THEORY OF WAVES CLOSE TO THE EXACT SOLUTIONS OF NONLINEAR ELECTRODYNAMICS AND OPTICS. II

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Asymptotic expansions of the nonlinear field solutions are set up near the singular wave-vector values for which the usual single-mode expansion in powers of the amplitude is impossible. In this case one either has to solve the nonlinear one-dimensional equation for the principal mode or, what is simpler, to employ several harmonic modes. Two-dimensional solutions that are close to the exact nonlinear periodic one-dimensional waves are set up. Regions of permissible values of the wave vectors are determined.

1. INTRODUCTION

As was already shown by us earlier[1], a productive method in the theory of nonlinear electromagnetic waves for two-dimensional (and three-dimensional) problems is that of solving the equations of nonlinear electrodynamics with the aid of asymptotic expansions in terms of a small nonlinearity. This has made it possible to construct a number of wave solutions of the nonlinear field equations, both for the case of small electric field amplitudes and for the case of a strong field, when the asymptotic expansions are constructed near the exact one-dimensional solutions. The theory of nonlinear waves developed in[1] was connected with power-law expansions of the solutions in terms of small amplitudes. In analogy with the ordinary oscillation theory[2], the elimination of the secularities called for the introduction of a dependence of the wave vectors on the wave amplitudes. The nonlinear waves investigated in[1] were characterized by asymptotic power expansions of the equations that determine the dependence of the wave vectors on the amplitudes. At the same time, it was observed in many cases that for certain expansion coefficients, singularities arise near definite values of the wave vectors (the so-called small divisors). This indicates that the simple power-law expansions used for the construction of the nonlinear waves in[1] cannot be used in the vicinity of such singular values of the wave vectors.

The present communication is devoted to the development of a theory of nonlinear electromagnetic waves, which makes it possible to construct asymptotic expansions of the solutions of the nonlinear field equations near the singular values of the wave vectors. As will be shown below, this affords two possibilities. First, the systems of linear equations obtained for the field amplitudes in the simple expansions in powers of the amplitudes give way, even for the fundamental amplitude of the harmonic expansion of the wave in terms of any particular variable, to a nonlinear differential equation with respect to another variable. In other words, the harmonic expansion of the fields in terms of all the variables is not productive near the singular values of the wave vectors. In the second case, to the contrary, the effectiveness of such harmonic expansions is retained, but the amplitudes of two (or more) modes become commensurate, and this is to some degree analogous to the situation characteristic of the intersection of molecular terms.

Just as in[1], we shall focus our attention on the study of wave solutions for an electric field in the form

$$E(\mathbf{r},\ t) = E(\mathbf{r})\sin[\omega t + \Psi(\mathbf{r})], \tag{1.1}$$

which are described by the equations[1,3]

$$\Delta E + [k_\omega{}^2 - (\nabla\Psi)^2 - \varkappa^2 + \varkappa^2(E^2/E_c{}^2)]E = 0, \tag{1.2}$$

$$\operatorname{div}[E^2\nabla\Psi] = 0. \tag{1.3}$$

Here $k_\omega^2 = (\omega/c)^2$, E_c is an electric field value characteristic of the nonlinear properties of the medium, and

$$\varkappa^2 = \frac{\omega^2}{c^2}[1-\varepsilon(\omega)], \tag{1.4}$$

where $\epsilon(\omega)$ is the ordinary linear dielectric constant. Just as in[1], we confine ourselves in the discussion to two-dimensional solutions, which make it possible to understand many essential features of three-dimensional solutions, too.

2. STANDING WEAKLY-LINEAR WAVES

The simplest case of deviation from the power-law expansions occurs in the case of small-amplitude standing waves (Ψ = const), when $E = aE_c e(x, z)$, and a can be regarded as a small quantity. Then Eq. (1.2) makes it possible to represent the first terms of the asymptotic expansion obtained in[1] in the following form:

$$e(x,\ z) = \cos k_\perp x\cos k_\parallel z + \frac{(a\varkappa)^2}{128}\Big[\frac{3}{k_\perp{}^2}\cos 3k_\perp x\cos k_\parallel z + \frac{3}{k_\parallel{}^2}\cos k_\perp x\cos 3k_\parallel z + \frac{1}{k_\omega{}^2-\varkappa^2}\cos 3k_\perp x\cos 3k_\parallel z\Big] + \ldots \tag{2.1}$$

The longitudinal and the transverse components of the wave vectors are connected by the relation

$$k_\perp{}^2 + k_\parallel{}^2 = k_\omega{}^2 - \varkappa^2 + \frac{9}{16}(a\varkappa)^2 + \frac{3(a\varkappa)^4}{2048}\Big[\frac{1}{k_\omega{}^2-\varkappa^2} + \frac{9}{k_\perp{}^2} + \frac{9}{k_\parallel{}^2}\Big] + \ldots \tag{2.2}$$

As was already indicated in[1], the appearance of small divisors (when $k_\perp \to 0$ or $k_\| \to 0$) signifies that the parameter a loses the meaning of the amplitude of the fundamental two-dimensional mode, since, for example when $k_\| \to 0$, not only $\cos k_\perp x \cos k_\| z$, but also the higher modes of the type $\cos nk_\| z \cos k_\perp x$ reduce to the fundamental two-dimensional mode. This statement actually means that at small values of $k_\|$ the field distributions in the medium must be sought in the form of one-dimensional harmonic expansions

$$e(x, z) = \sum_{n \geqslant 0} e_{2n+1}(z) \cos(2n+1) k_\perp x. \qquad (2.3)$$

Equation (1.2) yields for the functions e_{2n+1} an infinite system of one-dimensional nonlinear diffraction equations, which can be solved under the assumption that e_1 is much larger than the remaining amplitudes of the expansion (2.3). Namely, $e_{2n+1} \sim a^{2n}$. With this, in the region of small $k_\|$, which is of interest to us now, we have $k_\perp^2 \sim k_\omega^2 - \kappa^2$. Using this fact, we can obtain a system of successive-approximation equations.

In the first approximation we obtained for the amplitude e_1

$$e_1'' + (k_\omega^2 - \varkappa^2 - k_\perp^2) e_1 = -\frac{3}{4}(a\varkappa)^2 e_1^3. \qquad (2.4)$$

Retention of the nonlinear terms becomes necessary in the case of a slow z-dependence of the solutions, as is the case under the conditions when the coefficient preceding e_1 in the left side of (2.4) is small.

In the next approximation, the amplitude e_3 becomes different from zero, and the equation for it is

$$e_3'' + (k_\omega^2 - \varkappa^2 - 9k_\perp^2) e_3 = -{}^1/_4 (a\varkappa)^2 e_1^3. \qquad (2.5)$$

such linear inhomogeneous equations determine the amplitudes of the higher harmonics.

The first-approximation equation (2.4) can be readily seen to have the following solution

$$e_1 = \mathrm{cn}\left(z\sqrt{k_\omega^2 - \varkappa^2 - k_\perp^2 + {}^3/_4 (a\varkappa)^2}, k_\|\right), \quad k_\omega^2 - \varkappa^2 - k_\perp^2 > -{}^3/_8 (a\varkappa)^2. \qquad (2.6)$$

$$e_1 = \mathrm{dn}\left(\sqrt{{}^3/_8}\, a\varkappa z, k_\|^{-1}\right), \quad -{}^3/_8 (a\varkappa)^2 > k_\omega^2 - \varkappa^2 - k_\perp^2 > -{}^3/_4 (a\varkappa)^2. \qquad (2.7)$$

Here $\mathrm{dn}^2(y, k) \equiv 1 - k^2 \mathrm{sn}^2(y, k)$, where $\mathrm{sn}(y, k)$ and $\mathrm{cn}(y, k)$ are the Jacobi elliptic sine and cosine. The modulus of the elliptic functions is determined by the formula

$$k_\| = \sqrt{{}^3/_8}\, a \Big/ \sqrt{k_\omega^2 - \varkappa^2 - k_\perp^2 + {}^3/_4 (a\varkappa)^2}. \qquad (2.8)$$

Periodic solutions of (2.6) and (2.7) correspond to longitudinal components of the wave vector

$$k_\| = \frac{\pi}{2\mathrm{K}(k_\|)} \sqrt{k_\omega^2 - \varkappa^2 - k_\perp^2 + {}^3/_4 (a\varkappa)^2} \qquad (2.9)$$

$$k_\| = \pi \sqrt{3}\, a\varkappa \,/\, 2\mathrm{K}(k_\|^{-1}), \qquad (2.10)$$

where K is a complete elliptic integral of the first kind.

In the region of the small divisor $k_\omega^2 - \kappa^2 - k_\perp^2 \lesssim (a\kappa)^2$, and the z-dependence of the fundamental mode $e_1(z)$ is slow, while $k_\| \sim a\kappa$. When $k_\omega^2 - \kappa^2 - k_\perp^2 \to -({}^3/_8)(a\kappa)^2 \pm 0$, the solutions (2.6) and (2.7) go over into a bounded non-periodic solution

$$e_1(z) = \mathrm{ch}^{-1}(\sqrt{3}\, a\varkappa z / 2). \qquad (2.11)$$

The solution of the second-approximation equation (2.5) can be readily obtained and, for example for the case of (2.7) at small values of $k_\|$, we get

$$e_3(z) \sim \frac{(a\varkappa)^2}{4(9k_\perp^2 + \varkappa^2 - k_\omega^2)} \mathrm{cn}^3\left(z\sqrt{k_\omega^2 - \varkappa^2 + \frac{3(a\varkappa)^2}{4}}\right). \qquad (2.12)$$

The higher amplitudes of the asymptotic expansion are determined in similar fashion. It should be noted that the transition from the asymptotic expansions of the type (2.1), based on the use of the principal mode of the linear approximation, to the asymptotic expansions using the principal mode (2.6) and (2.7), actually correspond to separation of the terms that deviate from the expansion (2.1) at small values of $k_\|$ in the summation of the most important infinite sequence in this expansion.

3. WAVE IN A MEDIUM WITH A NONLINEAR PLANE WAVE

The problem of small divisors was raised in[1] also in a construction of solutions that are close to the exact solution, with constant amplitude of the electric field. Here

$$E(x, z) = \frac{E_c}{\varkappa} \sqrt{\varkappa^2 + k_\infty^2 - k_\omega^2}\, [1 + ae(x, z)], \qquad (3.1)$$
$$\Psi(x, z) = -k_\infty z + a\psi(x, z).$$

To construct a weakly-nonlinear non-one-dimensional field distribution that is suitable in the region of the small divisors (see[1]), we represent the sought amplitude and phase in the form

$$e(x, z) = \cos\chi_\perp \xi \cos\chi_\| \zeta + a_\perp \cos 2\chi_\perp \xi + a_\| \cos 2\chi_\| \zeta + \alpha(\xi, \zeta), \qquad (3.2)$$
$$\psi(x, z) = b \cos\chi_\perp \xi \sin\chi_\| \zeta + b_\| \sin 2\chi_\| \zeta + \beta(\xi, \zeta),$$

where

$$\xi = \sqrt{2(\varkappa^2 + k_\infty^2 - k_\omega^2)}\, x, \quad \zeta = \sqrt{2(\varkappa^2 + k_\infty^2 - k_\omega^2)}\, z, \ a\ 1,\ a_\perp,\ a_\|,\ b,\ b_\|$$

and 1, $a_\perp$, $a_\|$, b, and $b_\|$ are the total amplitudes of the separated modes, while the functions α and β are orthogonal to the separated modes. Substitution of (3.1) and (3.2) in (1.2) and (1.3) leads to a system of equations for the separated amplitudes and for the functions α and β. In the first approximation, the system determining the amplitudes of the separated modes leads to

$$a_\perp \approx -\frac{3}{8} \frac{a}{1 - 4\chi_\perp^2} \left[1 + \frac{16}{3} \frac{(\chi_\infty \chi_\| \chi_\perp)^2}{(\chi_\perp^2 + \chi_\|^2)^2}\right], \qquad (3.3)$$

$$a_\| \approx -\frac{3}{8} \frac{a}{1 + 4\chi_\infty^2 - 4\chi_\|^2} \left[1 + \frac{4}{3}\chi_\infty^2 - \frac{16}{3} \frac{(\chi_\infty \chi_\|^2)^2}{(\chi_\perp^2 + \chi_\|^2)^2}\right], \qquad (3.4)$$

$$b_\| \approx \frac{\chi_\infty}{\chi_\|} \left[a_\| + \left(1 - \frac{4\chi_\|^2}{\chi_\perp^2 + \chi_\|^2}\right) a\right]. \qquad (3.5)$$

Using (3.3)–(3.5), we obtain an equation for the connection between the projections of the wave vector:

$$1 - \chi_\perp^2 - \chi_\|^2 + 4\frac{(\chi_\infty \chi_\|)^2}{\chi_\perp^2 + \chi_\|^2} = \frac{9}{16} a^2 \left\{ \frac{1}{1 - 4\chi_\perp^2} \left[1 + \frac{16}{3} \frac{(\chi_\infty \chi_\| \chi_\perp)^2}{(\chi_\perp^2 + \chi_\|^2)^2}\right] + \frac{1}{1 + 4\chi_\infty^2 - 4\chi_\|^2} \left[1 + \frac{4}{3}\chi_\infty^2 - \frac{16}{3} \frac{(\chi_\infty \chi_\|)^2}{\chi_\perp^2 + \chi_\|^2}\right] \left[1 - \frac{4}{3}\chi_\infty^2 - \frac{16}{3} \frac{(\chi_\infty \chi_\|^2)^2}{(\chi_\perp^2 + \chi_\|^2)^2} + \frac{32}{3} \frac{(\chi_\infty \chi_\|)^2}{\chi_\perp^2 + \chi_\|^2}\right] \right\} + \frac{1}{4} a^2 \chi_\infty^2 \left(1 + \frac{4\chi_\|^2}{\chi_\perp^2 + \chi_\|^2}\right)^2. \qquad (3.6)$$

We note that in the derivation of (3.6) we did not use the assumption that the amplitudes $a_\perp$ and $a_\|$ of the separated modes are small compared with unity, i.e., compared with the amplitude of the fundamental one-dimensional mode. To determine the result of such a procedure, we shall investigate further, at $\chi_\infty = 0$, the region of the small divisor $\chi_\perp^2 \sim 3/4$, $\chi_\parallel^2 \sim 1/4$. Let $\chi_\perp^2 = 3/4 + \Delta_\perp$, $|\Delta_\perp| \ll 1$. Then (3.6) leads to

$$(\chi_\parallel^2)_\pm \approx {}^1/_4\{1 - {}^1/_2\Delta_\perp \pm \sqrt{{}^1/_4\Delta_\perp^2 + {}^9/_{64}a^2}\} \tag{3.7}$$

and to the following expression for the amplitude:

$$(a_\parallel)_\pm \approx -\frac{3}{4}\frac{a}{\Delta_\perp \mp (\Delta_\perp^2 + {}^9/_{16}a^2)^{1/2}}. \tag{3.8}$$

Let us consider one of the branches, for example the one corresponding to $(\chi_\parallel^2)_+$, $(a_\parallel)_+$. Outside the region of the small divisor we have $|\Delta_\perp| \gg a^2$, and when $\Delta_\perp < 0$ relations (3.7) and (3.8) lead to

$$(\chi_\parallel^2)_+ \sim {}^1/_4(1 - \Delta_\perp), \quad (a_\parallel)_+ \sim 3a/8\Delta_\perp. \tag{3.9}$$

When $\Delta_\perp > 0$, the amplitude $(a_\parallel)_+$ increases on going through the region of the small divisor, and $(\chi_\parallel^2)_+ \to 1/4$. In the vicinity of the small divisor, under the condition that $|\Delta_\perp| \ll a^2$, relations (3.7) and (3.8) lead to

$$(\chi_\parallel^2)_+ \sim {}^1/_4(1 + {}^3/_8 a), \ (a_\parallel)_+ \approx 1 \tag{3.10}$$

For the other branch, relations similar to (3.9) and (3.10), are of the form

$$\begin{gathered}(\chi_\parallel^2)_- \sim {}^1/_4(1 - \Delta_\perp), \quad (a_\parallel)_- \sim -3a/8\Delta_\perp \\ \text{for } \Delta_\perp > 0, \ |\Delta_\perp| \gg a^2, \quad (\chi_\parallel^2)_- \sim {}^1/_4(1 - {}^3/_8 a), \\ (a_\parallel)_- \sim -1 \quad \text{for } |\Delta_\perp| \ll 1.\end{gathered} \tag{3.11}$$

We note that the considered branches lie on opposite sides of the straight line $\chi_\perp^2 + \chi_\parallel^2 = 1$, which characterizes the linearized problem, and that the amplitudes grow in opposite directions.

The curve corresponding to the implicit function (3.6) at $\chi_\infty = 0$ is shown in Fig. 1. Outside the region of the small divisors, only one of the branches of the curve (3.6) is close to the straight line $\chi_\perp^2 + \chi_\parallel^2 = 1$ when $a^2 \ll 1$, whereas in the region of the small divisors a pair of branches of the curve (3.2) is simultaneously close to the aforementioned line. The curves shown in Fig. 1 indicates that, on passing through the region of the small divisor, the branch corresponding to $\chi_\perp^2 + \chi_\parallel^2 \gtrsim 1$ moves away from the straight line $\chi_\perp^2 + \chi_\parallel^2 = 1$ and approaches the asymptotes $\chi_\parallel^2 = 1/4$ and $\chi_\perp^2 = 1/4$. Such a behavior of this branch shows that on passing through the region of the small divisor the two-dimensional field distribution goes over into a distribution close to the exact one-dimensional periodic solution. The latter, of course, is determined far away from the region of the small divisor by the exact nonlinear equation (see below).

We present an expression for the first-approximation correction $\alpha(\xi, \zeta)$ to the distribution of three

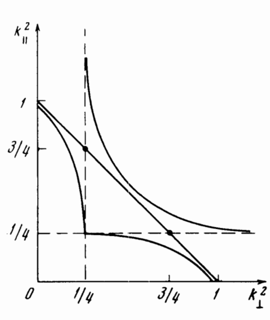

FIG. 1

modes ($\chi_\infty = 0$):

$$\begin{aligned}\alpha^{(1)}(\xi, \zeta) = -3a\Big\{&\frac{1}{8} + \frac{a_\perp^2 + a_\parallel^2}{4} + \Big(\frac{1}{8} + a_\perp a_\parallel\Big)\frac{\cos 2\chi_\perp\xi \cos 2\chi_\parallel\zeta}{1 - 4\chi_\perp^2 - 4\chi_\parallel^2} \\ &+ \frac{1}{4}a_\perp^2\frac{\cos 4\chi_\perp\xi}{1 - 16\chi_\perp^2} + \frac{1}{4}a_\parallel^2\frac{\cos 4\chi_\parallel\zeta}{1 - 16\chi_\parallel^2} \\ &+ \frac{1}{2}a_\perp^2\frac{\cos 3\chi_\perp\xi \cos \chi_\parallel\zeta}{1 - 9\chi_\perp^2 - \chi_\parallel^2} + \frac{1}{2}a_\parallel^2\frac{\cos \chi_\perp\xi \cos 3\chi_\parallel\zeta}{1 - \chi_\perp^2 - 9\chi_\parallel^2}\Big\}.\end{aligned} \tag{3.12}$$

From this we can readily determine the values of the wave vectors at which new small divisors appear. However, even for these, the solutions can be analyzed in the manner indicated above, the only difference being that the approach of other modes must be considered.

4. WAVES CLOSE TO A STANDING PERIODIC WAVE OF A STRONG FIELD

We now turn to non-one-dimensional field distributions close to the exact one-dimensional solutions of the nonlinear field equations. Accordingly, the one-dimensional solution satisfies the equation

$$e_\perp'' + e_\perp + {}^3/_2 A_\perp e_\perp^2 + {}^1/_2 A_\perp^2 e_\perp^3 = 0, \tag{4.1}$$

where $A_\perp$ can be called the amplitude of the exact one-dimensional distribution.

For the amplitudes $A_\perp \le \sqrt{2} - 1$, the solution of (4.1) is given by

$$e_\perp(\xi) = (1 + A_\perp)\,\mathrm{dn}[{}^1/_2(1 + A_\perp)\xi, \ k_\perp] - 1, \tag{4.2}$$

where the modulus of the elliptic function $k_\perp$ and the wave number $\chi_\perp$ characterizing the fundamental period of the field are determined by the expressions

$$k_\perp = \frac{\sqrt{2A_\perp(2 + A_\perp)}}{1 + A_\perp}, \quad \chi_\perp = \pi\frac{1 + A_\perp}{2K(k_\perp)} \tag{4.3}$$

In particular, when $A_\perp \ll 1$, relations (4.2) and (4.3) yield

$$e_\perp(\xi) \sim \cos\xi, \quad \chi_\perp \sim 1 + A_\perp. \tag{4.4}$$

In the limit as $A \to \sqrt{2} - 1 - 0$, when the modulus of the elliptic function tends to unity, we have $K(k_\perp) \sim \ln(4/\sqrt{1 - k_\perp^2})$ and $\chi_\perp \to 0$. Then the solution (4.2) degenerates into an aperiodic solution

$$\sqrt{2}\,\mathrm{ch}^{-1}(\xi/\sqrt{2}) - 1. \tag{4.5}$$

For the amplitudes $A_\perp > \sqrt{2} - 1$, the solution of (4.1) is

$$e_\perp(\xi) = (1 + A_\perp)\,\mathrm{cn}\left[\frac{\xi}{\sqrt{2}}\sqrt{(1 + A_\perp)^2 - 1}, \ k_\perp^{-1}\right] \tag{4.6}$$

and when $A_\perp \to \sqrt{2} - 1 + 0$ it also goes over into (4.5).

To construct non-one-dimensiona field solutions close to the one-dimensional solutions (4.2) and (4.6), we put

$$e(\xi, \zeta) = e_\perp(\xi) + a\varepsilon(\xi, \zeta). \tag{4.7}$$

In the linear approximation we get a solution of the type

$$\varepsilon(\xi, \zeta) = E(\xi, \chi_\parallel)\cos\chi_\parallel\zeta, \tag{4.8}$$

where $E(\xi, \chi_\parallel)$ is an eigenfunction of the equation

$$\left[\frac{d^2}{d\xi^2} + 1 - \chi_\parallel^2 + 3A_\perp e_\perp(\xi) + \frac{3}{2}A_\perp^2 e_\perp^2(\xi)\right]E(\xi, \chi_\parallel) = 0. \tag{4.9}$$

Of direct interest for our purposes are only those eigenfunctions of (4.9) which lead to non-negative values of the proper parameter $\chi_{\parallel}^2$. Assume for concreteness that $A_{\perp} < \sqrt{2} - 1$. Then, using the explicit form of the one-dimensional distribution (5.2) and going over to a new independent variable $y = (1 + A_{\perp})\xi/2$, we transform (4.9) to the Jacobi form for the Lame equation[4]

$$\left[\frac{d^2}{dy^2} + \gamma - 6k_{\perp}^2 \mathrm{sn}^2(y, k_{\perp}) \right] E(y, \gamma) = 0. \tag{4.10}$$

Here $\gamma/2 \equiv 3 - (1 + 2\chi_{\parallel}^2)/(1 + A_{\perp})^2$, and the condition that $\chi_{\parallel}^2$ be non-negative leads to the inequality $\gamma \leq 4 + k_{\perp}^2$, which determines that part of the spectrum of the eigenvalues of the Lame operator which leads to bounded solutions of the linearized problem in terms of the variable ζ.

Let us investigate the periodic solutions of (4.10). On the basis of the theory of the Lame equation[4] we reach the conclusion that equation (4.10) admits of five periodic solutions represented by polynomials of elliptic functions. Out of the five Lame polynomials, four satisfy the condition that $\chi_{\parallel}^2$ be non-negative. Using the recurrence relations given in[4], we obtain the explicit form of the Lame polynomials:

$$\begin{gathered} \mathrm{Ec}^- = 1 - \frac{H_-}{2k_{\perp}^2} + \frac{H_-}{k_{\perp}^2}\mathrm{sn}^2(y, k_{\perp}), \\ \mathrm{Es}^- = \mathrm{cn}(y, k_{\perp})\,\mathrm{dn}(y, k_{\perp}), \\ \mathrm{Es}^0 = \mathrm{sn}(y, k_{\perp})\,\mathrm{cn}(y, k_{\perp}), \\ \mathrm{Ec}^0 = \mathrm{sn}(y, k_{\perp})\,\mathrm{dn}(y, k_{\perp}), \end{gathered} \tag{4.11}$$

where

$$H_-/2 = 2 - k_{\perp}^2 - \sqrt{(2 - k_{\perp}^2)^2 + 3k_{\perp}^4}.$$

Further, it can be shown that when $A_{\perp} > \sqrt{2} - 1$, the periodic solutions represented by Lame polynomials coincide with the polynomials (4.11), but the modulus of the elliptic functions becomes in this case $k_{\perp}^{-1}$. The eigenvalues of the longitudinal wave number, corresponding to the eigenfunctions (4.11), are given by

$$\begin{gathered} \chi_{\parallel}^2(\mathrm{Ec}^-) = \tfrac{1}{2}\{1 + [4 - 6(1 + A_{\perp})^2 + 3(1 + A_{\perp})^4]^{1/2}\}, \quad A_{\perp} \gtrless \sqrt{2} - 1, \\ \chi_{\parallel}^2(\mathrm{Es}^-) = \tfrac{3}{4}(1 + A_{\perp})^2, \; A_{\perp} \gtrless \sqrt{2} - 1, \; \chi_{\parallel}^2(\mathrm{Es}^0) = 0, \; A_{\perp} < \sqrt{2} - 1, \\ \chi_{\parallel}^2(\mathrm{Ec}^0) = 0, \; A_{\perp} > \sqrt{2} - 1, \quad \chi_{\parallel}^2(\mathrm{Ec}^0) = \tfrac{3}{4}[2 - (1 + A_{\perp})^2], \\ A_{\perp} < \sqrt{2} - 1. \end{gathered} \tag{4.12}$$

In the plane $(\chi_{\parallel}^2, A_{\perp})$, the curves corresponding to (4.12) break up this plane into regions in which bounded solutions of (4.10) exist or do not exist. Besides the finite number of periodic eigenfunctions represented by the Lame polynomials, there exists, as is well known[4],

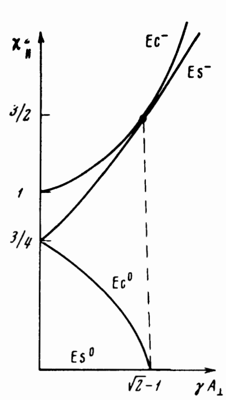

FIG. 2.

an infinite sequence of periodic transcendental Lame functions. However, the eigenvalues for the transcendental Lame functions are such that $\chi_{\parallel}^2 < 0$. In the plane $(\chi_{\parallel}^2, A_{\perp})$ there appear two allowed bands of the continuous spectrum of longitudinal wave numbers. In Fig. 2, one of the allowed bands lies between the curves Ec^- and Es^-, and the other lies between the curves Es^0 and Ec^0. Periodic solutions are realized on the boundaries of the allowed bands, and inside the bands we get bounded but non-periodic solutions.

We call attention to the fact that the degeneracy of the upper allowed bands when $A_{\perp} \to \sqrt{2} - 1 \pm 0$ into a point region (the point of tangency of the curves Ec^- and Es^-) reflects the appearance in the medium of a localized distribution—a plane waveguide layer. When $A_{\perp} \to \sqrt{2} - 1 \pm 0$ the Lame polynomials (4.11) degenerate into the eigenfunctions that appeared earlier[1] in the investigation of non-one-dimensional distributions close to a plane waveguide layer. When $A_{\perp} \ll 1$ and $a \ll 1$, the first terms of the asymptotic expansion of the non-one-dimensional field distribution corresponding to the choice of $\mathrm{Es}^-(y)\cos\chi_{\parallel}\zeta$ as the principal non-one-dimensional mode, are given by

$$\begin{aligned} E(x, z) &= A_{\perp}\{e_{\perp}(\xi) + a\,\mathrm{Es}^-(y)\cos\chi_{\parallel}\zeta + \ldots\} = \\ &= A_{\perp}\{\cos 2\chi_{\perp}\xi + a\cos\chi_{\perp}\xi\cos\chi_{\parallel}\zeta + \ldots\}, \end{aligned} \tag{4.13}$$

where $\chi_{\perp}^2 \approx 1/4$, $\chi_{\parallel}^2 \approx 3/4$, and the quantity $aA_{\perp}$ should be regarded as the amplitude of the principal non-one-dimensional mode. The obtained expression indicates that one of the non-one-dimensional field distributions close to the exact one-dimensional periodic field distribution is produced in the vicinity of the previously investigated small divisor $\chi_{\perp}^2 \approx 1/4$, $\chi_{\parallel}^2 \sim 3/4$.

5. WAVES IN A MEDIUM WITH A TRAVELING PERIODIC NONLINEAR WAVE

Let us consider the solution of the system of equations for the field phase and amplitude, determining the steady-state non-one-dimensional distributions of the field in the presence of an energy flux. The system of equations (1.2) and (1.3) admits of an exact one-dimensional periodic solution in the form

$$E/E_c \equiv e_{\perp}(x), \qquad \Psi = -k_{\infty}z. \tag{5.1}$$

Here $e_{\perp}(x)$ satisfies the equation

$$e_{\perp}'' + [k_{\omega}^2 - \varkappa^2 - k_{\infty}^2 + \varkappa^2 e_{\perp}^2]e_{\perp} = 0, \tag{5.2}$$

and k_{∞} determines the longitudinal energy flux in the medium. Putting

$$\begin{aligned} e(x, z) &= e_{\perp}(x) + aE(x)\cos k_{\parallel}z, \\ \Psi(x, z) &= -k_{\infty}z + a\frac{S(x)}{e_{\perp}(x)}\sin k_{\parallel}z \end{aligned} \tag{5.3}$$

and $k_{\omega}^2 - \kappa^2 - k_{\infty}^2 > 0$, we find that a non-one-dimensional steady-state field distribution close to the exact one-dimensional periodic distribution (5.1) is determined in the linear approximation by the solution of the problem for the eigenvalues of the longitudinal wave number $k_{\parallel}$

$$\begin{gathered} \left[\frac{d^2}{d\xi^2} + 1 - \chi_{\parallel}^2 + 3\gamma^2 e_{\perp}^2 \right] E + 2\chi_{\infty}\chi_{\parallel}S = 0, \\ 2\chi_{\infty}\chi_{\parallel}E + \left[\frac{d^2}{d\xi^2} + 1 - \chi_{\parallel}^2 + \gamma^2 e_{\perp}^2 \right] S = 0. \end{gathered} \tag{5.4}$$

We have used here the notation

$$\chi_{\|,\infty}^2 = \frac{k_{\|,\infty}^2}{k_\omega^2 - \varkappa^2 - k_\infty^2}, \quad \gamma^2 = \frac{\varkappa^2}{k_\omega^2 - \varkappa^2 - k_\infty^2}, \quad \xi = x\sqrt{k_\omega^2 - \varkappa^2 - k_\infty^2},$$

and the one-dimensional distribution is given by

$$e_\perp(\xi) = A_\perp \mathrm{cn}(\xi\sqrt{1+(\gamma A_\perp)^2},\ k_\perp), \tag{5.5}$$

$$k_\perp = \gamma A_\perp / \sqrt{2(1+(\gamma A_\perp)^2)}. \tag{5.5'}$$

Putting $\varphi = \xi\sqrt{1+(\gamma A_\perp)^2}$ and making the substitution $\mathrm{sn}\,\varphi = \cos\eta$, which is used to go over to the trigonometric form of the Lame equation[4], we rewrite (5.4) in the form

$$\begin{gathered}[1 - k_\perp^2\cos^2\eta]\frac{d^2E}{d\eta^2} + k_\perp^2\cos\eta\sin\eta\frac{dE}{d\eta} \\ + \left[\frac{1-\chi_\|^2}{1+(\gamma A_\perp)^2} + 6k_\perp^2\sin^2\eta\right]E + \frac{2\chi_\infty\chi_\|}{1+(\gamma A_\perp)^2}S = 0, \\ [1 - k_\perp^2\cos^2\eta]\frac{d^2S}{d\eta^2} + k_\perp^2\cos\eta\sin\eta\frac{dS}{d\eta} \\ + \left[\frac{1-\chi_\|^2}{1+(\gamma A_\perp)^2} + 2k_\perp^2\sin^2\eta\right]S + \frac{2\chi_\infty\chi_\|}{1+(\gamma A_\perp)^2}E = 0.\end{gathered} \tag{5.6}$$

The eigenfunctions of the system (5.6), and consequently also the fundamental non-one-dimensional mode of the linear approximation to the exact one-dimensional periodic field distribution with an energy flux can be represented when $k_\perp^2 \ll 1$ in the form of a series in powers of $k_\perp^2$:

$$E = E^{(0)}(\eta) + k_\perp^2 E^{(1)}(\eta) + \ldots, \quad S = S^{(0)}(\eta) + k_\perp^2 S^{(1)}(\eta) + \ldots,$$

$$\chi_\| = \chi_\|^{(0)} + k_\perp^2\chi_\|^{(1)} + \ldots \tag{5.7}$$

For the functions $E^{(0)}$ and $S^{(0)}$, the system (5.7) leads to

$$\begin{gathered}\ddot{E}^{(0)} + [1 - (\chi_\|^{(0)})^2]E^{(0)} + 2\chi_\infty\chi_\|^{(0)}S^{(0)} = 0, \\ 2\chi_\infty\chi_\|^{(0)}E^{(0)} + [1 - (\chi_\|^{(0)})^2]S^{(0)} + \ddot{S}^{(0)} = 0.\end{gathered} \tag{5.8}$$

One of the solutions of (5.8) is

$$E^{(0)} = a\cos 2n\eta, \qquad S^{(0)} = \pm a\cos 2n\eta. \tag{5.9}$$

The latter corresponds to the eigenvalues of the longitudinal wave number

$$\chi_\|^{(0)} = \chi_\infty \pm \sqrt{\chi_\infty^2 + 1 - (2n)^2}, \qquad \chi_\|^{(0)} = -\chi_\infty \pm \sqrt{\chi_\infty^2 + 1 - (2n)^2}. \tag{5.10}$$

Here n is an integer determining the number of oscillations of the eigenfunctions within an interval corresponding to the period 2π, and satisfies the obvious inequality

$$(2n)^2 < 1 + \chi_\infty^2. \tag{5.11}$$

Consequently, the number of oscillations characterizing the field distribution in the transverse dimension is determined by the magnitude of the longitudinal energy flux.

The system of equations for $E^{(1)}(\eta)$ and $S^{(1)}(\eta)$ has a solution in the form

$$\begin{gathered}E^{(1)} = A_{2(n+1)}\cos 2(n+1)\eta + A_{2(n-1)}\cos 2(n-1)\eta, \\ S^{(1)} = a_{2(n+1)}\cos 2(n+1)\eta + a_{2(n-1)}\cos 2(n-1)\eta,\end{gathered} \tag{5.12}$$

if $\chi_\|^{(1)}$ is determined from the condition that there be no secular terms in such a system of equations. The exclusion of the secular terms causes the asymptotic expansion of the eigenvalue of the longitudinal wave number $\chi_\|(n, k_\perp^2)$, corresponding to n-fold oscillations in terms of the transverse variables within the interval of the fundamental period, to assume at small values of $k_\perp^2$ the form

$$\chi_\|(n, k_\perp)^2 = \chi_\|^{(0)}(n) + \frac{1-3n^2}{\chi_\|^{(0)}(n) \pm \chi_\infty}k_\perp^2 + O(k_\perp^4). \tag{5.13}$$

the amplitudes $A_{2(n\pm1)}$ and $a_{2(n\pm1)}$ are determined by a simple system of algebraic equations, the determinant of which differs from zero at any integer value of n. The solution of the latter raises no difficulty.

The asymptotic expansion (5.13) shows that four points $\chi_\|^{(0)}(n)$ are located in the plane $(\chi_\|, k_\perp)$ on the axis $k^2 = 0$, symmetrically to the axis $\chi_\| = 0$, for any specified number of oscillations n satisfying the inequality (5.11) and for $k_\perp^2 \to 0$. When $k_\perp^2 \ll 1$, each of these points is the starting point of two diverging curves ("whiskers"), on which (Fig. 3) periodic solutions are realized.

In the absence of an energy flux ($k_\infty = 0$, $k_\omega^2 - \kappa^2 > 0$), the Lame equation, into which the equation for the field amplitude of the system (5.6) degenerates, can be investigated in greater detail. Namely, it is possible to ascertain that this equation admits of three periodic solutions represented by Lame polynomials. The latter coincide with the polynomials Ec^-, Es^-, and Ec^0, with the obviously replacement of the modulus of the elliptic functions by the quantity determined by relation (5.5′). On the other hand, the admissible values of the longitudinal wave numbers are determined by the expressions

$$\begin{gathered}\chi_\|^2(Ec^-) = -1 + [4 + 6(\gamma A_\perp)^2 + 3(\gamma A_\perp)^4]^{1/2}, \\ \chi_\|^2(Es^-) = \frac{3}{2}(\gamma A_\perp)^2, \quad \chi_\|^2(Ec^0) = 0\end{gathered} \tag{5.14}$$

These three solutions account for all the periodic Lame eigenfunctions that lead to the linearized-problem field distributions bounded in terms of the longitudinal variable. The curves shown in Fig. 4 delineate in the plane $(\chi_\|^2, A_\perp)$ regions in which bounded solutions exist or do not exist. The regions bounded by the curves Ec^- and Es^- correspond to the only band allowed in this case, that of the continuous spectrum of the eigenvalues of the longitudinal wave numbers $\chi_\|$.

We note that the solutions (5.3) can serve as the basis for the construction of asymptotic expansions. In this case, in particular, when $\chi_\infty = 0$, and $\chi_\|$ is determined by relations (5.14), it turns out that if the principal four-dimensional mode corresponds to the upper limit of the allowed band of the continuous spectrum $\chi_\|(Ec^{-1})$, then there are no divisors in the asymptotic expansion of the two-dimensional distribution in the field in terms of integer powers of the principal-mode

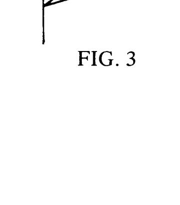

FIG. 3

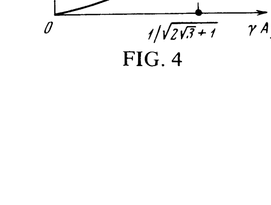

FIG. 4

amplitude. To the contrary, in the case of the lower limit of the allowed band of the continuous spectrum, there exist discrete values of the amplitudes of the one-dimensional distribution of the field $A_\perp$, for which small divisors arise in the asymptotic expansion. The corresponding values of the amplitudes are given by the formula

$$(A_\perp^2)_m = \frac{2}{\gamma^2}\frac{[(m^2-2)^2+3m^4-4]^{1/2}-m^2+2}{3m^4-4}, \qquad (5.15)$$

where $m \geq 2$ is an integer. It follows from this, in particular, that small divisors likewise do not arise when $A_\perp > (A_\perp)_{m=2}$. In the vicinity of the points (5.15), the construction of two-dimensional solutions can be carried out in accordance with the exposition in the second and third sections.

In conclusion it must be emphasized that in the more general case, when, unlike in our case, the nonlinear one-dimensional periodic wave solutions cannot be written in explicit form, a medium with such a periodic wave becomes nevertheless effectively inhomogeneous, owing to the nonlinear polarization. In this connection, bands of admissible values of wave vectors of waves close to the strong nonlinear wave will arise. The regions of existence and nonexistence of small divisors corresponding to intersection of terms can be revealed there in the same manner.

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