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FINITE AMPLITUDE ION ACOUSTIC WAVES IN AN UNSTABLE PLASMA

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Using hydrodynamic equations for the electrons and ions, we have obtained a van der Pol equation that describes the stationary laminar ion acoustic wave of finite amplitude that propagates across a weak magnetic field. If the ion viscosity is high, the amplitude of this wave is proportional to the square root of the linear growth rate. If the ion viscosity is small there exists a solution in the form of a sawtooth wave which describes the experimentally observed sharpening in the profile of the ion acoustic wave. It is also shown that this kind of wave can be established in a plasma with electrodes.

1. IN an earlier work of the present author[1] (cf. also [2]) consideration has been given to the nonlinear stationary drift wave that propagates in an unstable inhomogeneous plasma in a strong magnetic field. In the present work we study the case of a weak magnetic field in which excitation of ion acoustic waves is possible in the linear approximation; these waves propagate essentially across the magnetic field (cf. for example [3]). This kind of wave has recently been observed experimentally in a fully ionized plasma in a Q-machine.[4]

The principal results obtained in the hydrodynamic approximation are the following:

A. A nonlinear differential equation similar to the van der Pol equation is obtained; this equation describes stationary ion acoustic waves.

B. It is shown that if the ion viscosity is high, in which case higher harmonics of a given mode are damped in the linear approximation, the plasma supports an essentially harmonic ion acoustic wave that propagates essentially across the magnetic field. The amplitude of this wave is proportional to the square root of the linear growth rate for a given mode while the frequency is reduced by an amount proportional to the growth rate.

C. It is found that for the case of small ion viscosity the equation indicated in A allows a solution in the form of a stationary sawtooth wave which, in our opinion, describes the sharpening in the profile of the ion acoustic wave that is observed in Q-machine experiments.[4]

D. Consideration has also been given to the excitation effects associated with processes at the ends of the Q-machine in order to see the effect of these processes on the stationary finite-amplitude oscillations. It is found that such oscillations are possible only if there are large potential jumps in the double electrode sheath.

2. We first obtain a differential equation that relates the ion density and the electric potential φ for periodic perturbations of large amplitude in an inhomogeneous plasma in a weak magnetic field.

For reasons of simplicity the following assumptions are made: a) the electric field **E** is irrotational, that is, $\mathbf{E} = -\nabla\varphi$; b) the magnetic field **H** is uniform and constant and perturbations of the magnetic field can be neglected; c) the ion motion along **H** can be neglected; d) ion pressure effects are neglected (the ion temperature T_i is much lower than the electron temperature T_e); e) the frequencies in question ω lie appreciably above the ion-cyclotron frequency Ω_i (the ions are not magnetized); f) a plane geometry is assumed; g) the variation of the density n in the x-direction (perpendicular to **H**) is exponential:

$$\varkappa \equiv -\frac{1}{n}\frac{\partial n}{\partial x} = \text{const.} \tag{1}$$

Since we are interested in stationary waves it is assumed that the electric potential, density, and other physical quantities are periodic functions with period 2π of the argument $\xi \equiv k_y y - \omega t$, where k_y is the mean wave number. Thus, for example, $\varphi = \varphi(\xi, k_z z)$ where k_z is the mean wave number along the magnetic field, which is independent of the variable x. Separating the dependence of the perturbation on the variable z (along the magnetic field) makes it possible to treat nonlinear waves which can be traveling waves or standing waves along **H**.

The equations that describe a fluid made up of singly charged ions which perform one-dimensional motion along the y-axis assume the following form:

$$m\frac{\partial v}{\partial t} + mv\frac{\partial v}{\partial y} = -e\frac{\partial\varphi}{\partial y} + m\mu\frac{\partial^2 v}{\partial y^2}, \tag{2}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial(n_i v)}{\partial y} = 0, \tag{3}$$

where n_i is the ion density; $v \equiv v_{iy}$ is the velocity of the ion fluid; m is the ion mass; μ is the viscosity of the unmagnetized ion fluid:

$$\mu = 0{,}96 T_i \tau_{ii} / m, \tag{4}$$

τ_{ii} is the mean time for ion-ion collisions and the Boltzmann constant has been set equal to unity.

Making use of the periodicity assumption, we have from Eqs. (2) and (3):

$$-m\left(\omega - \frac{k_y v}{2}\right)v - m\mu k_y^2\frac{\partial v}{\partial\xi} = -ek_y\varphi, \tag{5}$$

$$-\omega n_i + k_y n_i v = -\omega n_0, \tag{6}$$

where n_0 is the unperturbed density. Substituting v from Eq. (6) in Eq. (5) we have

$$\frac{e\varphi}{m} = \frac{1}{2}\frac{\omega^2}{k^2}\left[1 - \left(\frac{n_0}{n_i}\right)^2\right] + \frac{\mu\omega n_0}{n_i^2}\frac{dn_i}{d\xi}, \tag{7}$$

or, introducing the notation $v_i \equiv \ln(n_i/n_0)$,

$$\psi = \frac{1}{2}\frac{\omega^2}{\omega_0^2}(1-e^{-2\nu_i}) + \frac{\omega_\mu\omega}{\omega_0^2}e^{-\nu_i}\frac{d\nu_i}{d\xi};$$
$$\psi \equiv e\varphi/T_e, \quad \omega_0 \equiv k_y(T_e/m)^{1/2}, \quad \omega_\mu \equiv \mu k_y^2. \tag{8}$$

3. In similar fashion it is possible to obtain an equation that relates the density of the electron fluid n_e with the potential. In addition to making the assumptions listed above we also assume that a) the frequencies in question lie well below the electron cyclotron frequency (fully magnetized electrons), the electron fluid is described by an isothermal equation of state, c) friction can be neglected for the electron flow perpendicular to **H**, as can the effects of electron inertia and the electron Larmor radius.

With these assumptions the relation being sought between the electron density and the potential can be obtained without difficulty (cf. [2]):

$$-\frac{\omega}{\omega_s}\frac{d\nu_e}{d\xi} + \frac{\omega_*}{\omega_s}\frac{d\psi}{d\xi} + e^{-\nu_e}(\psi_{zz} - \nu_{ezz}) = 0, \tag{9}$$

where $\nu_e \equiv \ln(n_e/n_0)$; $\omega_* \equiv \kappa T_e k_y/\Omega_i m$ is the mean drift frequency; $\Omega_i \equiv eH/mc$; $\omega_s \equiv D_e k_z^2$; $D_e \equiv T_e/m_e\nu_{ei}$ is the electron diffusion coefficient; ν_{ei} is the mean frequency of electron-ion collisions and c is the velocity of light. The subscript z in (9) indicates differentiation with respect to the argument $k_z z$.

4. In what follows it will be assumed that the motion in question is quasineutral, that is to say,

$$\nu_e = \nu_i \equiv \nu. \tag{10}$$

It is evident that when $\mu \to 0$ and $\nu_{ei} \to 0$ Eqs. (8)–(10) do not have solutions for an arbitrarily large amplitude. This result is reasonable because in the absence of dissipation processes for quasineutral motion we would expect to obtain a nonstationary solution characteristic of a simple Riemann wave.

Substituting Eq. (8) in Eq. (9), eliminating the potential ψ, and making use of Eq. (10) we obtain a wave equation for a single variable, the density ν. Thus

$$-\frac{\omega}{\omega_s}\nu' + \frac{\omega_*}{\omega_s}\left[\frac{1}{2}\frac{\omega^2}{\omega_0^2}(1-e^{-2\nu}) + \frac{\omega_\mu\omega}{\omega_0^2}e^{-\nu}\nu'\right]' + e^{-\nu}\left[\frac{1}{2}\frac{\omega^2}{\omega_0^2}(1-e^{-2\nu}) + \frac{\omega_\mu\omega}{\omega_0^2}e^{-\nu}\nu' - \nu\right]_{zz} = 0, \tag{11}$$

where the primes denote differentiation with respect to ξ.

We shall first consider solutions that correspond to infinitesimally small amplitudes. It will be assumed that the plasma is not bounded along **H** so that $\nu \propto \exp[i(k_y y + k_z z - \omega t)]$. In this case $\nu' = \nu_z = i\nu$ and linearizing Eq. (11), we obtain the dispersion equation

$$-i\frac{\omega}{\omega_s} + \frac{\omega_*}{\omega_s}\left(i\frac{\omega^2}{\omega_0^2} - \frac{\omega_\mu\omega}{\omega_0^2}\right) - \left(\frac{\omega^2}{\omega_0^2} + i\frac{\omega_\mu\omega}{\omega_0^2} - 1\right) = 0. \tag{12}$$

When

$$\omega_s \gg \omega_*, \quad \omega_0 \gg \omega_\mu \tag{13}$$

it is not difficult to obtain a solution of Eq. (12) in the form

$$\mathrm{Re}\,\omega = \omega_0, \quad \mathrm{Im}\,\omega = \frac{1}{2}\left(\frac{\omega_* - \omega_0}{\omega_s}\omega_0 - \omega_\mu\right). \tag{14}$$

If we now take

$$r_H\varkappa \gg 1, \tag{15}$$

where $r_H \equiv (T_e/m)^{1/2}/\Omega_i$ is the ion-Larmor radius computed with the electron temperature, we have

$$\mathrm{Re}\,\omega = \omega_0, \quad \mathrm{Im}\,\omega = \frac{1}{2}\left(\frac{\omega_*\omega_0}{\omega_s} - \omega_\mu\right). \tag{16}$$

It will be evident from Eq. (16) that modes with large k_y cannot be stabilized in the linear approximation by ion-ion collisions. The ion viscosity only leads to a reduction of the growth rate for an unstable mode with a given k_y. However, it does follow from Eq. (16) that one might expect to stabilize an unstable mode by virtue of wave interactions because it is evident from the expression for Im ω that starting at some given harmonic all higher harmonics will be damped.

By virtue of the approximations in (13) and (15), Eq. (11) can be written in the form

$$\frac{\omega_*\omega^2}{\omega_s\omega_0^2}\nu' + \left[\frac{1}{2}\frac{\omega^2}{\omega_0^2}(1-e^{-2\nu}) + \frac{\omega_\mu\omega}{\omega_0^2}\nu' - \nu\right]_{zz} = 0. \tag{17}$$

A further simplification of Eq. (17) is obtained under the assumption that

$$\nu(k_y y - \omega t, k_z z) = \nu(k_y y + k_z z - \omega t), \tag{18}$$

which means that the nonlinear wave is assumed to be a traveling wave in the direction of the magnetic field. This assumption means that the number of independent variables in Eq. (17) is reduced to one so that a single integration is possible. We will assume that the amplitude is not too large so that we can expand the nonlinear term in powers of ν and retain terms up to the cubic terms. For reasons of simplicity we neglect the quadratic term; this term does not have an effect on the qualitative results of the analysis since the contribution of this term reduces only to the introduction of an asymmetry between the peaks and valley of the wave profile.

As a result of these simplifications, we find that Eq. (17) is replaced by an expression of the form

$$\frac{\omega_s\omega_\mu}{\omega_*\omega}\frac{d^2\nu}{d\xi^2} - \frac{\omega_s}{\omega_*}\left(\frac{\omega_0^2}{\omega^2} - 1\right)\left(1 - \frac{2\omega^2\nu^2}{\omega_0^2 - \omega^2}\right)\frac{d\nu}{d\xi} + \nu = 0, \tag{19}$$

where now $\xi \equiv k_y y + k_z z - \omega t$. Equation (19) exhibits the structure of the van der Pol equation which, as is well known, has nontrivial periodic solutions for appropriate values of the parameters. We recall that the density ν must be a periodic function of the argument ξ with period 2π or with frequency $\omega = 1$. Introducing the new independent variable ζ

$$\zeta \equiv \xi(\omega\omega_*/\omega_\mu\omega_s)^{1/2}, \tag{20}$$

we reduce Eq. (19) to the standard form

$$\frac{d^2\nu}{d\zeta^2} - \varepsilon(1 - \lambda^2\nu^2)\frac{d\nu}{d\zeta} + \nu = 0, \tag{21}$$

where

$$\varepsilon \equiv (\omega_s/\omega_\mu\omega_*\omega^3)^{1/2}(\omega_0^2 - \omega^2), \tag{22}$$
$$\lambda^2 \equiv 2\omega^2/(\omega_0^2 - \omega^2) > 0. \tag{23}$$

It is well known[5] that values of the parameter $\epsilon \ll 1$ correspond to a solution of Eq. (21) in the form of harmonic oscillations with respect to ζ with frequency $\omega_\zeta = 1$: $\nu = \nu_0 \cos_\zeta \zeta$. On the other hand, values of the parameter $\epsilon \gg 1$ correspond to a solution in the form of a sharply nonsinusoidal sawtooth relaxation oscillation with period $\tau_\zeta = 1.614\,\epsilon$. In this connection we now consider the way in which the frequency ω (or the phase velocity $v_\varphi \equiv \omega/k_y$) varies for various values of the plasma parameters. In particular, we wish to examine solutions corresponding to the sinusoidal solution ($\epsilon \ll 1$) and the sawtooth solution ($\epsilon \gg 1$) for the traveling ion acoustic wave.

5. It is evident that a harmonic profile of the stationary wave will be obtained when

$$\varepsilon \ll 1. \tag{24}$$

In this case, as we have already noted, we find that $\omega_\zeta = 1$. However, in terms of the variable ξ the solution, by assumption, must be periodic with frequency $\omega_\xi = 1$; it follows that the coefficient in the conversion from the variable ξ to the variable ζ [in accordance with (20)] must be equal to unity:

$$(\omega\,\omega_* / \omega_\mu\omega_s)^{1/2} = 1,$$

Thus

$$\omega = \frac{\omega_\mu\omega_s}{\omega_*} \equiv \omega_0 - \frac{\omega_s}{\omega_*}\left(\frac{\omega_*\omega_0}{\omega_s} - \omega_\mu\right) \tag{25}$$

Taking account of Eqs. (22) and (25) we can write (24) in the form

$$\left(\frac{\omega_*\omega_0}{\omega_s} - \omega_\mu\right)\frac{1}{\omega_0} \ll \frac{1}{2}\left(\frac{\omega_\mu\omega_*^3}{\omega_0\omega_s^3}\right)^{1/2}, \tag{26}$$

and (23) reduces to the requirement

$$\omega_*\omega_0 / \omega_s - \omega_\mu > 0. \tag{27}$$

Thus, if (26) and (27) are satisfied we obtain a periodic solution of Eq. (21):

$$v \approx 2\left[\frac{\omega_s}{\omega_*\omega_0}\left(\frac{\omega_*\omega_0}{\omega_s} - \omega_\mu\right)\right]^{1/2} \cos(k_y y + k_z z - \omega t), \tag{28}$$

$$\omega \approx \omega_0 - \frac{\omega_s}{\omega_*}\left(\frac{\omega_*\omega_0}{\omega_s} - \omega_\mu\right).$$

The meaning of the conditions in (26) and (27) can be understood if we recall [cf. the solution of the dispersion equation (16)] that the quantity $[(\omega_*\omega_0)/\omega_s] - \omega_\mu/2$ represents the imaginary part of the frequency of the ion acoustic wave of infinitesimally small amplitude. Thus, the condition in (27) means that the wave must be a growing wave in the linear approximation. The condition in (26) means that the higher harmonics of a given unstable mode cannot be excited in the linear approximation (these are damped) so that as a result of the linear excitation of the given mode and the nonlinear damping in the higher harmonics there is established an essentially sinusoidal oscillation of the inhomogeneous plasma with an amplitude proportional to the square root of the linear growth rate. In this case the oscillation frequency is shifted (reduced) by an amount proportional to the growth rate.

6. We now wish to examine the conditions under which one might expect highly nonlinear waves, in which case the profile of the traveling wave will be approximately a sawtooth. As we have already noted this kind of relaxation oscillation with respect to the variable ζ arises when

$$\varepsilon \gg 1. \tag{29}$$

The period of these oscillations is[5]

$$\tau_\zeta = 1{,}614\,\varepsilon. \tag{30}$$

Assuming that

$$\tau_\zeta = (\omega\,\omega_* / \omega_\mu\omega_s)^{1/2}\tau_\xi, \tag{31}$$

where τ_ξ is the oscillation period with respect to ξ and

$$\tau_\xi = 2\pi, \tag{32}$$

making use of Eqs. (30), (31), (32), and (22) we find

$$2\pi = 1{,}614\frac{\omega_s}{\omega_*}\left(\frac{\omega_0^2}{\omega^2} - 1\right). \tag{33}$$

Thus

$$\omega = \omega_0\left(1 - \frac{\pi}{1{,}614}\frac{\omega_*}{\omega_s}\right), \tag{34}$$

$$\varepsilon = \frac{2\pi}{1{,}614}\left(\frac{\omega_0\omega_*}{\omega_\mu\omega_s}\right)^{1/2}. \tag{35}$$

The condition for the existence of a sawtooth profile (29) then becomes

$$\omega_\mu \ll \left(\frac{2\pi}{1{,}614}\right)^2 \frac{\omega_*\omega_0}{\omega_s} \tag{36}$$

and is physically clear since it means that in the linear approximation we not only have the excitation of a given mode, but also a large number of higher harmonics.

7. We now wish to show that a stationary ion acoustic wave of finite amplitude can, under certain conditions, also exist in an inhomogeneous plasma with electrodes that are oriented perpendicularly to the magnetic field and separated by a distance L. These waves have been considered in the linear approximation in an earlier work by the present author.[6] Here we consider the following simple model of the plasma which allows us to obtain some of the results in simple analytic form. As before we assume that the ions are described by Eqs. (2) and (3) so that the appropriate relation between the ion density and the potential is given by Eq. (8). In contrast with [6] now we assume that the electron fluid is ideal and governed by the equations

$$T_e\nabla n_e = en_e\nabla\varphi - \frac{en_e}{c}[\mathbf{v}_e\mathbf{H}], \tag{37}$$

$$\partial n_e / \partial t + \operatorname{div} n_e\mathbf{v}_e = 0. \tag{38}$$

Expressing the transverse flux by means of Eq. (37) and substituting it in the equation of continuity (38) we find that Eqs. (37) and (38) are replaced by the system

$$\partial\psi / \partial z = \partial\nu_e / \partial z, \tag{39}$$

$$\partial I_{ez} / \partial z = n_0 e^{\nu_e}(\omega\nu' - \omega_*\psi'), \tag{40}$$

where I_{ez} represents the z component of the electron flux.

$$I_{ez} \equiv n_e v_{ez}, \tag{41}$$

and, as before, the primes denote differentiation with respect to ξ. The solutions of Eqs. (39) and (40) must satisfy the electrode boundary conditions at the left ($z = -L/2$) and at the right ($z = L/2$); when $\varphi_p > 0$ (φ_p is the plasma potential with respect to the electrode) these conditions can be written in the form

$$\begin{gathered} I_{0e}^{e}\{1 - \exp(-[\psi(-L/2) - \nu_e(-L/2)]\} = I_{ez}(-L/2), \\ I_{0e}^{e}\{1 - \exp(-[\psi(L/2) - \nu_e(L/2)]\} = -I_{ez}(L/2), \end{gathered} \tag{42}$$

where I_{0e}^{e} is the fixed electron emission current from the electrode into the plasma.

It will be evident that perturbations of the density ν_e and potential ψ that are uniform along z and the antisymmetric current $I_{ez} = n_0(\exp \nu_e)(\omega\nu' - \omega_*\psi')z$ represent solutions of Eqs. (39) and (40). Substituting this solution in one of the boundary conditions (42) (in this case the second condition is satisfied automatically) we obtain the required relation between the electron density ν_e and the potential ψ:

$$-\frac{\omega}{\omega_s}\nu_e' + \frac{\omega_*}{\omega_s}\psi' - e^{-\nu_e}[1 - e^{-(\psi-\nu_e)}] = 0, \tag{43}$$

where now

$$\omega_s \equiv 2I_{0e}{}^{3} / n_0 L. \tag{44}$$

Substituting Eq. (8) in Eq. (43) and taking account of quasineutrality (10) we obtain a wave equation for the density:

$$-\frac{\omega}{\omega_s}\nu' + \frac{\omega_*}{\omega_s}\left[\frac{1}{2}\frac{\omega^2}{\omega_0^2}(1-e^{-2\nu}) + \frac{\omega_\mu\omega}{\omega_0^2}e^{-\nu}\nu'\right]' \tag{45}$$
$$- e^{-\nu}\left\{1 - \exp\left[-\frac{1}{2}\frac{\omega^2}{\omega_0^2}(1-e^{-2\nu}) - \frac{\omega_\mu\omega}{\omega_0^2}e^{-\nu}\nu' + \nu\right]\right\} = 0.$$

Linearizing Eq. (45) for small amplitude waves in the y-direction we obtain dispersion relation exactly like (12); now, however, the quantity ω_s is determined from Eq. (44). The solution of Eq. (12) in the form of Eq. (16) shows that the "end effect" instability being considered cannot be stabilized by ion-ion collisions in either the linear or nonlinear approximations; if any mode is unstable then all of its harmonics are also unstable.

When

$$\omega_*, \omega_0 \gg \omega_s \gg \omega_\mu \tag{46}$$

there is a solution of Eq. (12) in the form

$$\mathrm{Re}\,\omega = \omega_0\frac{\omega_0}{\omega_*}, \quad \mathrm{Im}\,\omega = \omega_s\left(1 - \frac{\omega_0^2}{\omega_*^2}\right) - \omega_\mu. \tag{47}$$

When $\omega_* \gtrsim \omega_0$ ion acoustic waves that are unstable for large values of k_y in the approximation of (46) can be stabilized by the nonmagnetic ion viscosity. However, a mode with $\mathrm{Im}\,\omega > 0$ in the linear approximation can be stabilized by wave interactions since the higher harmonics are damped.

Using the approximation of (46) and keeping in (45) only the cubic term in ν, which gives rise to the higher harmonics, we find an equation of the form

$$\frac{\omega_*\omega_\mu\omega}{\omega_s\omega_0^2}\nu'' + \left(-\frac{\omega}{\omega_s} + \frac{\omega_*\omega^2}{\omega_s\omega_0^2}\right)\nu' + \left(1 - \frac{\omega^2}{\omega_0^2}\right)\nu - \frac{2}{3}\frac{\omega^2}{\omega_0^2}\nu^3 = 0. \tag{48}$$

It is expected that if the amplitude is not too large the neglected terms will not change the qualitative features of the behavior in a plasma with electrodes.

The equation we have obtained (48) is in the form of the equation for an anharmonic oscillator with friction and with a "soft" restoring force. It will exhibit periodic solutions so long as the damping coefficient vanishes. This requirement allows us to obtain the oscillation frequency

$$\omega = \omega_0^2 / \omega_*, \tag{49}$$

which coincides with $\mathrm{Re}\,\omega$ of Eq. (47). Making use of (49) we can write Eq. (48) in the simple form

$$\nu'' + \alpha^2\nu - \rho^2\nu^3 = 0, \tag{50}$$

where

$$\alpha^2 \equiv \frac{\omega_s}{\omega_\mu}\frac{\omega_*^2 - \omega_0^2}{\omega_*^2}, \quad \rho^2 \equiv \frac{2}{3}\frac{\omega_s\omega_0^2}{\omega_\mu\omega^2}. \tag{51}$$

Equation (50) has a well-known periodic solution with respect to the variable ξ (cf. for example [7]):

$$\tau_\xi = 4\sqrt{2}\int_0^{\pi/2}\frac{d\theta}{[2\alpha^2 - \rho^2\nu_0^2 - \rho^2\nu_0^2\sin^2\theta]^{1/2}}, \tag{52}$$

where ν_0 is the amplitude. Since, by definition, the period in ξ is equal to 2π the relation in (52) determines the amplitude ν_0 in terms of the system parameters α^2 and ρ^2. In particular, at the oscillation threshold for the fundamental mode, in which case in Eq. (47)

$$\mathrm{Im}\,\omega \ll \mathrm{Re}\,\omega, \tag{53}$$

we can find the following relation without difficulty from Eqs. (52) and (53) (if the amplitude is assumed to be small):

$$\nu_0^2 = 2\frac{\omega_*}{\omega_s}\frac{\mathrm{Im}\,\omega}{\mathrm{Re}\,\omega}. \tag{54}$$

Thus, for sufficiently large positive jumps in the potential in the double sheath at the electrodes, in which case (53) is satisfied, the plasma can support essentially sinusoidal ion acoustic waves with amplitude proportional to the square root of the linear growth rate (47).

As the ion viscosity is reduced the wave profile becomes richer in higher harmonics, which are also excited in the linear approximation and the wave acquires a sawtooth shape. The maximum amplitude of these relaxation oscillations is found by setting the elastic force in Eq. (50) equal to zero:

$$\nu_{0\,max}^2 = \frac{\alpha^2}{\rho^2} = \frac{3}{2}\frac{\omega_*^2 - \omega_0^2}{\omega_0^2}, \tag{55}$$

which is also found to be proportional to the square root of the linear growth rate.

The nonlinear analysis presented here makes it possible to understand a number of experimental facts which can evidently not be explained within the framework of the linear theory. Here, we refer to experiments carried out in alkali-metal plasmas[4] which show steepening in the profile of an ion acoustic wave of large amplitude that travels in the azimuthal direction. A wave of this type corresponds to the sawtooth solutions obtained here in the limiting case of a nonmagnetic ion viscosity.

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