, 1970]).

2) It can be shown [7] that αL ceases to be a characteristic parameter of the expansion in the momentum region in which $\alpha L \gg 1$.

3) The $\alpha(\alpha L)^n$ asymptotic form of the part of the function G(p) which is odd in p_μ was considered earlier by Gor'kov [4]. His result, however, contains errors (compare with our formula (46)).

B(p), in terms of which the photon and electron Green's functions are expressed[4]:

$$D_{\mu\nu}(k)=\frac{d(k)}{ik^2}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right),\quad G^{-1}(p)=-\hat{p}A(p)+imB(p). \quad (3)$$

These relations are of the form[2,12]

$$\alpha d(k)=F\left(\varphi(\alpha)+\ln\frac{k^2}{m^2}\right), \quad (4)$$

$$A(p)=r_1(\lambda,\alpha)H_1\left(\varphi(\alpha)+\ln\frac{p^2}{m^2}\right),$$

$$B(p)=r_2(\lambda,\alpha)H_2\left(\varphi(\alpha)+\ln\frac{p^2}{m^2}\right),$$

$$k^2,\ p^2\gg m^2, \quad (5)$$

where F, φ, r_i, and H_i are unknown functions, and λ is a parameter introduced in order to eliminate the infrared divergence (for example, the photon "mass"). Relations (4) and (5), as will be shown below, make it possible to determine the functions d, A, and B with accuracy up to the $\alpha(\alpha L)^n$ approximation for the first four perturbation-theory terms for each of these functions (two terms of the form $(\alpha L)^n$ and two of the form $\alpha(\alpha L)^n$ at n = 0 and 1). (Thus using the results of perturbation theory, relations (4) and (5) assume the character of equations that makes it possible to additionally refine the form of the corresponding functions.)

For d(k), these four terms are[5]

$$d(k)=1+\frac{\alpha}{3\pi}\ln\frac{k^2}{m^2}-\frac{5\alpha}{9\pi}-\frac{13}{108}\frac{\alpha^2}{\pi^2}\ln\frac{k^2}{m^2}+\ldots. \quad (6)$$

Expanding formally the right side of relation (4) with respect to the parameter $\ln(k^2/m^2)$ up to the linear term and comparing the result with (6), we obtain two equations

$$F(\varphi(\alpha))=\alpha\left(1-\frac{5\alpha}{9\pi}+\ldots\right),\ \frac{d}{d\varphi(\alpha)}F(\varphi(\alpha))=\frac{\alpha^2}{3\pi}\left(1-\frac{13}{36}\frac{\alpha}{\pi}+\ldots\right), \quad (7)$$

which makes it possible to determine the form of the functions F and φ, and consequently also d(k), with the required accuracy. The result is of the

$$d(k)=\frac{1}{\eta}\left[1-\frac{5\alpha}{9\pi}\frac{1}{\eta}-\frac{3\alpha}{4\pi}\frac{\ln\eta}{\eta}\right],\quad \eta=1-\frac{\alpha}{3\pi}\ln\frac{k^2}{m^2}, \quad (8)$$

$$\varphi(\alpha)=-\frac{3\pi}{\alpha}-\frac{9}{4}\ln\frac{\alpha}{3\pi}. \quad (9)$$

Writing for the functions A and B expansions analogous to (6), in the form (here S = A, B)

$$S(p)=1+a_1\frac{\alpha}{\pi}\ln\frac{p^2}{m^2}+a_2\frac{\alpha}{\pi}+a_3\frac{\alpha^2}{\pi^2}\ln\frac{p^2}{m^2}+\ldots \quad (10)$$

and using expression (9) for $\varphi(\alpha)$, we can readily obtain on the basis of relations (5), in analogy with the procedure used for the function d(k), the following result:

$$S(p)=\xi^{-3a_1}\left\{1+\frac{\alpha}{\pi}\left[a_2+3\left(\frac{3}{4}a_1+a_1a_2-a_3\right)\left(1-\frac{1}{\xi}\right)-\frac{9a_1}{4}\frac{\ln\xi}{\xi}\right]\right\}$$

$$\xi=1-\frac{\alpha}{3\pi}\ln\frac{p^2}{m^2}. \quad (11)$$

The problem thus reduces to the calculation of the coefficients a_1, a_2, and a_3 for the functions A(p) and B(p). This problem is solved in the next section.

For the vertex function $\Gamma_\nu(p, q)$, the functional relations, analogous to (5), are of the form[12]

$$\Gamma^{(i)}(p,q)=R_i(\lambda,\alpha)\beta_i\left(\varphi(\alpha)+\ln\frac{p^2}{m^2},\ \varphi(\alpha)+\ln\frac{pq}{m^2},\ \varphi(\alpha)+\ln\frac{q^2}{m^2}\right),$$

$$p^2, q^2\gg m^2, \quad (12)$$

where $\Gamma^{(i)}$ is any of the scalar functions contained in $\Gamma_\nu(p, q)$. In the case $p^2 \gg q^2 \sim m^2$, Eq. (12) goes over into the simpler expression:

$$\Gamma^{(i)}(p,q)=R_i(\lambda,\alpha)\beta_i\left(\varphi(\alpha)+\ln\frac{p^2}{m^2},\ \varphi(\alpha)\right),\quad p^2\gg q^2\sim m^2. \quad (13)$$

Unlike the Green's functions considered above, the use of the results of perturbation theory for Γ_ν, even in the case (13), does not make it possible to refine the form of $\Gamma^{(i)}(p, q)$, since each term of the perturbation-theory series of the type (1) and (3), as can be readily verified, can be represented in the form (13). This is connected with the fact that the functions β_i in (13), unlike F and H_i in (4) and (5), depend not on one but on two arguments. To determine the asymptotic form of $\Gamma_\nu(p, q)$ it is necessary to use, consequently, more powerful methods, for example the Dyson-equation method.

3. CALCULATION OF THE COEFFICIENTS a_i^A AND a_i^B

In this section we obtain the coefficients a_i of the expansion of the form (10) for the functions A and B. In accordance with the assumed approximation, we represent any function f in the form

$$f=f_{(0)}+\tilde{f}, \quad (14)$$

where f(0) is the known contribution of the $(\alpha L)^n$ approximation, and $\tilde{f}$ is the sought contribution of the $\alpha(\alpha L)^n$ approximation, and to find a_i we shall need only the terms with n = 0 and 1. The functions A and B are connected with the renormalized self-energy part Σ_R by the relation

$$G^{-1}(p)=-\hat{p}A(p)+imB(p)=-\hat{p}+im-\Sigma_R(p). \quad (15)$$

The quantity Σ_R is expressed in terms of the corresponding non-regularized function $\Sigma(p)$ by the relation

$$\hat{p}-im+\Sigma_R(p)=Z_2[\hat{p}-im+\Sigma(p)-\Sigma(p_0)], \quad (16)$$

The renormalization constant Z_2, by virtue of the equality $Z_2 = Z_1$, is determined in terms of the vertex part Λ_μ, in accordance with

$$\Lambda_\mu(p_0,p_0)=\gamma_\mu(1-Z_2),\quad p_0^2=-m^2,\ \hat{p}_0=im. \quad (17)$$

The function $\Sigma(p)$ is determined by the integral

$$\Sigma(p)=\frac{\alpha}{4\pi^3}\int\gamma_\mu G(p-k)\Gamma_\nu(p-k,p)D_{\mu\nu}(k)\,d^4k. \quad (18)$$

Since (18) already contains the factor α, it suffices to substitute each of the functions G, Γ_ν, and $D_{\mu\nu}$ in the form (14), retaining in $f_{(0)}$ the terms ~1 and αL, and in $\tilde{f}$ only the term $\sim\alpha$. The contribution to $\Sigma(p)$ of interest to us will come from the following four integrals that result from this substitution:

[4]We adhere to the notation of the book [11]. The analysis is carried out in a transverse gauge (d_l = 0), since the functional relations (5) employed by us are valid for this case. The transition to an arbitrary gauge is effected is Sec. 4.

[5]The corresponding calculations were first carried out by Jost and Luttinger [13]. As a check, we calculated independently the coefficients in the expansion (6).

$$\Sigma(p) = I_0 + I_1 + I_2 + I_3, \quad (19)$$

$$I_0(p) = \frac{\alpha}{4\pi^3}\int \gamma_\mu G_{(0)}(p-k)\Gamma_{(0)\nu}(p-k,p)d_{(0)}(k)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{d^4k}{ik^2}, \quad (20)$$

$$I_1(p) = \frac{\alpha}{4\pi^3}\int \gamma_\mu \tilde{G}_{(1)}(p-k)\gamma_\nu\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{d^4k}{ik^2}, \quad (21)$$

$$I_2(p) = \frac{\alpha}{4\pi^3}\int \gamma_\mu S(p-k)\tilde{\Gamma}_{(1)\nu}(p-k,p)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{d^4k}{ik^2}, \quad (22)$$

$$I_3(p) = \frac{\alpha}{4\pi^3}\int \gamma_\mu S(p-k)\gamma_\nu \tilde{d}_{(1)}(k)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{d^4k}{ik^2}. \quad (23)$$

It is convenient to carry out the integration in four-dimensional spherical coordinates, going over to Euclidean space by rotating the contour[1] 6). The functions $d_{(0)}$, $D_{(0)}$, and $\Gamma_{(0)\nu}$ in the integral (20), taken accurate to terms with αL, are of the form[8]

$$d_{(0)}(k) = 1 + \frac{\alpha}{3\pi}\ln\frac{k^2}{m^2} + \ldots, \quad k^2 \gg m^2,$$

$$G_{(0)}(p-k) = [\hat{k} - \hat{p} + imB_{(0)}(p-k)]^{-1},$$

$$B_{(0)}(p-k) = 1 - \frac{3\alpha}{4\pi}\ln\frac{(p-k)^2}{m^2} + \ldots, \quad (p-k)^2 \gg m^2, \quad (24)$$

$$\Gamma_{(0)\nu}(p-k,p) = \begin{cases}\gamma_\nu, & k^2 \ll p^2\\ \gamma_\nu - \frac{\alpha}{8\pi k^2}[6imk_\nu + \hat{p}(\hat{k}\gamma_\nu - \gamma_\nu\hat{k})]\ln\frac{k^2}{p^2}, & k^2 \gg p^2 \gg m^2\end{cases}$$

Those terms of the integrand in (20), which contain $\ln k^2$ or $\ln(p-k)^2$, make a contribution $\sim\alpha^2 L$ only the regions $k^2 \gg p^2$ (in this case $\ln(k-p)^2 \to \ln k^2$) and $k^2 \ll p^2$ ($\ln(k-p)^2 \to \ln p^2$). In the region $k^2 \gg p^2$, integrals of the following type arise

$$\alpha\int_{k^2=p^2}^{\infty}\alpha\ln k^2\frac{d^4k}{k^5} = \alpha^2\ln p^2\int_{p^2}^{\infty}\frac{d^4k}{k^5} + O(\alpha^2), \quad (25)$$

and in the region $k^2 \ll p^2$, of the type

$$\alpha\int_0^{k^2=p^2}\alpha\ln k^2\frac{d^4k}{k^2} = \alpha^2\ln p^2\int_0^{p^2}\frac{d^4k}{k^2} + O(\alpha^2). \quad (26)$$

These considerations allow us to make effectively in the functions $G_{(0)}$, $\Gamma_{(0)\nu}$, and $d_{(0)}$ in (20) the substitution $\ln k^2 \to \ln p^2$. This is equivalent to replacing expressions (24) by the following expressions:

$$d_{(0)}(k) \to d_{(0)}(p) = 1 + \frac{\alpha}{3\pi}\ln\frac{p^2}{m^2} + \ldots,$$

$$G_{(0)}(p-k) \to [\hat{k} - \hat{p} + imB_{(0)}(p)]^{-1} \approx \frac{\hat{k} - \hat{p} - imB_{(0)}(p)}{(k-p)^2}, \quad p^2 \gg m^2,$$

$$\Gamma_{(0)\nu}(p-k,p) \to \gamma_\nu. \quad (27)$$

As a result of these simplifications, the integral (20) takes the form

$$I_0(p) = \frac{\alpha}{4i\pi^3}\int \gamma_\mu\frac{\hat{k} - \hat{p} - imB_{(0)}(p)}{(k-p)^2}\gamma_\nu\frac{d_{(0)}(p)}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)d^4k. \quad (28)$$

Further integration is simpler and leads to the result

$$I_0(p) = \frac{3\alpha}{4\pi}\left[\frac{1}{2}\hat{p}\left(1 + \frac{\alpha}{3\pi}\ln\frac{p^2}{m^2}\right) - im\left(\ln\frac{\Lambda^2}{m^2} + 1 - \frac{5\alpha}{12\pi}\ln\frac{p^2}{m^2}\right)\right]. \quad (29)$$

In the integrals (20)–(23), we denote by $\tilde{G}_{(1)}$, $\tilde{\Gamma}_{(1)\nu}$, and $\tilde{d}_{(1)}$ the terms $\sim\alpha$ in the corresponding functions. According to (6),

$$\tilde{d}_{(1)} = -5\alpha/9\pi, \quad (30)$$

whereas $\tilde{G}_{(1)}$ and $\tilde{\Gamma}_{(1)}$ must be calculated. The integrals (21)–(23) are proportional to α^2, and it is therefore necessary to separate in them the terms $\alpha^2 L$, which results from logarithmic integration in the region $k^2 \gg p^2 \gg m^2$. In this region the function

$$\tilde{G}_{(1)}(p-k) \equiv \left[(\hat{k} - \hat{p})\left(1 + a_2^A\frac{\alpha}{\pi}\right) + im\left(B_{(0)} + a_2^B\frac{\alpha}{\pi}\right)\right]^{-1} - [\hat{k} - \hat{p} + imB_{(0)}]^{-1}$$

takes the simpler form

$$\tilde{G}_{(1)}(p-k) \approx \frac{1}{k^2}\left[\left(\hat{p} - 2\frac{\hat{k}(kp)}{k^2} + 2im\right)a_2^A - ima_2^B\right]\frac{\alpha}{\pi}. \quad (31)$$

Substituting (31) in (21) and integrating in the region $p^2 \ll k^2 \ll \Lambda^2$ with logarithmic accuracy, we obtain

$$I_1(p) = \frac{3\alpha^2}{4\pi^2}im(2a_2^A - a_2^B)\ln\frac{\Lambda^2}{p^2}. \quad (32)$$

The function $\tilde{\Gamma}_{(1)\nu}$ in (22) is determined by the following integral (prior to regularization)

$$\Gamma_{(1)\nu}(p-k,p) = \frac{\alpha}{4\pi^3}\int \gamma_\sigma S(p-k-t)\gamma_\nu S(p-t)\gamma_\tau \times\left(\delta_{\sigma\tau} - \frac{t_\sigma t_\tau}{t^2}\right)\frac{d^4t}{it^2}. \quad (33)$$

Calculating this integral in the region $k^2 \gg p^2 \gg m^2$ with accuracy to terms linear in p/k and m/k, we obtain

$$\tilde{\Gamma}_{(1)\nu}(p-k,p) = \frac{\alpha}{4\pi}\left\{\frac{1}{2}\gamma_\nu + \frac{1}{k^2}\left[\hat{k}\gamma_\nu\hat{k} - \gamma_\nu\hat{p}\hat{k} + \frac{k_\nu\hat{k}\hat{p}\hat{k}}{k^2} + \frac{im}{2}\left(\frac{\hat{p}\hat{k}\gamma_\nu\hat{p}}{p^2} - \hat{k}\gamma_\nu\right)\right]\right\}. \quad (34)$$

The regularization reduces to subtraction of the quantity

$$\tilde{\Lambda}_{(1)\nu}(p_0,p_0) = \frac{\alpha}{4\pi^3}\int \gamma_\sigma S(p_0-t)\gamma_\nu S(p_0-t)\gamma_\tau\left(\delta_{\sigma\tau} - \frac{t_\sigma t_\tau}{t^2}\right)\frac{d^4t}{it^2} = \gamma_\nu\frac{\alpha}{4\pi}\left(\frac{3}{2} - \sigma_0\right), \quad (35)$$

$$\sigma_0 = \begin{cases} -3\left(\ln\frac{\lambda^2}{m^2} + 1\right)\\ -6\left(\ln\frac{\Delta^2}{m^2} + 1\right)\end{cases}$$

The quantity σ_0 has a different form, depending on whether the infrared divergence is eliminated by introducing the proton "mass" $\lambda \ll m$ or by introducing the "deviation off the mass shell" $\Delta^2 \equiv p_0^2 + m^2 \ll m^2$ is introduced.

Substituting the regularized expressions for $\tilde{\Gamma}_{(1)\nu}$ in (22) and carrying out logarithmic integration in the region $p^2 \ll k^2 \ll \Lambda^2$, we obtain

$$I_2(p) = \frac{3\alpha^2}{16\pi^2}\left[im\left(\frac{3}{2} - \sigma_0\right) - \frac{1}{2}\hat{p}\right]\ln\frac{\Lambda^2}{p^2}. \quad (36)$$

Analogously, taking (30) into account, we get for $I_3(p)$

$$I_3(p) = \frac{5}{12}\frac{\alpha^2}{\pi^2}im\ln\frac{\Lambda^2}{p^2}. \quad (37)$$

Substituting (29), (32), (36), and (37) in (19), we obtain

$$\Sigma(p) = \frac{3\alpha}{4\pi}\left\{\frac{\hat{p}}{2}\left[1 + \frac{\alpha}{3\pi}\ln\frac{p^2}{m^2} - \frac{\alpha}{4\pi}\ln\frac{\Lambda^2}{p^2}\right] - im\left[\ln\frac{\Lambda^2}{p^2} + 1 - \frac{5\alpha}{12\pi}\ln\frac{p^2}{m^2} - \left(2a_2^A - a_2^B - \frac{\sigma_0}{4} - \frac{67}{72}\right)\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{p^2}\right]\right\}. \quad (38)$$

The regularization of $\Sigma(p)$ is in accordance with formula (16). Taking into account the result (35) and the definition (17), we can write, with the required ac-

6) This can be done in the case of the space-like vector p_μ. The transition to the temporal p_μ is by analytic continuation.

curacy,

$$Z_2=1+\frac{\alpha}{4\pi}\left(\sigma_0-\frac{3}{2}\right)+X\frac{\alpha^2}{\pi^2}\ln\frac{\Lambda^2}{m^2}, \qquad (39)$$

and analogously for $\Sigma(p_0)$ (the term $\sim\alpha$ was taken from[11], formula (47.26))

$$\Sigma(p_0)=-\frac{3\alpha}{4\pi}im\left(\ln\frac{\Lambda^2}{m^2}+\frac{1}{2}\right)+im\,Y\frac{\alpha^2}{\pi^2}\ln\frac{\Lambda^2}{m^2}, \qquad (40)$$

where X and Y are certain numbers, which we determine henceforth from the renormalizability condition.

Substituting (38)–(40) in (16), we readily obtain

$$\Sigma_R(p)=(\hat{p}-im)\frac{\alpha}{4\pi}\sigma_0+\hat{p}\frac{\alpha^2}{\pi^2}\left[\frac{7}{32}\ln\frac{p^2}{m^2}+\left(X-\frac{3}{32}\right)\ln\frac{\Lambda^2}{m^2}\right]$$
$$+im\frac{3\alpha}{4\pi}\left\{\left[1+\frac{\alpha}{\pi}\left(\frac{\sigma_0}{2}-\frac{8}{9}-2a_2{}^A+a_2{}^B\right)\right]\ln\frac{p^2}{m^2}\right.$$
$$\left.+\left[2a_2{}^A-a_2{}^B-\frac{\sigma_0}{4}+\frac{67}{72}-\frac{4}{3}(X+Y)\right]\frac{\alpha}{\pi}\ln\frac{\Lambda^2}{m^2}\right\}. \qquad (41)$$

The first term in (41) yields, upon comparison with (10), $a_2^A = a_2^B = \sigma_0/4$. We now must substitute these values in (41). The numbers X and Y obviously should be such as to cause the terms $\sim\ln(\Lambda^2/m^2)$ in (41) to vanish, i.e.,

$$X={}^3/_{32},\qquad Y={}^{29}/_{48}\ {}^{7)}. \qquad (42)$$

The final expression for Σ_R is

$$\Sigma_R(p)=im\frac{3\alpha}{4\pi}\ln\frac{p^2}{m^2}+(\hat{p}-im)\frac{\alpha}{4\pi}\sigma_0$$
$$+\left[\hat{p}\frac{7}{32}-im\left(\frac{2}{3}-\frac{3}{16}\sigma_0\right)\right]\frac{\alpha^2}{\pi^2}\ln\frac{p^2}{m^2}. \qquad (43)$$

Comparing (43) with (15) and (10), we get

$$a_1{}^A=0,\quad a_2{}^A={}^1/_4\sigma_0,\quad a_3{}^A={}^7/_{32}, \qquad (44)$$

$$a_1{}^B=-{}^3/_4,\quad a_2{}^B={}^1/_4\sigma_0,\quad a_3{}^B={}^2/_3-{}^3/_{16}\sigma_0. \qquad (45)$$

Substituting (44), (45), in (11), we arrive at the result

$$A(p)=1+\frac{\alpha}{4\pi}\left[\sigma_0+\frac{21}{8}\left(\frac{1}{\xi}-1\right)\right], \qquad (46)$$

$$B(p)=\xi^{3/4}\left\{1+\frac{\alpha}{4\pi}\left[\sigma_0+\frac{59}{4}\left(\frac{1}{\xi}-1\right)+\frac{27}{4}\frac{\ln\xi}{\xi}\right]\right\}. \qquad (47)$$

4. GAUGE TRANSFORMATION OF THE FUNCTION G(p)

The results (46) and (47) were obtained in a transverse gauge. We now generalize with the case of an arbitrary gauge, when the renormalized photon Green's function is given by

$$D_{\mu\nu}(k)=\frac{d(k)}{ik^2}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+d_l\frac{k_\mu k_\nu}{ik^4}. \qquad (48)$$

(For simplicity we shall henceforth assume that d_l is a number independent of k.) For the renormalized electronic function in an arbitrary gauge we shall write, as before,

$$G^{-1}(p)=-\hat{p}A(p)+imB(p). \qquad (49)$$

We need to find first the gauge transformation for G(p). We start from the well known transformation[14]

$$\delta G^{(0)}(p)=\frac{i\alpha_0}{4\pi^3}\int\left[G^{(0)}(p)-G^{(0)}(p-k)\right]\frac{d^4k}{k^2(k^2+\lambda^2)}\delta d_l^{(0)} \qquad (50)$$

for the corresponding renormalization function $G^{(0)}_{(p)}$ connected with G(p) by the Dyson relation

$$G^{(0)}(p)=Z_2G(p). \qquad (51)$$

In (50), α_0 is a renormalized coupling constant, and $d^{(0)}_{(l)}$ is a "renormalized" longitudinal function connected with d_l in (48) by the equation

$$\alpha_0 d_l^{(0)}=\alpha d_l. \qquad (52)$$

Although the integral (50) converges in the region of small k, we have introduced infrared regularization, since an infrared divergence appears in the gauge transformation for the normalized function G(p), as we shall presently show. The explicit form of the transformation for G(p), consequently, depends on the method of infrared regularization. This transformation turns out to be simplest in the case of regularization with the aid of the photon "mass" λ. When λ is introduced, as is well known, the singularity of the function $G^0_{(p)}$ on the mass shell becomes a pole (and not a branch point as when λ = 0):

$$G^{(0)}(p)\approx Z_2(im-\hat{p})^{-1},\qquad \hat{p}\approx im. \qquad (53)$$

This circumstance makes it possible to write, on the basis of (50), neglecting the non-pole term $G^{(0)}(p-k)$, a simple gauge transformation of the renormalization constant Z_2[15]:

$$\delta Z_2=Z_2\frac{i\alpha_0}{4\pi^3}\int\frac{d^4k}{k^2(k^2+\lambda^2)}\delta d_l^{(0)} \qquad (54)$$

From (50)–(54) we get the following form of the transformation for G(p):

$$\delta G(p)=\frac{\alpha}{4i\pi^3}\int G(p-k)\frac{d^4k}{k^2(k^2+\lambda^2)}\delta d_l. \qquad (55)$$

The integration in (55), accurate to the $\alpha(\alpha L)^n$ terms considered by us, can be carried out in explicit form, after which we can change over from (55) to the corresponding integral transformation. The logarithmic integration in (55) arises in the region $k^2\ll p^2$, in which $G(p-k)\approx G(p)$, and is taken outside the integral sign. It suffices to carry out the nonlogarithmic (exact) integration with the $(\alpha L)^n$ part of the function G, which takes the form

$$G_{(0)}(p-k)=[(\hat{k}-\hat{p})A_{(0)}(p-k)+imB_{(0)}(p-k)]^{-1},$$

where $A_{(0)}(p-k)$ and $B_{(0)}(p-k)$ are functions of the argument $\alpha\ln[(p-k)^2/m^2]$. In the case of nonlogarithmic integration, this argument can be replaced effectively by $\alpha\ln(p^2/m^2)$, in analogy with the procedure used for the integral (20). The foregoing makes it possible to substitute in the integral (55), in place of the function G(p − k), the following effective form:

$$G(p-k)\to G(p)\Theta(p^2+k^2)+G'_{(0)}(p-k), \qquad (56)$$

where Θ is the usual step function,

$$G'_{(0)}(p-k)=[(\hat{k}-\hat{p})A_{(0)}(p)+imB_{(0)}(p)]^{-1}\approx$$
$$\approx\frac{(\hat{k}-\hat{p})A_{(0)}(p)-imB_{(0)}(p)}{(k-p)^2A^2_{(0)}(p)},\qquad p^2\gg m^2, \qquad (57)$$

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7) These numbers can, of course, be obtained also by an independent method, by calculating directly the terms $\sim\alpha^2\ln(\Lambda^2/m^2)$ in Z_2 and $\Sigma(p_0)$.

and in the integration of (57) it is necessary to separate only the $\alpha(\alpha L)^n$ terms, discarding the $(\alpha L)^n$ terms, since they are already fully taken into account for by the first terms of expression (56).

Substituting (56) in (55) and carrying out the integration in the indicated manner, we obtain (taking δd_l to be a number)

$$\delta G(p) = \frac{\alpha}{4\pi}\delta d_l\left\{G(p)\ln\frac{p^2}{\lambda^2} - \frac{im}{p^2}\frac{B_{(0)}(p)}{A^2_{(0)}(p)}\right\}. \qquad (58)$$

Owing to the presence of the factor α in (58), the quantities $B_{(0)}(p)$ and $A_{(0)}(p)$ can be replaced, without loss of accuracy, by $B(p)$ and $A(p)$. We then get from (58) and (49) the relations

$$\frac{\delta A(p)}{A(p)} = -\frac{\alpha}{4\pi}\ln\frac{p^2}{\lambda^2}\delta d_l, \quad \frac{\delta B(p)}{B(p)} = -\frac{\alpha}{4\pi}\left(\ln\frac{p^2}{\lambda^2} - 1\right)\delta d_l, \qquad (59)$$

integration of which with $\alpha(\alpha L)^n$ accuracy yields

$$A(p) = \exp\left\{-\frac{\alpha}{4\pi}d_l\ln\frac{p^2}{m^2}\right\}\left(1 + \frac{\alpha}{4\pi}d_l\ln\frac{\lambda^2}{m^2}\right)A(p)\Big|_{d_l=0} \qquad (60)$$

$$B(p) = \exp\left\{-\frac{\alpha}{4\pi}d_l\ln\frac{p^2}{m^2}\right\}\left[1 + \frac{\alpha}{4\pi}d_l\left(\ln\frac{\lambda^2}{m^2} + 1\right)\right]B(p)\Big|_{d_l=0}, \qquad (61)$$

Where A and B with $d_l = 0$ are given by formulas (46) and (47). It is convenient to represent the final result in the form

$$A(p) = A_{(0)}(p)\left\{1 + \frac{\alpha}{4\pi}\left[\sigma + \frac{21}{8}\left(\frac{1}{\xi} - 1\right)\right]\right\}, \qquad (62)$$

$$B(p) = B_{(0)}(p)\left\{1 + \frac{\alpha}{4\pi}\left[\sigma + d_l + \frac{59}{4}\left(\frac{1}{\xi} - 1\right) + \frac{27}{4}\frac{\ln\xi}{\xi}\right]\right\}, \qquad (63)$$

where

$$A_{(0)}(p) = \exp\left\{-\frac{\alpha}{4\pi}d_l\ln\frac{p^2}{m^2}\right\} = \exp\left\{-\frac{\alpha}{4\pi}\int_{m^2}^{p^2} d_l\frac{dk^2}{k^2}\right\},$$

$$B_{(0)}(p) = A_{(0)}(p)\xi^{9/4}, \quad \xi = 1 - \frac{\alpha}{3\pi}\ln\frac{p^2}{m^2},$$

$$\sigma = \sigma_0 + d_l\ln\frac{\lambda^2}{m^2} = (d_l - 3)\ln\frac{\lambda^2}{m^2} - 3. \qquad (64)$$

When the infrared divergence is eliminated with the aid of the parameter Δ^2 (see (35)), the functions A(p) and B(p) take on the same form (62) and (63), but the expression (64) for σ is now replaced by

$$\sigma = 2(d_l - 3)\ln\frac{\Delta^2}{m^2} - 6 + 3d_l. \qquad (64')$$

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Translated by J. G. Adashko
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