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ON GRAVITATIONAL WAVES IN A GAS NEAR THE CRITICAL POINT FOR THE GAS-LIQUID TRANSITION

V. D. KHAIT

Moscow State University

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Appearance of gravitational waves is possible in a single-phase liquid near the critical point for the gas-liquid transition as a consequence of a considerable density gradient. The applicability of the existing theory of gravitational waves to such a medium is justified. The problem is solved of waves in a liquid with an exactly critical temperature. It is shown that the velocity of propagation of gravitational waves is determined by the derivative $(\partial^3 p/\partial\rho^3)_c$ while the limiting frequency (the "Väisälä frequency") is determined by the derivative $(\partial^2 p/\partial\rho\partial T)_c$. The subscript "c" means that the derivatives are taken at the critical point.

1. As is well known, the compressibility of a medium increases greatly near the critical point for the vapor-liquid transition. A consequence of this is the appreciable vertical density gradient in the liquid[1]. It is therefore natural to expect even in a single-phase system the appearance of gravitational waves brought about by the deviation of the density from its hydrostatic equilibrium distribution.

Gravitational waves in an inhomogeneous liquid are described by a well known theory[2-4]. The main assumptions of this theory are: absence of viscosity, smallness of oscillations, incompressibility of the liquid. Under these conditions the gravitational waves are described by the system of equations

$$\rho_0\frac{\partial \mathbf{V}}{\partial t} = -\nabla p' + \rho' \mathbf{g}, \quad \frac{\partial \rho'}{\partial t} + (\mathbf{V}\nabla)\rho_0 = 0, \quad \operatorname{div}\mathbf{V} = 0. \qquad (1)$$

Here $\mathbf{V}$ is the velocity of the liquid, $\rho_0(\mathbf{r})$ is its equilibrium density (ρ_0 is the solution of the hydrostatic equation $(\partial p/\partial\rho)_T\nabla\rho_0 = \rho_0 \mathbf{g}$), $\rho' \equiv \rho - \rho_0$, $p' \equiv p - p_0$ are respectively the deviations of the density and of the pressure from their equilibrium values, $\mathbf{g}$ is the acceleration of free fall.

If the positive z axis is directed vertically upwards the propagation of a horizontal monochromatic wave along the x axis will be given, for example, for the vertical component of the velocity V_z by the expression

$$V_z(z, x, t) = V(z)\exp\left[i(\omega t - kx)\right].$$

The amplitude of the wave $V(z)$ then satisfies the equation[2]

$$\frac{d^2V}{dz^2} + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{dV}{dz} - k^2\left(1 + \frac{g}{\omega^2}\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right)V = 0. \qquad (2)$$

On the rigid surfaces bounding the volume of the liquid the normal component of velocity must vanish.

Equation (2) can be solved for a few special cases of the dependence of $\rho_0(z)$ on z. In[3], a quite general distribution of density (Fig. 1) is investigated which on the whole reflects the real dependence of $\rho_0(z)$ on z for a compressible liquid in a gravitational field.

As can be seen from Fig. 1, the liquid has a layer (of thickness a) of considerable variation of density outside of which the density rapidly approaches its asymptotic values ρ_1 and ρ_2.

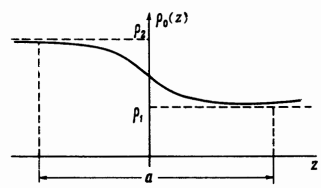

FIG. 1

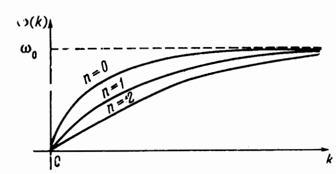

FIG. 2

In[3], the case of an unbounded medium was considered. It was shown that gravitational waves are localized within the layer in which density varies significantly. The amplitude of the waves outside this layer falls off exponentially to zero over a distance of the order of a wavelength.

In contrast to the well investigated case of two liquids with different, but constant densities (this case is obviously the limiting case a = 0) an infinite number of wave modes exists in a liquid with the density distribution described above. In such a case the zero mode (the velocity amplitude V(z) has no zeros) corresponds approximately to vibration as a whole of the layer in which the density varies significantly. In the limiting case of zero thickness of this layer this mode describes the oscillation of the surface separating the two liquids. The remaining modes correspond to oscillations of a number of horizontal layers into which the whole layer of thickness a can be divided.

Moreover, it has turned out that the frequencies of all the modes are smaller than a certain limiting frequency ω_0 (the Väisälä frequency) related to the maximum logarithmic gradient of the density of the medium:

$$\omega_0^2 = g\max\left|\frac{1}{\rho_0}\frac{d\rho_0}{dz}\right|.$$

The dispersion curves $\omega_n(k)$ are shown in Fig. 2.

We now apply these results to a medium the temperature of which differs from the critical temperature by a small amount $\Delta T = T - T_c > 0$, while at z = 0 the critical density ρ_c is attained. Then the hydrostatic equation and the expansion of the equation state near the critical point[5] yield:

$$\omega_0^2 = \frac{g^2}{(\partial^2 p/\partial\rho\partial T)_c \Delta T} \sim \frac{\mu g^2}{RT_c \Delta T/T_c} \sim \frac{10^{-2}}{\Delta T/T_c}\frac{1}{\sec^2},$$

where μ is the gram-molecular weight, R is the gas constant. For $\Delta T/T_c \sim 10^{-4}$, $\omega_0 \sim 10\ \sec^{-1}$.

2. As has been shown above, the theory under consideration utilizes the approximation of an incompressible liquid. The possibility of such an approximation near the critical point for the vapor-liquid transition requires a special investigation. In taking into account the compressibility of the medium on the assumption of the adiabatic nature of the oscillations the second equation of the system (1) must be replaced by the equation

$$\frac{d\rho}{dt} = \frac{1}{c_S^2}\frac{dp}{dt},$$

where c_S is the adiabatic velocity of sound in the medium. Taking the hydrostatic equation into account this equation may be rewritten in the following manner:

$$\frac{\partial\rho'}{\partial t} - \frac{1}{c_T^2}\rho_0 g V_z = \frac{1}{c_S^2}\frac{\partial p'}{\partial t} - \frac{1}{c_S^2}\rho_0 g V_z, \qquad (3)$$

where C_T is the isothermal sound velocity.

It can be easily seen that when the well known condition for the incompressibility of a homogeneous liquid

$$\omega / k \ll c_S,$$

is satisfied, it is possible to neglect the first term on the right hand side of (3) in comparison with its left hand side. Indeed, the left hand side in virtue of the equation of continuity is $\rho_0 \operatorname{div} \mathbf{V}$ and therefore contains a term of the order $\rho_0 k V_X$. But the quantity $\partial p'/\partial t$ as a result of the equation of motion is of order $\rho_0 \omega^2 V_X/k$ and, consequently, the ratio of the terms under consideration is of order $\omega^2/k^2 c_S^2$. The ratio of the second term on the right hand side of (3) to the second term on the left hand side is c_T^2/c_S^2. As is well known, this ratio near the critical point is the smaller compared to unity the nearer the medium is to critical conditions. Thus, when the conditions

$$\omega / k \ll c_S, \quad c_T^2 / c_S^2 \ll 1,$$

are satisfied Eq. (3) has the form:

$$\frac{d\rho}{dt} \equiv \frac{\partial\rho}{\partial t} + (\mathbf{V}\nabla)\rho = 0.$$

Thus, the applicability of the approximation of an incompressible liquid near the critical point is determined by the smallness of the adiabatic compressibility compared with the isothermal one.

We now consider the possibility of neglecting the viscosity. The viscous term $\Delta\mathbf{V}$ in the Navier-Stokes equation in the region in which the waves are localized is obviously given in order of magnitude by $\nu(k^2 + n^2/a^2)V$ (n is the number of the mode). Comparing this term with the $\partial\mathbf{V}/\partial t$ term in the same equation we obtain the condition for being able to neglect the viscosity:

$$k^2\nu / \omega \ll 1, \quad n^2\nu / \omega a^2 \ll 1.$$

Near the critical conditions of interest to us the viscosity of water is of the order of 10^{-3} cm^2/sec. Setting $\omega \sim 10\ \sec^{-1}$ and $a \sim 1$ cm we obtain $\lambda \sim 1/k \gg 10^{-2}$ cm, $n \ll 100$.

3. We will now consider the case of exactly critical temperature of the medium and we will assume that at z = 0 the critical density is attained. In this case the Väisälä frequency becomes infinite. In a real experiment, of course, such conditions are never realized. The Väisälä frequency always remains finite. However, if the frequencies of the gravitational waves under consideration are much smaller than the Väisälä frequency, this frequency in such a case can be regarded as infinite, and this corresponds to the ideal conditions under consideration.

We solve the problem in a bounded layer of liquid of thickness 2d. In equation (2) we go over to a new independent variable $\rho = \rho_0(z)$. Then taking into account the hydrostatic equation

$$\frac{1}{\rho}\frac{d\rho}{dz} = -\frac{g}{(\partial p/\partial\rho)_T}$$

we obtain

$$\frac{d^2V}{d\rho^2} + \left[\frac{2}{\rho} - \frac{(\partial^2 p/\partial\rho^2)_T}{(\partial p/\partial\rho)_T}\right]\frac{dV}{d\rho} + k^2\left[\frac{(\partial p/\partial\rho)_T}{\rho^2\omega^2} - \frac{(\partial p/\partial\rho)_T^2}{\rho^2 g^2}\right]V = 0.$$

We shall assume that the layer is sufficiently thin so that the term $2/\rho$ can be neglected in comparison with the term

$$\frac{(\partial^2 p/\partial\rho^2)_T}{(\partial p/\partial\rho)_T} \sim \frac{2}{\rho - \rho_c}$$

(utilizing an expansion in the neighborhood of the critical point), and ρ^2 in the term with V can be treated as a constant equal to ρ_c^2. Moreover, we assume that ω is sufficiently low so that we can neglect the term involving $(\partial p/\partial\rho)_T^2$ compared with the term involving $(\partial p/\partial\rho)_T$. For this it is necessary that

$$\omega^2 \ll \min\frac{g^2}{(\partial p/\partial\rho)_T} \sim \frac{10^{-3}}{(\Delta\rho/\rho_c)^2}\frac{1}{\sec^2},$$

where $\Delta\rho$ is the difference between the densities at the boundaries of the layer. According to reference[1] we have

$$\Delta\rho = \left[\frac{6gd}{\rho_c^2(\partial^3 p/\partial\rho^3)_c}\right]^{1/3}. \qquad (4)$$

Then the equation assumes the form

$$\frac{d^2V}{d\rho^2} - \frac{2}{\rho - \rho_c}\frac{dV}{d\rho} + \frac{k^2}{2\rho_c^2\omega^2}\left(\frac{\partial^3 p}{\partial\rho^3}\right)_c(\rho - \rho_c)^2 V = 0.$$

Setting $V = t^{3/4}\psi(t)$, where $t = \lambda(\rho - \rho_c)^2/\rho_c^2$, and $\lambda^2 = (k^2\rho_c^2/2\omega^2)(\partial^3 p/\partial\rho^3)_c$ we obtain a Bessel equation for the function ψ:

$$\psi'' + \frac{1}{t}\psi' + \left[1 - \frac{(3/4)^2}{t^2}\right]\psi = 0.$$

From this we obtain two types of solutions for $V(\rho - \rho_c)$, an even and an odd one. The general solution must satisfy the condition that V should vanish over the boundaries of the liquid, i.e.

$$\left\{A(\rho - \rho_c)^3\left(\frac{J_{3/4}[\lambda(\rho - \rho_c)^2/2\rho_c^2]}{(\rho - \rho_c)^{3/2}}\right) + B\left(\frac{J_{-3/4}[\lambda(\rho - \rho_c)^2/2\rho_c^2]}{(\rho - \rho_c)^{-3/2}}\right)\right\}_{\rho = \rho_c \pm \Delta\rho} = 0.$$

Here $J_{3/4}$ and $J_{-3/4}$ are Bessel functions of the first kind, while the expressions in parentheses are even functions of $\rho - \rho_c$. From this we obtain that either $A \neq 0$, then $B = 0$ and $J_{3/4}(\lambda\Delta\rho^2/2\rho_c^2) = 0$; or $B \neq 0$, then $A = 0$ and $J_{-3/4}(\lambda\Delta\rho^2/2\rho_c^2) = 0$. From this we obtain the dispersion law

$$\omega^2 = k^2 \frac{\rho_c^2}{8\xi_n^2} \left(\frac{\Delta\rho}{\rho_c}\right)^4 \left(\frac{\partial^3 p}{\partial\rho^3}\right)_c.$$

Here ξ_n are zeros of the Bessel function $J_{+3/4}(\xi_n) = 0$.

The square of the velocity of propagation of the wave u^2 is equal, after (4) has been taken into account, to:

$$u^2 = \left(\frac{\Delta\rho}{\rho_c}\right)^4 \left(\frac{\partial^3 p}{\partial\rho^3}\right)_c \frac{\rho_c^2}{8\xi_n^2} = \frac{1}{8\xi_n^2} \left[\frac{6gd}{\rho_c^{7/2}(\partial^3 p/\partial\rho^3)_c^{1/4}}\right]^{4/3}.$$

Its maximum value (corresponding to the minimum value of ξ_n) is equal to:

$$u_{max}^2 = 0{,}11 \left[\frac{6gd}{\rho_c^{7/2}(\partial^3 p/\partial\rho^3)_c}\right]^{4/3},$$

where we have taken into account the fact that the lowest zero of the function $J_{-3/4}$ is 1.06[6]. In order of magnitude we have:

$$u_{max}^2 \sim \left(\frac{\Delta\rho}{\rho_c}\right)^4 \frac{RT_c}{\mu},$$

or, if we assume that $\Delta\rho/\rho_c \sim 10^{-2}$, then $u_{max} \sim 10$ cm/sec. The adiabatic velocity of sound, at least in the practically accessible critical region, is much greater than this velocity of propagation of the waves. Thus, the approximation of an incompressible liquid is justified and, consequently, the applicability of the results obtained is also justified in the neighborhood of the critical point.

It should be noted that the results of experiments on the determination of the Väisälä frequency and of the velocity of propagation of gravitational waves in a liquid near its critical point could yield values of the quantities $(\partial^2 p/\partial\rho\partial T)_c$ and $(\partial^3 p/\partial\rho^3)_c$ which are important for the theory of the critical state.

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