CONCERNING THE "SUPERHEATED" STATE OF TYPE-I SUPERCONDUCTORS

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A method is proposed for observing superheated states of type-I superconductors which are ordinarily in the instability region.

1. The magnitude of the superheating field for bulk type-I superconductors has been obtained in a series of papers on the basis of the Ginzburg-Landau equations. These calculations were carried out either numerically[1] or by asymptotic matching of solutions written down separately for large and small x[2] (superconducting semispace in a parallel magnetic field with the x axis directed into the superconductor). In this paper we present a solution which utilizes only the smallness of the parameter of the Ginzburg-Landau theory $\varkappa \ll 1$. This solution contains no matching with which inaccuracies of the order of unity in the numerical coefficients are possible.

The initial system of one-dimensional Ginzburg-Landau equations is of the form

$$d^2f/dx^2 = \varkappa^2[-(1-A^2)f + f^3], \quad \varkappa \ll 1, \tag{1}$$

$$d^2A/dx^2 = f^2A, \tag{2}$$

with

$$df/dx|_{x=0} = 0, \quad f|_{x\to+\infty} \to 1,$$
$$dA/dx|_{x=0} = H_e, \quad A|_{x\to+\infty} \to 0;$$

here f(x) is the modulus of the order parameter, and A(x) is the vector potential. The first integral of Eqs. (1) and (2) is

$$\left(\frac{dA}{dx}\right)^2 = \frac{1}{2} - \frac{1}{\varkappa^2}\left(\frac{df}{dx}\right)^2 - (1-A^2)f^2 + \frac{f^4}{2}. \tag{3}$$

The problem consists in finding a connection between the external magnetic field H_e and the value of the order parameter of the superconductor f_0 on the surface of a superconductor.

Let us start from Eq. (2) which by the substitution of $u = A'/A$ takes on the form (the primes denote derivatives with respect to x)

$$u' = f^2 - u^2, \quad u|_{x\to+\infty} \to 1.$$

The solution of this equation has the following structure:

$$u(x) = -f(x) - \varphi(x), \quad f(x) \gg \varphi(x),$$

where $\varphi(x)$ satisfies the following equation

$$\varphi' - 2f\varphi - \varphi^2 = -f', \quad \varphi|_{x\to+\infty} \to 0.$$

By virtue of the inequality $f \gg \varphi$ which is fulfilled for all x (so far this inequality is an assumption but below it is confirmed for a sufficiently large range of values f_0) one should in the exact equation omit the term φ^2 compared with $2f\varphi$; following this, the equation is solved and yields

$$\varphi(x) = \exp\left[2\int_0^x f\,dx\right]\int_x^\infty f'(x)\exp\left[-2\int_0^x f\,dx\right]dx.$$

Taking into account the obvious inequality $f'(x) \geq 0$, we find that $\varphi_0 > 0$

$$\varphi_0 \equiv \varphi(0) = \int_0^\infty f'(x)\exp\left[-2\int_0^x f\,dx\right]dx. \tag{4}$$

Utilizing now the definition $u \equiv A'/A = -f - \varphi$ and the corresponding boundary conditions, we write Eq. (3) at the point x = 0:

$$H_e^2 = \frac{1}{2} - \left[1 - \frac{H_e^2}{(f_0+\varphi_0)^2}\right]f_0^2 + \frac{f_0^4}{2},$$

whence

$$H_e^2 = \frac{f_0}{4\varphi_0}(1-f_0^2). \tag{5}$$

Relation (5) shows that in order to solve the set problem it is sufficient to express φ_0 from (4) in terms of H_e and f_0. To this end we shall first obtain the function $f'(x)$ in the neighborhood of $x \sim f_0^{-1}$. In the indicated region of x, Eq. (1) reduces to the equality

$$f'' = \varkappa^2 A^2 f.$$

Assuming again that $f'/f = R(x)$, we have

$$R' = \varkappa^2 A^2 - R^2.$$

In this equation one can (as can be readily confirmed) omit the R^2 term. In addition,

$$A(x)|_{x\sim f_0^{-1}} = -\frac{H_e}{f_0}\exp(-f_0 x).$$

All the utilized simplifications separate the principal terms in the parameter $\varkappa \ll 1$. This accuracy is sufficient in itself. In addition the sought function $f'(x)$ will then be integrated, a fact which improves further the accuracy of the approximation.

Solving the simplified equation

$$R' = \varkappa^2 H_e^2 f_0^{-2}\exp(-2f_0x), \quad R(0) = 0$$

and taking into account the equality $f' = R(x)f_0$ valid with the same accuracy, we have

$$f'(x) = \frac{\varkappa^2 H_e^2}{2f_0^2}[1 - \exp(-2f_0x)].$$

Substituting this value of $f'(x)$ in (4), we find $\varphi_0 = \varkappa^2 H_e^2/8f_0^3$, after which (5) takes on the form

$$H_e = 2^{1/4}\varkappa^{-1/2}f_0\sqrt{1-f_0^2}. \tag{5a}$$

This expression coincides exactly with the corresponding result of Galaĭko.[2]

With the aid of (5a) one can readily confirm that all the assumed approximations [$\varphi(x) \ll f(x)$ and $R^2(x) \ll \kappa^2 A^2(x)$] are valid as long as $f_0 \gg \kappa$ which is quite sufficient for determining the maximum superheating fields for which $f_0 = 1/\sqrt{2}$. The graphical dependence of (5a) is shown in Fig. 1 where formula (5a) corresponds to the entire branch a—b and to a certain portion of the branch b—c in that region in which the inequality $f_0 \gg \kappa$ is fulfilled. On further decreasing f_0 the correct solution should tend to $H_c = 1/\sqrt{2}$.

2. On going over from the branch a—b to the branch b—c the solution of the system (1)–(3) for the semispace in a given external field becomes unstable, a fact which can be confirmed by the use of the results of Galaĭko.[2] There exists, however, an artificial assumption described below with the aid of which some of the states of the branch b—c can be rendered observable. In this connection it becomes essential to solve the system (1)–(3) in the neighborhood of the point c where $H_e \gtrsim H_c$ and $f_0 \lesssim \kappa$.

Let us start from the solution of the auxiliary problem in which there is no external surface and the boundary conditions are of the form

$$f(x)|_{x\to-\infty} \to 0, \quad f(x)|_{x\to+\infty} \to 1,$$
$$A'(x)|_{x\to-\infty} \to H_c, \quad A(x)|_{x\to+\infty} \to 0.$$

This problem was solved by Ginzburg and Landau[3] for calculating the coefficient of surface tension on the ns boundary. However, they did not consider the region of small values of the order parameter since this range of values of f(x) is unimportant for determining the coefficient of surface tension and one can restrict oneself to the appropriate estimates. For us, on the other hand, it is precisely the region of small values of f(x) which is important.

Let us rewrite Eq. (3) expressing in it the parenthesis $(1 - A^2)$ with the aid of (1) in terms of f and f'':

$$\varkappa^{-2}[(f')^2 - ff''] = {}^1/_2 - (A')^2 - {}^1/_2 f^4. \qquad (6)$$

In this exact equation starting from $-\infty$ up to certain values of x the right-hand side is a small quantity compared with the left-hand side of (6). As the characteristic point up to which one can expect this smallness to be preserved one can specify the turning point of the function $A'(x)$, i.e., the turning point of the coordinate dependence of the magnetic field. The third derivative of the vector potential vanishes at this point, $A'''(x) = 0$. Calculating by means of (2) the derivative $A'''(x)$, setting it equal to zero, and placing the origin at the turning point, we find that at the origin the following condition should be fulfilled:

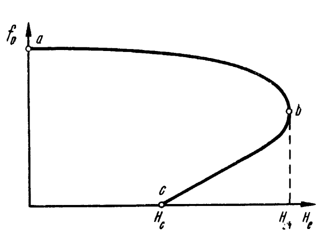

FIG. 1

$$2f_0'/f_0 = -A_0'/A_0; \qquad (7)$$

$f_0 \equiv f(0)$ and the primes denote the first derivatives.

Thus the proposed structure of the solution is the following. In the region of positive x the functions f(x) and A(x) have the usual form

$$f(x) = \operatorname{th}[\varkappa x/\sqrt{2} + \xi], \quad \operatorname{th}\xi = f_0,$$
$$A(x) = A_0 \exp[-f_0 x], \quad x > 0. \qquad (8)$$

For negative x the magnetic field is practically constant and the order parameter is described by Eq. (6) with the right-hand part discarded:

$$A'(x) \approx 1/\sqrt{2}, \quad A = (x + \sqrt{2}A_0)/\sqrt{2},$$
$$(f')^2 - ff'' = 0, \quad f(x)|_{x\to-\infty} \to 0, \quad f(x)|_{x=0} = f_0. \qquad (6a)$$

The functions f(x) and A(x) and their first derivatives should match at the origin. In addition, condition (7) should be fulfilled.

Equation (6a) is solved in the following way: we write the identity

$$f'' = \frac{1}{2}\frac{d}{df}\left(\frac{df}{dx}\right)^2$$

and introduce the notation $(df/dx)^2 = R(f)$, after which (6a) takes on the form

$$R(f) - \frac{f}{2}\frac{dR}{df} = 0.$$

Solving this equation and furthermore the equation which determines R(f), we find that

$$f(x) = f_0 e^{Cx}, \quad x < 0,$$

where C is an arbitrary constant. Using this expression for f(x), we carry out the required matching including condition (7). As a result

$$f_0 = 2^{1/4}\sqrt{\varkappa}, \quad A_0 = -1/\sqrt{2}f_0, \quad C = 2^{-3/4}\sqrt{\varkappa}. \qquad (9)$$

The order of magnitude of f_0 from (9), $f_0 \sim \kappa^{1/2}$, attests to the sensible nature of the chosen approximation. On the one hand, for such values of f_0 the definition of f(x) of the form $f(x)|_{x>0} = \tanh(\kappa x/\sqrt{2} + \xi)$ is still valid at the limit of applicability. On the other hand, the value of f_0 is sufficiently small so that at the limit of applicability one can assume for $x < 0$ that the magnetic field is constant (for details see the Appendix).

Let us now go over to the case of a semispace; to this end we introduce a free surface and transfer onto it the origin. The conditions

$$A'(x)|_{x=0} = H_e > 1/\sqrt{2}, \quad f'(x)|_{x=0} = 0.$$

should be fulfilled on this surface. In order to satisfy the conditions which have been set up, one must solve Eq. (6) and not (6a). This can be done if account is taken of the fact that for small deviations of H_e from H_c ($H_e^2 - {}^1/_2 \ll 1$, see the Appendix) in the region up to the matching point $x = \lambda$ which is separated from the free surface by some so far unknown distance λ one can assume H_e constant, as in the previous problem. Here the replacement $(df/dx)^2 = R(f)$ again turns out to be effective and the solution (6) is written as follows:

$$f(x) = f_0 \operatorname{ch}\gamma x, \quad \gamma = \varkappa(H_e^2 - {}^1/_2)^{1/2}/f_0, \quad x \leqslant \lambda. \qquad (10)$$

Matching the solutions (8) and (10) for $x = \lambda$ where the matching point λ is determined by the same condition (7) which is however already fulfilled at the point $x = \lambda$, we find

$$f(\lambda) = 2^{1/4}\varkappa^{1/2}, \quad f_0 = 2^{3/4}\varkappa^{1/2}(H_e^2 - 1/2)^{1/2},$$
$$\lambda = \frac{1}{2^{1/4}\varkappa^{1/2}} \ln \frac{2}{H_e^2 - 1/2}, \quad \gamma = \frac{\varkappa^{1/2}}{2^{3/4}}. \tag{11}$$

None of the numerical factors in (11) have any particular significance and should be assumed to be of the order of unity.

From the cited formulas (11) we note the expression for $f_0(H_e)$ which together with (5a) attests to the conservation of the sign of the derivative $\partial f_0/\partial H_e > 0$ on the entire line b—c (on the line a—b we have $\partial f_0/\partial H_e < 0$). We also note that the order of magnitude of λ from (11), $\lambda \gg \kappa^{-1/2}$ ($\kappa^{-1/2}$ is the distance at which the magnetic field after the point λ is attenuated practically to zero), indicates that the obtained solution of (11) corresponds to a well formed ns boundary separated from the free surface of the superconductor by a distance λ.

3. Let us go over to the problem of the possibility of observing states corresponding to the branch b—c. As has been noted above, for a semispace with a given external magnetic field these states are unstable. According to Galaĭko,[2] the fluctuation shifts of the ns boundary as a whole into the superconductor are the most dangerous in this instance. In order to describe qualitatively the development of this instability, it is convenient to speak of the magnetic pressure of the external magnetic field H_e on the superconducting semispace. Say we are at some point H_e, f_0 on the line b—c. The value of f_0 characterizes the degree of deformation of the order parameter required to balance the magnetic pressure. Now let f_0 decrease by means of a fluctuation by Δf_0, $\tilde{f}_0 = f_0 - \Delta f_0$, i.e., the ns boundary shifts somewhat into the superconductor. Smaller f_0 on the b—c curve correspond to lower magnetic pressures which they can resist. Therefore the initial pressure $H_e^2/8\pi$ for the system after the fluctuation becomes excessive, as a result of which the ns boundary moves even further into the superconductor. This will lead to a further decrease of f_0, that is for a given H_e to an even greater nonequilibrium situation, etc. Two factors are important for the development of a given type of instability: the sign of the derivative $\partial f_0/\partial H_e > 0$ on b—c and the constancy of the external magnetic field H_e. The first of these factors is unavoidable since it is determined by the properties of the Ginzburg-Landau equations, but the second one which follows from the boundary conditions is in general not essential and can be removed if desirable.

Let us consider, for example, a system of two superconducting semispaces separated by a vacuum gap d. In such a system one can consider not the magnetic field in the gap to be given but the value of the magnetic flux $\Phi = H_e\tilde{\lambda}(H_e)$ [$\tilde{\lambda}(H_e)$ is the effective gap size including the geometric gap d and the region of penetration of the field into the superconductors]. A change of Φ leads to a change of H_e in the gap. Therefore, just as in the case of one semispace, by increasing H_e we finally enter the region of instability. However, whereas in the case of the semispace the instability developed without bound, this does not take place in this case. The appearance of the instability corresponds to the nonequilibrium motion of the ns boundaries into the superconductors [to an increase of $\tilde{\lambda}(H_e)$]. For a given total flux Φ this motion will automatically be accompanied by a decrease of the field H_e. However, the field in the gap cannot take on values smaller than H_c. [According to (11), the field $H_e = H_c$ corresponds to an infinite λ, i.e., an infinite displacement of the ns boundaries from the free surfaces. For a finite field in the gap $H_e = H_c$ this would correspond to infinite flux in the gap which contradicts the initial assumption about the finite value of Φ. Consequently, for finite Φ, λ should also be finite, and this is possible only when $H_e > H_c$.] It can thus be stated that the development of the instability will in this case cease for some finite $\tilde{\lambda}(H_e)$ and $H_e > H_c$.

4. Let us now consider certain properties of the function $H_e(\Phi)$ in the described system consisting of two superconductors separated by a gap d. The relation between Φ and H_e is readily determined with the aid of the formulas obtained above.

In the region in which the value of f_0 is not small, where relation (5a) is valid

$$\Phi = H_e\left[\frac{d}{\delta} + 2f_0^{-1}\right],$$

where δ is the London penetration depth, $\delta \sim 10^{-5}$ cm, or

$$\Phi = H_e\left[\frac{d}{\delta} + \frac{2}{1/2 + (1/4 - \varkappa H_e^2/\sqrt{2})^{1/2}}\right], \tag{12a}$$

$$\Phi = H_e\left[\frac{d}{\delta} + \frac{2}{1/2 - (1/4 - \varkappa H_e^2/\sqrt{2})^{1/2}}\right]. \tag{12b}$$

Formula (12a) is for the a—b branch and (12b) is for b—c near b. In the vicinity of the point c

$$\Phi = H_e[d/\delta + 2(\lambda + f^{-1}(\lambda))].$$

Taking into account the expression for λ from (11) and the inequality $\lambda \gg f^{-1}(\lambda)$, equivalent to $H_e^2 - H_c^2 \ll 1$, we have hence

$$\Phi = H_e\left[\frac{d}{\delta} + \frac{2^{3/4}}{\sqrt{\varkappa}} \ln \frac{2}{H_e^2 - 1/2}\right] \approx H_e\left[\frac{d}{\delta} + \frac{2}{\sqrt{\varkappa}} \ln \frac{2}{H_e^2 - 1/2}\right]. \tag{13}$$

Figure 2 shows the graphic dependence $H_e(\Phi)$.

The line A—B corresponds to formula (12a) and the line B—C is partly described by formula (12b) and partly by (13); the region of matching on B—C is notched. The curve C—∞ is fully described by expression (13).

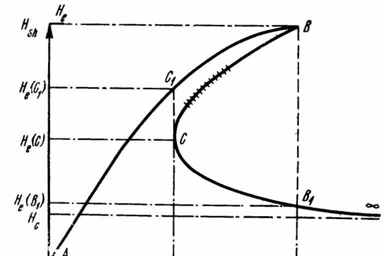

FIG. 2

With increasing Φ the representative point H_e moves along the curve A—B, reaches its maximum value H_{sh} at the point B (H_{sh} is the maximum superheating field), and then decreases abruptly to a value $H_e(B_1)$ determined by the equality

$$\Phi_{sh}+\delta\Phi\Big|_{\delta\Phi\to+0}=H_e(B_1)\left[\frac{d}{\delta}+\frac{2}{\sqrt{\varkappa}}\ln\frac{2}{H_e^2(B_1)-{}^1/_2}\right].$$

Subsequently H_e moves along $B_1-\infty$ approaching H_c asymptotically.

In moving in the opposite direction H_e moves along the line ∞—C up to the point C given by the condition $\partial\Phi/\partial H_e = 0$:

$$H_e(C)=\frac{1}{\sqrt{2}}\left(1+\frac{2\delta}{d\sqrt{\varkappa}}\right),\quad \frac{2\delta}{d\sqrt{\varkappa}}<1,$$

$$\Phi(C)=\frac{1}{\sqrt{2}}\left[\frac{d}{\delta}+\frac{2}{\sqrt{\varkappa}}\ln\frac{d\sqrt{\varkappa}}{2\delta}\right] \tag{14}$$

At the point C a decrease of Φ ceases to be consistent with the existence of the system in a state of weak superheating (this is how we refer to states lying on the line C—∞). Therefore for a further decrease of Φ it is essential to carry out the work in carrying the system over from the state of weak superheating (point C) to the state of strong superheating (point C_1; the states of strong superheating are located on the line A—B). At the same time the magnetic field in the system increases abruptly from $H_e(C)$ to $H_e(C_1)$:

$$H_e(C_1)=\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{\varkappa}}\frac{\delta}{d}\ln\frac{d\sqrt{\varkappa}}{2\delta},$$

after which it follows the line C_1—A. Thus the cyclic variation of Φ ($0\to\Phi>\Phi_{sh}\to 0$) is accompanied by a hysteresis behavior of $H_e(\Phi)$.

The $H_e(\Phi)$ dependences presented for monotonically increasing or decreasing Φ do not touch the states on the line B—C of Fig. 2. In order to realize these states a nonmonotonic change of Φ is obviously required. Let us rise along the line A—B into the vicinity of the point B. In this region the states corresponding to A—B and B—C differ little from one another, and it is here that fluctuation transitions from A—B to B—C and vice versa become therefore noticeable. However the line B—C is somewhat more convenient, since for given Φ it corresponds to lower magnetic fields than A—B. Therefore, if one now starts to decrease Φ, then the system will proceed with greater probability along B—C than along A—B. In addition to these two possible paths, there is in the vicinity of the point B a considerable probability of a break into states of low superheating onto the line C—∞ which increases exponentially on approaching the point B. If nevertheless on decreasing Φ the system turns out to be on the line B—C, a fact which is readily established from the values of H_e intermediate between A—B and C—∞, then further decrease of Φ should bring it to the point C.

In concluding this section let us consider the point C in somewhat more detail. According to (14) the location of this point is essentially determined by the ratio d/δ. The requirement $2\delta/d\sqrt{\kappa}<1$ cited in (14) and which bounds the possible values of d/δ from below is connected with the approximate solution of Eq. (6) and is therefore in principle not mandatory. In the general case the problem concerning small superheating also remains correct for $2\delta/d\sqrt{\kappa}>1$. However, the approximations for describing the behavior of the point C in this instance utilized in this paper become rather unsuitable.

5. Let us summarize. The results obtained above show that as regards type-I superconductors it makes sense to speak of two types of superheating states. States of strong superheating are marked by small deformation of the order parameter and values of the superheating field large compared to H_c. In addition, in the problem of the semispace in an external field these states are stable. In Fig. 1 these states correspond to the line a—b, in Fig. 2 to the line A—B.

Strong deformation of the order parameter and weak superheating fields $H_e - H_c \ll 1$ are characteristic of states of small superheating. In addition, these states do not have the usual stability and require for their existence certain artificial conditions. In Fig. 1 the states of small superheating are located in the neighborhood of the point c, in Fig. 2—on the line C—∞. When Φ is monotonically increased or decreased both types of superheating are sharply separated from one another by instability regions (the lines B—B_1 and C—C_1 in Fig. 2).

In addition to these states there exists an intermediate region (the line B—C). However, in the system which we have described the observation of these states, although possible in principle, is apparently difficult.

The author is sincerely grateful to V. P. Galaĭko for his attention to this work and for useful remarks.

APPENDIX

1. The solution for f(x) in the form

$$f(x)\,|_{x\geqslant 0}=\operatorname{th}\left(\frac{\varkappa}{\sqrt{2}}x+\xi\right)$$

is valid so long as $f'(x)<f^2(x)$. For $\xi\sim\kappa^{1/2}$ [see Eq. (9)] the inequality $f'<f^2$ is fulfilled at the limit of applicability up to x = 0.

2. Let us write Eq. (2) for A(x) in the form

$$u'=f^2(x)-u^2,\quad u=A'(x)/A(x).$$

The solution of this equation in the form

$$\frac{dA}{dx}=\text{const}=\frac{1}{\sqrt{2}},\quad A=(x+\sqrt{2}A_0)\frac{1}{\sqrt{2}},\quad x\leqslant 0$$

corresponds to the assumption $f(x)<u(x)$. The solution (9) obtained satisfies this inequality at the limit of applicability. In fact at the origin $f(x)\sim\kappa^{1/2}$ and $u(x)\sim\kappa^{1/2}$, i.e., $f_0\sim u_0$. On the other hand, in the region of negative x the function f(x) decreases exponentially for $x\to-\infty$, whereas the ratio $u\equiv A'/A$ decreases in stepwise fashion, i.e., $f(x)<u(x)$.

3. In the case when $H_e>1/\sqrt{2}$ the solution (10) and (11) again assumes that in the region up to the matching point $x=\lambda$ the inequality $f<u$ is fulfilled. At the matching point, just as when $H_e=1/\sqrt{2}$, the values of f_λ and u_λ are of the same order of magnitude. Therefore the inequality $f<u$ will be fulfilled practically in the entire range $0\le x\le\lambda$, if it is required to be fulfilled on the surface of the superconductor, i.e., $f_0<u_0$. Substituting

in this inequality the expressions for f_0 and u_0 from (11), we find that the inequality

$$(H_e^2 - {}^1/_2)^{1/2} \ln \frac{2}{H_e^2 - {}^1/_2} < 1,$$

which is valid for $H_e^2 - {}^1/_2 < 1$ should be fulfilled.

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