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SIMPLE WAVES IN THE KINETICS OF A RAREFIED PLASMA

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A general class of nonstationary solutions in which the distribution function depends only on the velocity and electric field potential is derived on the basis of self-similar solutions of the kinetic equation for a collisionless quasi-neutral plasma that were obtained previously. These solutions are similar to the Riemann solution for simple waves in ordinary hydrodynamics.

IN previous researches of the authors and Pariĭskaya, self-similar solutions were investigated for the collisionless kinetic equation for the case of a quasi-neutral plasma.[1-3] In the present work we shall show that one can construct even more general solutions on this basis, solutions which correspond to the Riemann solutions for simple waves in ordinary hydrodynamics (see, for example,[4], Sec. 94).

We note that simple waves in a plasma were investigated previously only for the case of cold ions, when it was possible to describe them in terms of the equations of hydrodynamics.[5-7] We shall consider the general kinetic case when the ions are described by distribution functions.

In the one-dimensional case the kinetic equation for the ion distribution function has the form

$$\frac{\partial g}{\partial t}+u\frac{\partial g}{\partial \xi}-\frac{1}{2}\frac{\partial g}{\partial u}\frac{\partial \psi}{\partial \xi}=0. \tag{1}$$

Here the variables

$$\xi=x\sqrt{\frac{M}{2T_e}};\quad u=v\sqrt{\frac{M}{2T_e}},\quad \psi=\frac{e\varphi}{T_e};\quad g=\frac{f_i}{N_0}\sqrt{\frac{2\pi T_e}{M}},$$

have been introduced for convenience; M is the mass of the ions, v their velocity, f_i the ion distribution functions, N_0 the undisturbed ion concentration, φ the electric field potential, T_e the temperature of the electrons in energy units. If all the quantities change slowly over distances of the order of the Debye radius, then the plasma is quasi-neutral:

$$N_e=N_i. \tag{2}$$

If the electrons are described by the Boltzmann distribution, which is the usual case, then it follows from (2) that

$$\psi=\ln\frac{N_i}{N_0}=\ln\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} g\,du\right). \tag{3}$$

The assumption of a Boltzmann distribution for the electrons has no significance in principle and is used only for definiteness. Actually, the solution can be obtained for any stationary distribution of the electrons in which N_e is locally connected with φ. This problem has been analyzed in detail in[3,8].

The solutions which we shall seek are characterized by the fact that the function g in them depends on ξ and t only through the intermediate variable ψ:

$$g(\xi,t,u)=g[\psi(\xi,t),u]. \tag{4}$$

(In self-similar solutions, where g and ψ depend only on the combination $\tau = \xi/t$, it is obvious that this is the case.) Substituting (4) in (1), we obtain

$$\frac{\partial g}{\partial \psi}\left[\frac{\partial \psi}{\partial t}+u\frac{\partial \psi}{\partial \xi}\right]-\frac{1}{2}\frac{\partial g}{\partial u}\frac{\partial \psi}{\partial \xi}=0,$$

or

$$\frac{\partial\psi/\partial t}{\partial\psi/\partial\xi}=\frac{1}{2}\frac{\partial g/\partial u}{\partial g/\partial\psi}-u. \tag{5}$$

The right side of Eq. (5) by assumption depends only on ψ and u, and the left side, only on ξ and t. It is therefore clear that these expressions are certain functions of ψ only, which we shall denote by $-\tau(\psi)$, so that

$$\frac{\partial\psi/\partial t}{\partial\psi/\partial\xi}=-\tau(\psi) \tag{6}$$

and

$$(u-\tau)\frac{\partial g}{\partial\psi}-\frac{1}{2}\frac{\partial g}{\partial u}=0. \tag{7}$$

Changing in (7) to differentiation with respect to τ, we get

$$(u-\tau)\frac{\partial g}{\partial\tau}-\frac{1}{2}\frac{\partial g}{\partial u}\frac{d\psi}{d\tau}=0. \tag{8}$$

This is the same equation which was obtained in[1] for g as a function of u and $\tau = \xi/t$. In our case, however, the dependence of τ on ξ and t can have a more general character. To make this dependence clear, we return to (6). We have

$$\tau(\psi)=-\frac{(\partial\psi/\partial t)_\xi}{(\partial\psi/\partial\xi)_t}=\left(\frac{\partial\xi}{\partial t}\right)_\psi=\left(\frac{\partial\xi}{\partial t}\right)_\tau;$$

whence

$$\xi=\tau t+p(\tau), \tag{9}$$

where p is an arbitrary function. If we set p = 0, then $\tau = \xi/t$, so that we are returned to the special case of self-similar motion.

Finally, we see that in the class of solutions of Eq. (1), (3) found by us, the distribution function is an arbitrary solution of the self-similar equation in the variables u and τ, where τ is connected with the physical variables ξ and t by Eq. (9), which contains the arbitrary function $p(\tau)$. In accord with (9), a straight line in the (ξ, t) plane corresponds to each value of τ. On this straight line, the function g has a constant value, g(τ, u). (Naturally, in a problem with initial conditions, we must consider only t > 0.)

If we solve the problem with initial conditions then, for t = 0,

$$\xi=p(\tau)$$

and $\tau = q(\xi)$, where q is a function inverse to p. Thus, our solution corresponds to the problem in which the distribution function at the initial instant is

$$g[t=0, \xi, u] = g_a[q(\xi), u],$$

where g_a is an arbitrary solution of the self-similar equation (8), and q is an arbitrary function of ξ. If we recall that g_a is determined by two arbitrary functions of the velocity u (the boundary conditions for $\tau \to \pm\infty$), then we see that our solution depends on three arbitrary functions, two of u and one of ξ.

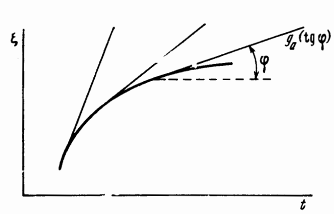
As in hydrodynamics, the relation (9) cannot have a meaning over the entire (ξ, t) plane. Actually, let there be an inflection point τ_0 on the curve $p(\tau): p''(\tau_0) = 0$. Then, beginning at the instant of time $t_0 = -p'(\tau_0)$, the dependence of τ on ξ becomes non-single-valued. In reality, for $\tau \approx \tau_0$, $t \approx t_0$, (9) can be represented in the form

$$\xi^* - \tau_0 t^* = \tau^* t^* - \alpha\tau^{*3},$$
$$\tau^* = \tau - \tau_0, \quad t^* = t - t_0, \quad \xi^* = \xi - \tau_0 t_0 - p(\tau_0), \quad \alpha = -{}^1/_6 p'''(\tau_0). \tag{10}$$

We assume that $\alpha > 0$, considering the solutions that were single-valued at the initial instant of time, and only became non-single-valued later. Equation (10) has one root for $\tau(\xi, t)$ for $t^* > 0$ and three for $t^* > 0$. At the same time $t^* = 0$, $\tau^* = -(\xi^*/\alpha)^{1/2}$ so that the solution has the form

$$g = g_a[\tau_0 - (\xi^*/\alpha)^{1/3}, u];$$
$$\frac{\partial g}{\partial \xi} \to \infty \quad \text{for} \quad \xi^* \to 0.$$

In hydrodynamics at the time $t = t_0$ at the points $\xi^* = 0$, a shock wave is formed, the intensity of which increases with increase in t. The hydrodynamic solution for $t > t_0$ in the entire space ξ no longer has the form of a simple wave. In similar fashion, in the kinetics for $t > t_0$, the solution loses the form of a simple wave. Although its character for $t > t_0$ is not completely clear at the present time, it is certain that the erosion of the sharp front, associated with the thermal motion of the ions, has an important value in kinetics. We note that the very fact of the formation of a front of infinite curvature in collisionless kinetics in the presence of a distribution of the ions over the velocity u is quite noteworthy.



In conclusion, we consider a problem with boundary conditions as an example of the use of simple waves. Let a piston move in a plasma with velocity $u_0(t)$. Ion recombination takes place on the surface of the piston. The ion distribution function is described by Eq. (1). The boundary condition here is

$$g = 0 \quad \text{for} \quad u < u_0 = d\xi_0/dt \tag{11}$$

($\xi = \xi_0(t)$ is the law of motion of the piston). The condition (11) expresses the absence of ions reflected from the body. The solution of Eqs. (1) and (3) with such boundary conditions can be represented in the form of a simple wave. Actually, the solution of the self-similar equation (8) $g_a(\tau, u)$ obtained in[2] possesses the property

$$g_a = 0 \quad \text{for} \quad u < \tau.$$

The function $g = g_a$ satisfies Eqs. (1) and (3). The boundary conditions (11) for them are satisfied if $\tau = u_0$ on the surface of the piston, or

$$\frac{d\xi_0}{dt} = \tau; \quad \xi_0(t) = t\tau + p(\tau). \tag{12}$$

The latter equations in parametric form (parameter t) determine the function $p(\tau)$. Then Eq. (9) expresses the value of g at each point of the plane (ξ, t) in terms of the solution of the self-similar equation.

What has been said has a very simple geometric meaning. We construct the curve of motion of the piston $\xi = \xi_0(t)$ in the (ξ, t) plane. We now draw the tangent to this line in the direction $t > 0$. On these tangent functions, the distribution will have the value $g_a(\tau, u)$, where τ is the tangent of the angle of inclination of the tangent to the t axis (see the figure).

We note that just such a construction permits us to obtain a solution of the problem of the ion distribution close to the surface of a metallic body moving rapidly in the plasma. If the curve of motion of the piston or the contour of the moving body has a singularity, then the solution in the (t, ξ) plane will have a weak discontinuity. The structure of such a discontinuity is studied in[3].

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