STABILITY OF SOLITARY ELECTROACOUSTIC WAVES

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The stability of electroacoustic solitons is investigated in liquids with negative dielectric constant. It is shown that low-amplitude solitons are stable against small perturbations.

IT has been shown earlier[1] that the nonlinear interaction between a high-frequency electromagnetic wave and density oscillations leads to the possibility of propagation of nonlinear electroacoustic waves in a medium with a negative dielectric constant $\epsilon < 0$; these are carried with the electromagnetic fields itself (in the linear approximation, propagation of these waves is not possible in such media). The simplest example of an electroacoustic wave in a medium characterized by $\epsilon < 0$ is a stationary individual wave (soliton). By analogy with other cases in which solitons appear, one expects that the latter should play an important role in the dynamics of nonstationary electroacoustic processes.

In the present note we consider the stability of electroacoustic solitons in fluids. We shall limit our investigation to low-amplitude solitons. Under these conditions the basic equations for electroacoustic waves [cf.[1], Eqs. (2.24) and (3.7)] can be written in the form

$$a_{xx} - \mu^2\gamma^{-2}(\gamma^2 + \nu)a = 0, \tag{1}$$

$$\nu_t + c_s\nu_x = -\frac{c_s}{2E_c^2}(a^2)_x + O\left(\frac{a^4}{E_c^4}\right), \tag{2}$$

$$(a^2)_t + c_s(a^2)_x = O\left(\frac{a^4}{E_c^4}\right). \tag{3}$$

Here, ν is the relative deviation of the fluid density ρ from the equilibrium value ρ_0: $\nu = (\rho - \rho_0)/\rho_0$; $E_c^2 = 16\pi c_s^2/|\partial\epsilon_0/\partial\rho_0|$, c_s is the acoustic velocity in the medium, $\epsilon_0 = \epsilon(\omega, \rho_0)$ is the dielectric constant of the medium at equilibrium density ρ_0 ($\epsilon_0 < 0$); a is the amplitude of the electric field, ($\mathscr{E} = (1/2)[a(x, t)e^{-i\omega t} + \text{c.c.}]$

$$\mu^2 = -\omega^2\varepsilon_0/c^2, \quad \gamma^2 = -\varepsilon_0/|\rho_0\partial\varepsilon_0/\partial\rho_0|. \tag{4}$$

The quantity μ^{-1} represents the penetration distance of an electromagnetic field in a medium characterized by $\epsilon = \epsilon_0 < 0$ in the linear approximation, that is to say, the dimension of the skin depth.

From Eqs. (1—3) we can obtain conservation relations for waves that damp at $x \to \pm\infty$:

$$\frac{d}{dt}\int_{-\infty}\nu\,dx = 0, \tag{5}$$

$$\frac{d\Phi}{dt} = 0, \quad \Phi = \frac{\mu}{\gamma^4}\int_{-\infty}^{\infty}\nu^2\,dx, \tag{6}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty}a^2\,dx = 0. \tag{7}$$

The relation in (5) has the meaning of mass conservation while those in (6) and (7) represent the conservation of elastic energy and electric energy respectively in the wave.

The stationary solution of Eqs. (1)—(3), which describes a solitary wave, is of the form

$$\nu(x, t) = -2\gamma^2\,\text{sech}^2\,\mu(x - wt), \tag{8}$$

$$a(x, t) = a_m\,\text{sech}\,\mu(x - wt), \tag{9}$$

$$a_m = 2E_c(1 - M)^{1/2}\gamma, \tag{10}$$

where $M = w/c_s$ is the electroacoustic Mach number, which is always smaller than unity. The condition that the amplitude be small, which is necessary if Eqs. (2) and (3) are to be valid, reduces to the form[1)]

$$1 - M \ll 1. \tag{11}$$

We now introduce a new variable $z = \tanh\mu(x - wt)$ and represent all quantities as functions of z and t; in these variables the soliton equation becomes

$$a_{(s)} = a_m\sqrt{1 - z^2}, \quad \nu_{(s)} = -2\gamma^2(1 - z^2).$$

We now assume that a perturbation exists on the soliton background so that

$$a(z, t) = a_m\sqrt{1 - z^2}[1 + \varphi(z, t)], \tag{12}$$

where $\varphi(z, 0) \ll 1$. Expressing ν in terms of a by means of Eq. (1), we have

$$\nu = \gamma^2(1 - z^2)\left[\frac{\hat{L}\varphi}{1 + \varphi} - 2\right], \tag{13}$$

$$\hat{L} = (1 - z^2)^{-1}\frac{d}{dz}\left[(1 - z^2)^2\frac{d}{dz}\right]. \tag{14}$$

Taking account of the form of the operator $\hat{L}$ we can expand the perturbation $\varphi(z, t)$ in Gegenbauer polynomials $C_n^{3/2}(z) = dP_{n+1}/dz$ (P_n is the Legendre polynomial) which are characteristic functions of the operator $\hat{L}$: $\hat{L}C_n^{3/2} = -n(n + 3)C_n^{3/2}$

$$\varphi(z, t) = \sum_{n=0}\alpha_n(t)C_n^{3/2}(z). \tag{15}$$

$$\alpha_n(t) = h_n^{-1}\int_{-1}^{1}\varphi(z, t)C_n^{3/2}(z)(1 - z^2)\,dz, \tag{16}$$

$$h_n = 2(n + 1)(n + 2)/(2n + 3). \tag{17}$$

Thus, the problem reduces to the investigation of the time behavior of a specified number of generalized coordinates $\alpha_n(t)$ which characterize the perturbation. In

1)The solution in (8)—(10) differs from the soliton solution obtained in [1] on the basis of more exact equations (cf. Eqs. (3.19)—(3.21) of [1]) in the fact that $1-M^2 \approx 2(1-M)$.

order to investigate this behavior we make use of the basic ideas of the Lyapunov method (cf. for example[2]). In carrying out this procedure we consider the quantity Φ, which is expressed as a functional of $\varphi(z, t)$ by Eqs. (6) and (13). The increment $\delta\Phi$ due to the perturbation of the soliton is

$$\delta\Phi \equiv \Phi\{\varphi(z,t)\} - \Phi\{0\} = 4\int_{-1}^{1}(1-z^2)\varphi\hat{L}\varphi\,dz + \int_{-1}^{1}(1-z^2)(\hat{L}\varphi)^2dz. \quad (18)$$

Substituting (15) in (18) we have

$$\delta\Phi = \sum_{n=0}^{\infty}\alpha_n^2(t)n(n+3)[n(n+3)-4]h_n. \quad (19)$$

The first two terms in this sum vanish so that $\delta\Phi$ is a positive-definite quadratic form of the generalized coordinates $\alpha_n(t)$ (n = 2, 3, ...). Assuming that $\Phi(t)$ is an integral of the motion and assuming that the initial perturbation is small, we find that $\delta\Phi(t) = \delta\Phi(0) < \epsilon^2$, where ϵ is specified beforehand to be a small number. It then follows that

$$\alpha_n^2(t) < \frac{\varepsilon^2}{n(n+3)[n(n+3)-4]h_n}, \quad n \geqslant 2. \quad (20)$$

Thus, the coefficients $\alpha_n(t)$ with n = 2, 3, ... remain small if they are small at the initial time.

We now investigate the quantities $\alpha_0(t)$ and $\alpha_1(t)$. Substituting Eqs. (12) and (15) in Eq. (7) we have

$$\sum_{n=0}^{\infty}\alpha_{2n}(t) = \sum_{n=0}^{\infty}\alpha_{2n}(0). \quad (21)$$

It follows from Eq. (21) and (20) that

$$\alpha_0(t) = \alpha_0(0) + O(\varepsilon). \quad (22)$$

Without loosing generality we can assume that

$$\alpha_0(0) = 0, \quad (23)$$

because if this is not the case, as is evident from Eqs. (12) and (15) we would be considering the problem of stability of a soliton with amplitude $\tilde{a}_m = a_m[1 + \alpha_0(0)]$ with the initial perturbation[2)]

$$\tilde{\varphi}(z, 0) = [\varphi(z, 0) - \alpha_0(0)]/[1 + \alpha_0(0)]. \quad (24)$$

[2)]We would simultaneously have to redefine Z, taking Z = tanh $\mu(x - c_s\tilde{M}t)$.

In order to obtain an equation that describes the variation of $\alpha_1(t)$ we substitute Eqs. (12), (13) and (15) in Eq. (2) and expand both sides in Gegenbauer polynomials. We then obtain the following system of equations for the quantity $\alpha_n(t)$:

$$\dot{\alpha}_1 = -{}^2/{}_3\mu c_s(1-M)(\alpha_0 + {}^{27}/{}_7\alpha_2), \quad (25)$$

$$n(n+3)\dot{\alpha}_n = -\mu c_s(1-M)\left\{\alpha_{n-1}[4-(n-1)(n+2)]\frac{n(n+1)}{(2n+1)} - \alpha_{n+1}[4-(n+1)(n+4)]\frac{(n+2)(n+3)}{(2n+5)}\right\}, \quad (26)$$

$$n = 2, 3, \ldots$$

Attention is directed to the fact that Eq. (26) does not contain the quantities $\alpha_0(t)$ and $\alpha_1(t)$, that is to say, this system of equations represents a closed system with respect to the parameters $\alpha_2(t)$ and $\alpha_3(t)$. Under these conditions the function Φ defined by Eq. (6) is the Lyapunov function for the system in (26).

It follows from Eqs. (20), (22), and (25) that

$$d\alpha_1/dt \approx O(\varepsilon). \quad (27)$$

It follows from Eq. (27) that for sufficiently small values of ϵ the characteristic time for a significant change in the coefficient $\alpha_1(t)$ will be much greater than the time in which terms that have been neglected in Eq. (1—3) become important. Thus, within the limits of the accuracy used here it can be assumed that an electroacoustic soliton of low amplitude is stable against small perturbations.

[1]V. Ts. Gurovich and V. I. Karpman, Zh. Eksp. Teor. Fiz. 56, 1952 (1969) [this issue, p. 1048].

[2]I. G. Malkin, Teoriya ustoĭchvosti dvizheniya (Theory of Stability) Fizmatgiz, 1966.

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