THE DISTURBANCES OF THE METRIC WHEN A COLLAPSING SPHERE PASSES BELOW THE SCHWARZSCHILD SPHERE

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A proof is given of the fact that in the collapse of a finite sphere the disturbances of the metric remain small at the surface of the sphere in a comoving reference system when the surface of the sphere passes below the Schwarzschild sphere.

It is well known that for a distant observer the collapse of a spherical body leads to a "frozen" pattern when the radius of the body becomes equal to the gravitational radius, $r_{sphere} = r_g = 2GM/c^2$, where M is the mass of the body, G is Newton's constant, and c is the speed of light.

The problem of the collapse of a body in which the distribution and motion of the matter deviate slightly from spherical symmetry has been solved in[1]. In that paper it is shown that in this case also the pattern for an external observer approaches a "frozen" condition, and the external gravitational field approaches a stationary field (all $\partial g_{\mu\nu}/\partial t \to 0$). The proof in[1] makes essential use of the fact that in the contraction of a sphere with initially small deviations from spherical symmetry the disturbances of the metric at its boundary r_{sphere} in a comoving reference system remain small for $r_{sphere} = r_g$. In this connection we referred to a paper by E. Lifshitz,[2] in which it is shown that the disturbances in a uniform medium increase without bound only when the density becomes infinite. Thorne[3] emphasizes that this reference is not really cogent, since E. Lifshitz was considering an unbounded uniform medium, whereas in collapse we are dealing with a bounded body and the effect of the surface may be important.

From a physical point of view, in the collapse of a finite body there is no reason to expect that in a comoving reference system the disturbance of the metric will increase without bound at $r_{sphere} = r_g$, since in this system the unperturbed solution has no singularities in the $g_{\mu\nu}$ or their derivatives (for a smooth fall of the density ρ at the boundary of the sphere), nor in the density or motion of the material. Nevertheless, the assertion calls for a mathematical treatment not based on such intuitive physical arguments. A proof of this kind is given in the present paper.

Let us consider a contracting sphere. For simplicity we assume dustlike matter, p = 0, and that the velocities of all its points are zero at infinity.[1)] In Lagrangian coordinates the solution of the Einstein equations for this problem can be written in the form[4]

$$ds^2 = d\tau^2 - e^\lambda dR^2 - r^2(R,\tau)(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$e^\lambda = (r')^2, \quad r = (3/2)^{2/3}F^{1/3}(R)(\tau_*(R) - \tau)^{2/3},$$
$$\varepsilon = F'(R)/r'r^2. \qquad (1)$$

Here we have set $8\pi G = 1$, and the prime denotes $\partial/\partial R$. The arbitrary function $\tau_*(R)$ can be fixed by the choice of scale of the coordinate R. We shall assume that the arbitrary function F(R), which describes the distribution of the matter in the sphere, is sufficiently smooth at the boundary (is differentiable a sufficient number of times at the point R_0 where the density becomes zero).

We are interested in the properties of the solution in the neighborhood of the world lines of the boundary of the sphere. We shall show that in the collapse of a body with initial small deviations from sphericity an arbitrary point on a world line of the boundary of the sphere remains nonsingular until the sphere contracts to a point. In other words, disturbances which are small for some τ = const, smaller than a given τ_0, do not become especially large in a finite neighborhood of τ_0 (with respect to R and τ and for all θ and φ). We are naturally primarily interested in the instant τ_0 at which the surface of the sphere in the unperturbed solution passes below the gravitational radius. For definiteness, let us speak of this instant.

The idea of the proof is as follows. We take the instant τ_0 in which we are interested, in the unperturbed solution. We shall now suppose that small perturbations are imposed on this solution at the instant τ_0 in the spatial region $R_0 \pm \Delta R$ and for all θ and φ. If the result of solving the Einstein equations for these perturbations shows that: 1) they remain small in a finite neighborhood, $\tau_0 \pm \Delta\tau$, $R_0 \pm \Delta R$, all θ and φ, and 2) they admit the necessary functional arbitrariness for the prescription of an arbitrary small perturbation, then the assertion will be proved. In fact, in this case arbitrary small perturbations at $\tau_1 = \tau - \Delta\tau$ in the region of $R_0 \pm \Delta R$ and all θ and φ remain small also at $\tau = \tau_0$ and at $\tau_2 = \tau \pm \Delta\tau$.

We now present the proof.

We denote the small corrections to the metric of the unperturbed solution (1) by h^i_k:

$$g^i_k = g^i_{k(0)} + h^i_k; \quad i, k = 0, 1, 2, 3. \qquad (2)$$

We require that after the disturbance the reference system still be a synchronous one[5]: $h^0_k = 0$. We shall

[1)]The proof can be extended to the case of a dustlike sphere with an arbitrary initial rate of contraction, and also to that of a sphere of matter with nonzero pressure.

look for the solution for the disturbances near a regular point in the form of series

$$h_\mu^\nu = a_\mu^\nu + b_\mu^\nu t + c_\mu^\nu t^2 + \ldots,$$
$$\mu, \nu = 1, 2, 3, \quad t = \tau - \tau_0. \tag{3}$$

The quantities a_μ^ν, b_μ^ν, c_μ^ν are (small) functions of the space coordinates.

We write

$$a = a_\alpha^\alpha, \quad b = b_\alpha^\alpha, \quad c = c_\alpha^\alpha, \quad \varkappa_{\alpha\beta} = \partial g_{\alpha\beta(0)}/\partial t, \quad \varkappa_\alpha^\beta = g_{(0)}^{\beta\gamma} \varkappa_{\alpha\gamma}.$$

The Einstein equations for the perturbations can be written in the form[5]

$$\delta R_k^i = \delta T_k^i - {}^1/_2 \delta_k^i \delta T. \tag{4}$$

For the perturbations δT_k^i we have

$$\delta T_0^0 = -\delta\varepsilon, \quad \delta T_\alpha^0 = u^0 \varepsilon \delta u_\alpha, \quad \delta T_\alpha^\beta = 0. \tag{5}$$

Substituting (3) and (5) in (4) and using the expressions for δR_k^i given in[5], we find for the terms of zeroth order in t:

$$2c + \varkappa_\alpha^\beta b_\beta^\alpha = -\delta\varepsilon, \tag{6}$$

$${}^1/_2 b_{;\alpha} - {}^1/_2 (b_\alpha^\beta + a_\alpha^\gamma \varkappa_\gamma^\beta - a_\gamma^\beta \varkappa_\alpha^\gamma)_{;\beta} + {}^1/_4 (\varkappa_\gamma^\beta a_{\beta;\alpha}^\gamma - \varkappa_\alpha^\beta a_{;\beta}) = u^0 \varepsilon \delta u_\alpha, \tag{7}$$

$$\delta P_\alpha^\beta + {}^1/_2 \{2c_\alpha^\beta + {}^1/_2 \varkappa_\alpha^\beta b - \varkappa_\alpha^\gamma b_\gamma^\beta + \dot\varkappa_\gamma^\beta b_\alpha^\gamma + \dot\varkappa_\gamma^\beta a_\alpha^\gamma - \varkappa_\alpha^\gamma a_\gamma^\beta$$
$$+ {}^1/_2 \varkappa (b_\alpha^\beta - \varkappa_\alpha^\gamma a_\gamma^\beta + \varkappa_\gamma^\beta a_\alpha^\gamma)\} = {}^1/_2 \delta\varepsilon \delta_\alpha^\beta. \tag{8}$$

Here P_α^β is the three-dimensional Ricci tensor, and $\delta\epsilon$ and δu_α are the perturbations of the energy ϵ and the velocity u_α at the instant τ_0. The semicolon indicates covariant differentiation, and the dot, differentiation with respect to τ.

Contracting (8) and substituting in (6) instead of c, we get a differential relation for a_α^β, b_α^β, and $\delta\epsilon$. From (8) we can determine all the c_α^β in terms of a_α^β, b_α^β, and $\delta\epsilon$. The expressions (7) can be written in a form solved for the derivatives of b_1^3, b_2^3, and b_2^2 with respect to R:

$$\frac{\partial b_1^3}{\partial R} = F_1, \quad \frac{\partial b_2^3}{\partial R} = F_2, \quad \frac{\partial b_2^2}{\partial R} = F_3. \tag{9}$$

Here F_1, F_2, and F_3 are functions of a_μ^ν, b_μ^ν, their first derivatives [but not of the right members of (9)], and δu_α. Equation (6) can be solved for $\partial^2 a_2^2/\partial R^2$:

$$\partial^2 a_2^2 / \partial R^2 = F_4, \tag{10}$$

where F_4 depends on the a_μ^ν and their derivatives (but not on $\partial^2 a_2^2/\partial R^2$). Thus if we prescribe arbitrarily, for example, the functions a_1^1, a_1^2, a_1^3, a_3^2, b_1^1, b_1^2, b_3^3, $\delta\epsilon$, δu_α, then from (9) and (10) we get a system of the Cauchy-Kowalewski type. The functions b_1^3, b_2^3, b_2^2, and a_2^2 can be found in this way from (9) and (10) in the neighborhood of R_0 for all θ and φ.

We have obtained a solution of the problem which depends on twelve arbitrary functions. An arbitrariness by four functions is due to the arbitrariness in the choice of the perturbed reference system with $h_k^0 = 0$ (for the formulas see[5]). There remains an arbitrariness by eight functions of the three spatial coordinates. As is well known, a solution containing eight different arbitrary functions is a general solution.

Accordingly, we have shown that arbitrary small disturbances of the matter and field near the surface of the sphere remain small in the comoving perturbed solution for the reference system as the sphere passes below the Schwarzschild surface. The proof that in this case they always remain small in vacuum near $r = r_g$ and in the entire external space $r > r_g$ is given in[1].

[1] A. G. Doroshkevich, Ya. B. Zel'dovich, and I. D. Novikov, Zh. Eksp. Teor. Fiz. **49**, 170 (1965) [Sov. Phys.-JETP **22**, 122 (1966)].

[2] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **16**, 587 (1946).

[3] K. S. Thorne, Gravitational Collapse, Preprint, California, 1968.

[4] L. D. Landau and E. M. Lifshitz, Classical Theory of Fields, Reading, Mass., Addison-Wesley, 1962.

[5] E. M. Lifshitz and I. M. Khalatnikov, Usp. Fiz. Nauk **80**, 391 (1963) [Sov. Phys.-Uspekhi **6**, 495 (1964)].

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