RELAXATION OF A QUANTUM OSCILLATOR IN THE PRESENCE OF AN EXTERNAL FORCE

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A solution of the kinetic equation for a damped quantum oscillator on which an external classical force f(t) acts is obtained. A number of exact solutions for various initial conditions are considered and the spectrum of the radiation emitted by the oscillator is found.

1. INTRODUCTION

IN previous work[1] we considered a kinetic equation for the density matrix $\hat{\rho}(t)$ describing the relaxation of a quantum oscillator to the state of thermodynamical equilibrium[1)]. This problem is of interest for the quantum theory of the laser and the photostatistics of laser radiation, and also as an example of a problem from the quantum theory of irreversible processes which permits an exact solution. The results obtained in I give a description of the relaxation itself; however, the process of excitation of the oscillator was not considered there. In the present work we discuss a more complicated case when the quantum oscillator in the relaxation process is subjected to the action of an external (classical) force f(t) which depends in an arbitrary way on the time t.

The kinetic equation with the inclusion of a force f(t) is given in I and has the form I, (60). Methods for its solution and a number of concrete examples are discussed in Secs. 2 and 3. We note that the formulas obtained can describe the statistics of photocounts in the process of optical heterodyning[2)].

If we neglect the relaxation (i.e., put $\gamma = 0$, where γ is the damping constant), Eq. I, (60) describes the action of a force f(t) on an isolated oscillator, not interacting with a dissipative subsystem (thermostat). The kinetic equation in this case reduces to the usual Liouville equation. Such a problem was considered in the well-known papers of Feynman[3] and Schwinger[4].

The solution found in these papers allows us to investigate excitation of the quantized field (i.e., of a set of oscillators) by an external classical current. The formulas obtained in Sec. 3 of the present work give a generalization of these results for a quantum oscillator with damping. These formulas have a specially simple form for $\nu \to 0$ (i.e., $\hbar\omega_0 \gg kT$, where ω_0 is the eigenfrequency of the oscillator and T is the temperature of the thermostat), which is fulfilled for the optical range. We emphasize that the results obtained are exact, since no perturbation theory in the force f(t) is used.

The action of the external force f(t) on the oscillator causes the population densities of the levels, $w_n(t)$ to depend on time even when $\gamma t \gg 1$; this induces transitions between the levels. If we consider a charged quantum oscillator, dipole radiation arises. The spectrum of this radiation is determined in Sec. 4.

2. THE KINETIC EQUATION AND ITS SOLUTION

We consider an oscillator, acted upon by an external force f(t). Such a model describes, for example, the interaction of a field oscillator with a classical current or with the polarization of a medium in a resonator (in the linear approximation); the current or polarization induces buildup of the vibrations of the oscillator. As shown in I, the relaxation of an oscillator in the absence of an external medium has a "one-dimensional" character; each diagonal of the density matrix ρ_{mn} ($m - n$ = const) relaxes independently of the others. If we include the force f(t), different diagonals begin to mix and the problem becomes substantially more complicated (in particular, the method of generating functions, developed in I, is no longer applicable in the presence of the force). We consider two different methods of solving the kinetic equation I, (60).

1. The method of characteristic functions $\chi(\eta)$ and distributions of quasi-probabilities $W(\alpha)$. A general solution of the kinetic equations for $\chi(\eta, t)$ and $W(\alpha, t)$ is given by the formulas I, (65) and I, (69); using these, it is not difficult to obtain a picture of the evolution of the initial state on the α-plane (in particular, in the α-representation of Glauber[2] for the density matrix $\hat{\rho}$). However, the derivation of the population densities $w_n(t) = \rho_{nn}(t)$ of the different levels in this method requires the calculation of complicated integrals.

2. The method of Fock and Bargmann[5]. This is more convenient for the determination of the population densities $w_n(t)$. The density matrix of the oscillator in this representation has the following form:

$$R(z_1, z_2; t) = \sum_{m,\, n=0}^{\infty} \rho_{mn}(t) \frac{z_1^m \bar{z}_2^n}{(m!n!)^{1/2}} \qquad (1)$$

[1)] The paper [1] is referred to below as I; e.g. I, (10) signifies a reference to formula (10) in the paper I.

[2)] As already noted in I, the model of the quantum oscillator with damping is related directly to photocount statistics (for an electromagnetic field, coherent in first order; see Glauber [2]). Analogously, the relaxation of an oscillator under the action of an external force f(t) describes the photocount statistics for the sum of two signals (heterodyning). To see this, it is sufficient to assume that the optical heterodyne field is quasi-classical and corresponds to the coherent state. Then, a "superposition," in the sense of [2], of the amplitude of this coherent state with the amplitude of the light field being studied, corresponds completely to the result of relaxation in the presence of an external force (the role of this force is played by the heterodyne field).

(here, z_1, z_2 are independent complex variables; the bar denotes the complex conjugate). The function $R(z_1, z_2)$ is proportional to the matrix element of $\hat{\rho}$ in the coherent states:

$$R(\alpha,\beta)=\exp\{{}^1/_2(|\alpha|^2+|\beta|^2)\}\langle\bar{\alpha}|\hat{\rho}|\beta\rangle. \tag{2}$$

We recall that the coherent state $|\alpha\rangle$ has the form

$$|\alpha\rangle=\exp\Big(-\frac{1}{2}|\alpha|^2\Big)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

It is clear from (1) that the function $R(z_1, z_2; t)$ contains information about all the matrix elements $\rho_{mn}(t)$. The kinetic equation I, (60) takes upon transformation from $\hat{\rho}$ to R the form:

$$\frac{\partial R}{\partial t}=\gamma\Big[(\nu+1)\frac{\partial^2 R}{\partial z_1\partial\bar{z}_2}-\Big(\nu+\frac{1}{2}\Big)\Big(z_1\frac{\partial R}{\partial z_1}+\bar{z}_2\frac{\partial R}{\partial\bar{z}_2}\Big)+\nu(z_1\bar{z}_2-1)R\Big]-\frac{i}{\sqrt{2\omega_0}}\Big[f(t)e^{i\omega_0 t}\Big(\frac{\partial}{\partial\bar{z}_2}-z_1\Big)-\bar{f}(t)e^{-i\omega_0 t}\Big(\frac{\partial}{\partial z_1}-\bar{z}_2\Big)\Big]R, \tag{3}$$

where ω_0 is the eigenfrequency of the oscillator, γ is the damping constant, T is the temperature of the thermostat, and

$$\nu=(e^{\hbar\omega_0/kT}-1)^{-1}. \tag{4}$$

It is possible to show[6] that the solution of equation (3) under arbitrary initial conditions is of the type:

$$R(z_1,z_2;t)=\int d\mu(\zeta_1)\,d\mu(\zeta_2)\,G(z_1,z_2;\zeta_1,\zeta_2|t)\,R(\zeta_1,\zeta_2;0); \tag{5}$$

here[3] $d\mu(\zeta)=\pi^{-1}\exp(-|\zeta|^2)d^2\zeta$ and the Green function G is equal to

$$G(z_1,z_2;\zeta_1,\zeta_2|t)=\frac{1}{1+q\nu}\exp\Big\{\frac{F(z_1,z_2;\zeta_1,\zeta_2|t)}{1+q\nu}\Big\}, \tag{6}$$

where

$$F=q\nu z_1\bar{z}_2+q(\nu+1)\bar{\zeta}_1\zeta_2+p^{1/2}(z_1\bar{\zeta}_1+\bar{z}_2\zeta_2-\bar{v}\bar{\zeta}_1-v\zeta_2)+\nu z_1+\bar{v}\bar{z}_2-|v|^2 \tag{6a}$$

and we use throughout the notation

$$p=\exp(-\gamma t),\quad q=1-\exp(-\gamma t), \tag{7a}$$

$$v(t)=\frac{i}{\sqrt{2\omega_0}}\int_0^t f(t')\exp\Big[-\frac{\gamma}{2}(t-t')+i\omega_0 t'\Big]dt' \tag{7b}$$

(the quantity v(t) coincides with the complex amplitude of the forced vibrations of a classical operator excited by a force f).

The formula (5), in principle, solves the problem of the evolution of an arbitrary initial state. To obtain the population densities $w_n(t)$ of the different levels and the non-diagonal elements of the density matrix $\rho_{mn}(t)$, we have only to expand the function $R(z_1, z_2; t)$ in powers of z_1 and z_2.

We note that the evolution of the density matrix $\hat{\rho}(t)$, according to Eq. (5), may be described as the result of three consecutive transformations on the initial density matrix $\hat{\rho}(0)$:

1) Relaxation with zero thermostat temperature and $f(t)\equiv 0$. This process corresponds to attenuation of the complex amplitude on the Glauber α-plane by a factor of $p^{-1/2}$. The corresponding Green function is obtained from (6) with $\nu=\nu=0$.

2) Increase of Gaussian fluctuations of the amplitude (with $f(t)\equiv 0$ and without consideration of the attenuation of the amplitude). The Green function corresponding to this transformation is obtained from (6) after the replacements $p\to 1$, $q\to 0$, $q\nu\to\nu(1-e^{-\gamma t})$, $\nu\to 0$.

3) The unitary transformation:

$$\hat{\rho}'=D(v)\hat{\rho}D^{-1}(v),\quad D(v)=\exp(va^+-\bar{v}a), \tag{7c}$$

i.e., the displacement of the complex amplitude of the oscillator by a vector v(t). The transformation (7c) is identical in form with the change of $\hat{\rho}(t)$ for an oscillator without damping under the influence of an external force (the attenuation γ appears in D(v) only through the definition of v(t), cf. formula (7b)).

The transformations (1) and (2), in contrast to (3), are non-unitary; this is characteristic for a relaxation process[4]. It is natural to combine them into the one (also non-unitary) transformation considered in detail in I.

3. SOLUTIONS FOR PARTICULAR CASES

We consider the time evolution of a number of initial states of physical interest.

1. Let

$$R(z_1,z_2;0)=\frac{1}{1+\nu_0}\exp\Big[\frac{\nu_0 z_1\bar{z}_2+\alpha z_1+\bar{\alpha}\bar{z}_2-|\alpha|^2}{1+\nu_0}\Big], \tag{8}$$

where $\nu_0\geq 0$ and α is an arbitrary complex number. Such a density matrix corresponds[5] to a characteristic function $\chi_N(\eta)=\exp(-\nu_0|\eta|^2+\bar{\alpha}\eta-\alpha\bar{\eta})$, i.e., to a superposition of the coherent state $|\alpha\rangle$ and Gaussian noise (a Planck distribution with the parameter $\nu_0=[\exp(\hbar\omega_0/kT_0)-1]^{-1}$).

As Glauber has shown, a superposition of two states corresponds to combination of electromagnetic fields created by independent sources. Therefore, the example (8) is related to the case, often encountered in quantum optics, when the incoming field consists of a determined signal (the coherent state $|\alpha\rangle$) on which random thermal noise is imposed.

Putting (8) into (5), we find

$$R(z_1,z_2;t)=\frac{1}{1+\mu}\exp\Big[\frac{\mu z_1\bar{z}_2+\beta z_1+\bar{\beta}\bar{z}_2-|\beta|^2}{1+\mu}\Big], \tag{9}$$

where

$$\mu=p\nu_0+q\nu,\quad \beta=\beta(t)=p^{1/2}\alpha+v(t) \tag{9a}$$

(p, q and v(t) are defined in (7)). Comparison of (9) and (8) demonstrates the invariance of the form of the given state in the process of its relaxation; only the parameters β and μ change. In this case, $\mu(t)$ can be identified with the average energy of an oscillator relaxing in the

[3] The integration in (5) is carried out over the whole plane, $d^2\zeta=d\zeta_1 d\zeta_2$, where $\zeta=\zeta_1+\zeta_2$ ($-\infty<\zeta_1,\zeta_2<\infty$). We note that the measure $d\mu(\zeta)$ is normalized to unity: $\int d\mu(\zeta)=1$.

[4] We note that in the case of negative temperature of the thermostat (buildup of the vibrations, $\gamma<0$), the transformation (1) alone, i.e., without (2), does not conserve the non-negative character of the density matrix.

[5] We point out here the connection between R and the normal characteristic function:

$$R(z_1,z_2)=\frac{1}{\pi}e^{z_1 z_2}\int d^2\eta\,\chi_N(\eta)\exp\{-(|\eta|^2+z_1\eta-z_2\eta)\}.$$

absence of an external force, and $\beta(t)$ is the amplitude of the vibrations of the classical oscillator with the initial condition $\beta(0) = \alpha$.

From (9) we obtain the formulas for the population densities:

$$w_n(t) = \frac{\mu^n}{(1+\mu)^{n+1}} \exp\left\{-\frac{|\beta|^2}{1+\mu}\right\} L_n\left(-\frac{|\beta|^2}{1+\mu}\right), \quad (10)$$

and also those for the non-diagonal elements of the density matrix:

$$\rho_{mn}(t) = R_{mn}(\beta, \mu). \quad (10a)$$

Here $L_n(x)$ is the Laguerre polynomial normalized according to the tables[7] (we note that $L_n(x) > 0$ for $x \leq 0$), and the function $R_{mn}(\beta, \mu)$ is defined for $m \geq n$ by formula I, (34a). In this case $R_{nm}(\beta, \mu) = R_{mn}(\bar{\beta}, \mu)$.

In the classical limit ($|\beta|^2 \gg 1$), expression (10) is simplified:

$$w_n(t) \approx \frac{1}{[2\pi(1+2\mu)n_{cl}]^{1/2}} \exp\left\{-\frac{(n-n_{cl})^2}{2(1+2\mu)n_{cl}}\right\}, \quad (10b)$$

where $n_{cl} = E_{cl}/\hbar\omega_0 = |\beta(t)|^2$ (E_{cl} is the energy which the classical oscillator has at time t).

2. Putting $\alpha = 0$ we arrive at formulas describing the relaxation of a Planck distribution (with parameter ν_0 and initial temperature T_0). In this case $\beta(t) \equiv v(t)$. If we neglect the relaxation, then $\mu(t) = \nu_0$ and we must put $\gamma = 0$ in the expression (7b) for v(t). After these simplifications, the expression (10) goes over to the formula obtained by Schwinger[8].

3. In the particular case of $\nu_0 = 0$ the formulas of subsection 1 of this section describe the relaxation of the coherent state $|\alpha\rangle$ (in this case $\mu = q\nu$). An especially simple result is obtained with zero thermostat temperature[6] ($\nu = 0$), namely, $\hat{\rho}(t) = |\beta(t)\rangle\langle\beta(t)|$, i.e., the oscillator at any moment of time is in the coherent state $|\beta(t)\rangle$. This is the only case where the relaxation leaves the state pure in spite of the interaction with a dissipative subsystem (in the given case, with vacuum fluctuations of the electromagnetic field). This fact again underlines the quasiclassical nature of the coherent states.

4. The values $\alpha = \nu_0 = 0$ correspond to the nonexcited oscillator, $\hat{\rho}(0) = |0\rangle\langle 0|$. As before, $\rho_{mn}(t)$ is correctly given by the formulas (10) and (10a), where now $\mu = q\nu$, $\beta(t) = v(t)$. With $\nu = 0$, a Poisson distribution is obtained for the population densities $w_n(t)$:

$$w_n(t) = e^{-\lambda}\lambda^n/n!, \quad \lambda = \lambda(t) = |v(t)|^2. \quad (11)$$

This formula was obtained by Feynman[3] for an oscillator without damping. It is interesting to note that with $\nu = 0$, inclusion of the damping affects only the value of the amplitude v(t) and does not change the form of the distribution (11). For $\nu > 0$, this is no longer so.

5. A more complicated case is when the initial state of the oscillator is an N-quantum state, $\rho_{mn}(0) = \delta_{mN}\delta_{nN}$. Evaluation of the integrals (5) gives[6]

$$R(z_1, z_2; t) = \frac{[q(1+\nu)]^N}{(1+q\nu)^{N+1}} \times L_N\left(-\frac{p(z_1-\bar{v})(\bar{z}_2-v)}{q(1+\nu)(1+q\nu)}\right) \exp\left\{\frac{qvz_1\bar{z}_2 + vz_1 + \bar{v}\bar{z}_2 - |v|^2}{1+q\nu}\right\}. \quad (12)$$

From this are derived the following expressions for the elements of the density matrix[7]:

$$w_n(t) = \sum_{k=0}^{\infty} w'_{N\to k}(t)\, w''_{k\to n}(t), \quad \rho_{mn}(t) = \sum_{k=0}^{\infty} w'_{N\to k}(t)\, \rho''_{mn}(t; k). \quad (13)$$

Here $w'_{N\to k}(t)$ is the probability of the transition $N \to k$ during the time t, if the force f(t) is absent ($w'_{N\to n}$ is given by formula I, (23)); $w''_{k\to n}(t)$ is the probability of the transition $k \to n$ under the action of a force f(t), if the relaxation is neglected but the expression for v(t) is taken with inclusion of the attenuation γ (cf. (7b)). Finally, $\rho''_{mn}(t; N)$ is the density matrix of an oscillator in an N-quantum state at t = 0, under the same conditions as for $w''_{N\to n}$. An explicit expression for $w'_{N\to n}$ may be obtained within the framework of ordinary quantum mechanics (without considering the relaxation); cf.[3] and especially[4]. A slight generalization of these calculations leads to the formula:

$$\rho''_{mn}(t; N) = (-1)^{m+n} \frac{\sqrt{m!n!}}{N!} e^{-\lambda}\lambda^{N-(m+n)/2} L_m^{N-m}(\lambda) L_n^{N-n}(\lambda) \times e^{i(m-n)\varphi}, \quad (14)$$

$$v(t) = \sqrt{\lambda}e^{i\varphi}, \quad w''_{k\to n}(t) \equiv \rho''_{nn}(t; k).$$

The expressions (13) for w_n and ρ_{mn} have, generally, a complex form. We shall indicate a few simple cases.

A. Putting $z_1 = z_2 = 0$ in (12), we find the population density of the ground level:

$$w_0(t) = \frac{[q(1+\nu)]^N}{(1+q\nu)^{N+1}} \exp\left\{-\frac{|v|^2}{1+q\nu}\right\} L_N\left(-\frac{p|v|^2}{q(1+\nu)(1+q\nu)}\right). \quad (15)$$

B. For $\nu = 0$ (zero thermostat temperature) the expression for $w'_{N\to k}(t)$ is markedly simplified, as the relaxation is only downwards ($k \leq N$). In this case

$$w_n(t) = e^{-\lambda} \sum_{k=0}^{N} \frac{n_<!\, N!}{n_>!\, k!\, (N-k)!} p^k q^{N-k} \lambda^m |L_{n_<}^m(\lambda)|^2, \quad (16)$$

where

$$n_< = \min(k, n), \quad n_> = \max(k, n), \quad m = |k-n|, \quad \lambda = |v(t)|^2$$

In particular, for N = 0, the formula (16) goes over to (11), while for N = 1 we have

$$w_n(t) = e^{-\lambda}\frac{\lambda^{n-1}}{n!}[p(n-\lambda)^2 + q\lambda], \quad w_n(0) = \delta_{n,1}. \quad (11a)$$

C. The generating function for the population densities $w_n(t)$ for $\nu = 0$ has the form

$$G_N(z, t) = \sum_{n=0}^{\infty} w_n(t) z^n = (1-p\zeta)^N e^{-\lambda\zeta} L_N\left(-\frac{p\lambda\zeta^2}{1-p\zeta}\right), \quad (17)$$

$$\zeta = 1 - z.$$

D. Let the initial state be an incoherent mixture of n-quantum states, $\rho_{mn}(0) = \delta_{mn}w_n(0)$, where $w_n(0)$ is arbitrary. Then the average energy of the oscillator and its dispersion vary according to the rules

[6] In optics, ν is practically equal to zero. Thus for the light of a ruby laser ($\hbar\omega_0 = 1.78$ eV), $\nu \sim 10^{-30}$ for T = 300°K. Values of $\nu \sim 1$ are attained in the radio-frequency range.

[7] The possibility of representing $\rho_{mn}(t)$ in the form (13) is a result of the separation, indicated in Sec. 2, of the evolution of $\hat{\rho}(t)$ into the simpler transformations (1)–(3). In this case $\rho'(t)$ corresponds to the transformation (1) + (2), and ρ'' to the transformation (3). Formula (14) follows from (7c) if we take note of the form of the matrix elements:

$$D_{mn}(\eta) = (n!/m!)^{1/2}\eta^{m-n}L_n^{m-n}(|\eta|^2)\exp(-\tfrac{1}{2}|\eta|^2).$$

$$\bar{n}(t) = \bar{n}_0 p + \nu q + \lambda,$$
$$\overline{\Delta n^2}(t) = \overline{\Delta n_0^2} p^2 + (2\nu+1)(\bar{n}_0 - \nu) pq + \nu(\nu+1)(1-p^2) + \lambda[1 + 2(\bar{n}_0 p + \nu q)]. \quad (18)$$

In the absence of the force, $\lambda = 0$, and these formulas go over into I, (10).

E. We find the limiting expressions for the population densities $w_n(t)$ when $\gamma t \ll 1$ and $\gamma t \gg 1$. In the first case the relaxation is still insignificant, and $w'_{N\to k}(t) \to \delta_{Nk}$. Therefore, from (13) and (14) we obtain

$$w_n(t) = \frac{n_<!}{n_>!} \lambda^m |L_{n_<}^m(\lambda)|^2 e^{-\lambda}, \quad (19)$$

where

$$n_< = \min(N, n), \quad n_> = \max(N, n), \quad m = |N - n|,$$

and we must take $\gamma = 0$ in the formula (7b) for $\nu(t)$. The expression (19) coincides with Schwinger's result[4]. It is investigated in more detail in Appendix A and its range of applicability is discussed in Appendix B.

In the opposite case $\gamma t \gg 1$ the relaxation has already been completed and the memory of the initial state disappears. The function $R(z_1, z_2; t)$ takes the form (9) with values of the parameters $\mu = \nu$ and $\beta = \nu(t)$.

The intermediate case $\gamma t \sim 1$ is much more complicated. To understand the qualitative picture we assume that $N, \lambda \gg 1$ (a strongly excited oscillator). Then in (13), (14), and (19) we may use a quasi-classical asymptotic form for the polynomials $L_n^m(\lambda)$. The relevant formulas are given in Appendix B.

4. THE RADIATION SPECTRUM OF THE OSCILLATOR

We now discuss the question of the spectrum of quanta radiated by a charged oscillator on which a force f(t) acts. Since the external force induces transitions between the levels $E_n = N\hbar\omega_0$, it would seem at first sight that the radiation spectrum should consist of harmonics $\omega_n = n\omega_0$, independently of the form of the force. In fact, just as in the case of a classical oscillator, the radiation spectrum is completely defined by the spectrum of the acting force. The reasoning given above is erroneous, since it does not take into account the equal spacing of the spectrum of the oscillator, which leads to interference of the quanta emitted during the transitions between different levels (compare with the discussion of the so-called "harmonic oscillator paradox" in I).

To consider this question we take the Louisell[9] model, in which we add the interaction of the oscillator with the external force to the Hamiltonian H_0; $H = H_0 + H_1$, where

$$H_0 = \omega_0 \hat{a}^+ \hat{a} + \sum_j \omega_j \hat{b}_j^+ \hat{b}_j,$$
$$H_1 = \sum_j (f_j \hat{a}^+ \hat{b}_j + f_j^* \hat{a} \hat{b}_j^+) - \frac{1}{\sqrt{2\omega_0}} (f(t)\hat{a}^+ + f^*(t)\hat{a}). \quad (20)$$

Here $\hbar = m = 1$, j labels the field oscillator, f_j is the interaction constant, and $\hat{a}$ and $\hat{b}_j$ are the operators in the Heisenberg representation of the charged oscillator and of the j-th field oscillator respectively. Going over to the interaction representation,

$$\hat{a}(t) = a(t) e^{-i\omega_0 t}, \quad \hat{b}_j(b) = b_j(t) e^{-i\omega_j t}, \quad (21)$$

we obtain from (20) equations for the operators a(t) and $b_j(t)$:

$$a = -i \sum_j f_j \exp[i(\omega_0 - \omega_j)t] b_j + \frac{i}{\sqrt{2\omega_0}} f(t) \exp(i\omega_0 t),$$
$$\dot{b}_j = -i f_j^* \exp[i(\omega_j - \omega_0)t] a. \quad (22)$$

It is essential that these equations are linear; the force f(t) occurs in the equation for a(t) as an additional c-number term. Therefore, the solution for a(t) has the form (in the Weisskopf-Wigner approximation)

$$\hat{a}(t) = e^{-i\Omega t} \hat{a}(0) + \sum_j v_j(t) \hat{b}_j(0) + e^{-i\omega_0 t} v(t), \quad (23)$$

where

$$\Omega = \omega_0 - i\gamma/2, \quad v_j(t) = f_j (e^{-i\omega_j t} - e^{-i\Omega t}) / (\omega_j - \Omega),$$

and v(t) is defined by (7b). For $f(t) \equiv 0$ the expression (23) coincides with that found by Louisell[9].

The spectrum of the emitted quanta is given by formula I, (74), in which the correlation function $\langle \hat{a}^+(t+\tau)\hat{a}(t)\rangle$ occurs. From (23), we find[8)]

$$\langle \hat{a}^+(t_2)\hat{a}(t_1)\rangle = \exp[-i\omega_0(t_1 - t_2)]\{\bar{n}(0)\exp[-\gamma(t_1+t_2)/2] + v(t_1)v^*(t_2)\}, \quad (24)$$

where $\bar{n}(0) = \langle a^+(0)a(0)\rangle$ and it is assumed that $\langle a(0)\rangle = \langle a^+(0)\rangle = 0$ and $\nu = 0$ (in the case $\nu > 0$ the question arises of discerning the quanta emitted by the oscillator against the infinite thermal background). Putting (24) into I, (74) we find, after certain transformations

$$\frac{dE}{d\omega} = g(\omega) = \frac{\gamma}{2\pi[(\omega - \omega_0)^2 + \gamma^2/4]} \{\bar{n}(0) + |f_\omega|^2\}, \quad (25)$$

where

$$f_\omega = \frac{1}{\sqrt{2\omega_0}} \int_0^\infty dt\, f(t)\, e^{i\omega t}.$$

The first term in the curly brackets describes the deexcitation of the initial excitation of the oscillator, and the second the radiation under the action of the force f(t). If f(t) is a periodic function, $|f_\omega|^2$ for $\gamma t \gg 1$ contains a term which increases linearly with t.

The assertions made at the beginning of this section follow from (25). If the oscillator is excited by a monochromatic line of frequency ω, then, basically (i.e., for a time $t \gg t/\gamma$), it radiates the same monochromatic line (the intensity contains a resonance factor $[(\omega - \omega_0)^2 + \gamma^2/4]^{-1}$). The radiation spectrum (25) of a quantum oscillator is the same as in the classical problem. Here the "harmonic oscillator paradox" appears.

In conclusion we note that essential in the preceding work was the fact that linear relaxation of the oscillator, even in the presence of an external force f(t), does not mix the creation (a^+, b_j^+) and annihilation (a, b_j) operators amongst themselves. In contrast to this, for an oscillator with parametrically changing frequency $\omega(t)$, the operators a and a^+ are mixed in the process of time-evolution. As a result such an oscillator can intensify its zero fluctuations and radiate an amplified field[10].

8)From (23) it follows also that $[\hat{a}(t), \hat{a}^+(t')] = \exp[-\gamma|t - t'|/2 - i\omega_0(t - t')]$. The attenuation of the commutator when $\gamma|t - t'| \gg 1$ arises because of the spontaneous radiation which introduces an uncontrolled phase difference between the operators $\hat{a}(t)$ and $\hat{a}^+(t)$.

APPENDIX A

We shall analyze formula (19) in two limiting cases, $\lambda = |v(t)|^2 \ll 1$ (the complex amplitude of v(t) is much smaller than the amplitude of the vacuum fluctuations) and $\lambda \gg 1$. In the first case we have

$$w_{N\to n}(t) = \begin{cases} 1-(2N+1)\lambda + O(\lambda^2) & n = N, \\ \dfrac{n_>!}{n_<!\, m!}\lambda^m & n \neq N \end{cases} \quad (A.1)$$

The transition probabilities $w_{N\to n}$ rapidly decrease with increase of $m = |N - n|$.

The opposite case $N, \lambda \gg 1$, corresponds to a strongly excited oscillator. We use the quasi-classical approximation

$$w_{N\to n} = \frac{2\cos^2\Phi_n}{\pi\sqrt{R_n}}, \quad (A.2)$$

$$R_n = [(\sqrt{N}+\sqrt{n})^2-\lambda][\lambda-(\sqrt{N}-\sqrt{n})^2] = (n-n_1)(n_2-n),$$

$$\Phi_n = \frac{1}{2}\int_{x_1}^{\lambda}\frac{[(x-x_1)(x_2-x)]^{1/2}}{x}dx - \frac{\pi}{4}.$$

Here $x_{1,2} = (\sqrt{N} \pm \sqrt{n})^2$, $n_{1,2} = (\sqrt{N} \pm \sqrt{\lambda})^2$. The distribution of the transition probabilities $w_{N\to n}$ lies mainly in the range $n_1 < n < n_2$, beyond the limits of which $w_{N\to n}$ falls away exponentially. On averaging over a rapidly-varying phase Φ_n, we obtain

$$w^{cl}_{N\to n} = 1/\pi[(n-n_1)(n_2-n)]^{1/2}. \quad (A.3)$$

which corresponds to the purely classical picture[9]. The quantum effects are contained in the factor $\cos^2\Phi_n$, which leads to oscillations of $w_{N\to n}$ about its mean value. Taking account of the analogy between expression (19) for $w_{N\to n}$ and the form of radial wave functions in a Coulomb potential, we apply the same parametrization as in the Coulomb problem to calculate the phase Φ_n[11]:

$$x = a(1 - e\cos\xi), \quad 0 \leqslant \xi \leqslant \pi, \quad (A.4)$$

$$a = \frac{x_1+x_2}{2} = N+n, \qquad e = \frac{x_2-x_1}{x_2+x_1} = \frac{2\sqrt{Nn}}{N+n}. \quad (A.5)$$

From (A.2) we then obtain a fairly simple expression for Φ_n:

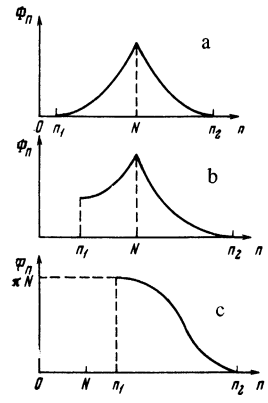
$$\Phi_n = \frac{a}{2}\left\{\xi + e\sin\xi - \sqrt{1-e^2}\arccos\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\right\} - \frac{\pi}{4}. \quad (A.6)$$

ξ is defined from the equation: $\lambda = a(1 - e\cos\xi)$, whence

$$n = N(\cos\xi \pm \sqrt{\lambda/N - \sin^2\xi})^2. \quad (A.7)$$

For $\lambda < N$ here we must take both signs, and ξ varies in the interval $0 \leq \xi \leq \arcsin\sqrt{\lambda/N}$; for $\lambda > N$ in (A.7) we must retain only the upper sign, with $0 \leq \xi \leq \pi$. The dependence of the phase Φ_n on the state number n is set completely by equations (A.5)–(A.7). The explicit expression for Φ_n in terms of n is extremely cumbersome.

We require the dependence of the phase Φ_n on n when the values of N and λ are fixed. It is difficult to assess



The dependence on n of the phase Φ_n in the classically allowed region $n_1 < n < n_2$; a) $\lambda < N$, b) $N < \lambda < 4N$, c) $\lambda > 4N$. Here N is the initial excitation of the oscillator (when t = 0 the oscillator is in an N-quantum state) and $\lambda = |v(t)|^2$ is the energy (in units of $\hbar\omega_0$) acquired by the oscillator under the influence of the external force.

this as all three parameters a, e, and ξ vary with change of n. However,

$$\frac{\partial\Phi_n}{\partial n} = \frac{1}{2}\int_{x_1}^{\lambda}\frac{x+N-n}{x[(x-x_1)(x_2-x)]^{1/2}}dx = \frac{1}{2}\left[\xi \pm \arccos\left(\frac{\cos\xi - e}{1-e\cos\xi}\right)\right], \quad (A.8)$$

where the plus sign is taken in the case $n < N$ and the minus in the case $n > N$ (in calculating the integral it is convenient to make the substitution (A.4)). Hence it follows that $\partial\Phi/\partial n > 0$ when $n < N$, and $\partial\Phi/\partial n < 0$ when $n > N$ ($e = 1$, $\xi = \xi_0 = \arccos(1 - \lambda/2N)$ and $\Phi_N = N(\xi_0 + \sin\xi_0)$ correspond to the equality $n = N$). The form of the curve $\Phi_n = \Phi(n)$ depends on the relation between N and λ (see the figure). When $n = N$, there is a break; however, if $\lambda > 4N$, then $n_1 > N$ and the curve for Φ_n is smooth within the limits of the classically-allowed range $n_1 \leq n \leq n_2$.

APPENDIX B

In the limiting case of large quantum numbers the expression (13) for $w_n(t)$ may be simplified if we use the approximate formulas I, (43) and (A.2) for $w'_{N\to k}$ and $w''_{k\to n}$. Going over in (13) from a sum over k to an integration, we have

$$w_n(t) = \frac{2}{\pi\sqrt{2\pi\sigma^2}}\int_{k_1}^{k_2}dk\frac{\cos^2\Phi_k}{[(k-k_1)(k_2-k)]^{1/2}}\exp\left\{-\frac{(k-Np)^2}{2\sigma^2}\right\}, \quad (B.1)$$

where $\sigma^2 = Npq(1 + 2\nu)$, $k_{1,2} = (\sqrt{n} \pm \sqrt{\lambda})^2$. The factor $\cos^2\Phi_k$ leads to oscillations of the distribution w_n, which are rapidly smoothed away in the relaxation process. This takes place when $\sigma \gtrsim \Delta n$, where Δn is the average distance between the zeros of the function $\cos\Phi_n$. An estimation of Δn, obtained by the use of formula (A.6), gives $n \sim (n_2 - n_1)/D$, where D is the number of zeros of the function $\cos\Phi_n$ lying between n_1 and n_2:

$$D \sim \begin{cases} \sqrt{N\lambda}, & \lambda \ll N \\ N, & \lambda \gtrsim N \end{cases}. \quad (B.2)$$

Hence it follows that the oscillations of $w_n(t)$ are appreciable only for very small times $t \ll (1 + \lambda/N)t_1$, where $t_1 = [N\gamma(1 + 2\nu)]^{-1}$ is the time of mixing (cf. Sec. 4[11]). When $t \sim (1 + \lambda/N)t_1$ the oscillations of the distribution (19) are already obscured by the relaxation; at the same time $p = e^{-\gamma t} \approx 1$, i.e., the relaxation process is still only just beginning. In this case, $\cos^2\Phi_k \to \frac{1}{2}$ and the expression (B.1) takes the form

$$w_n(t) = \frac{1}{\sqrt{2\pi\sigma^2}}F(\alpha,\beta), \qquad \alpha = \left[\frac{2n\lambda}{Npq(1+2\nu)}\right]^{1/2},$$

[9] The action of the external force f(t) leads to the displacement of the initial distribution $W(\alpha, t = 0)$ by the vector v(t) (cf., e.g., formula I, (69)). Choosing $W(\alpha, 0) = \pi^{-1}\delta(|\alpha|^2 - N)$, we arrive at (A3).

$$\beta = \frac{n + \lambda - Np}{[2Npq(1+2\nu)]^{1/2}}, \tag{B.3}$$

where

$$F(\alpha, \beta) = \frac{1}{\pi}\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} e^{-(\alpha x - \beta)^2} = \begin{cases} \exp(-\beta^2) & \alpha = 0 \\ \exp(-\alpha^2/2) I_0(\alpha^2/2) & \beta = 0 \end{cases} \tag{B.4}$$

We note that the parameter α characterizes the relation of the width of the distribution (19), brought about by the action of the force f(t), to the width σ, arising from the relaxation. In the case $\alpha \ll 1$ we may expand the function $F(\alpha, \beta)$ in a rapidly convergent series:

$$F(\alpha, \beta) = e^{-\beta^2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{2^{2n}(n!)^2} H_{2n}(\beta), \tag{B.5}$$

where $H_{2n}(\beta)$ are Hermite polynomials. In the opposite case, $\alpha \gg 1$, the distribution of the population densities w_n is similar to (A.3) and is concentrated mainly in the range $n_1 \leq n \leq n_2$. Outside this region, w_n decreases exponentially.

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