REMOVAL OF INFRARED DIVERGENCES IN RADIATIVE CORRECTIONS

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The effect of multiple Coulomb scattering on the radiative corrections to arbitrary processes with one charged relativistic particle and an arbitrary number of neutral particles in the initial and final states, is investigated quantitatively. The removal of infrared divergences in the radiative corrections is analyzed in detail.

1. The occurrence of infrared divergences in quantum electrodynamics is connected with the circumstance that the perturbation expansion is in fact an expansion in powers of $e^2\ln(E/\omega)$, where E is the characteristic energy of the charged particle, ω is the frequency of the emitted quantum, and $\hbar = c = 1$.[1] For $\ln(E/\omega) \sim 137$ perturbation theory is no longer applicable. However, as was shown by Landau and Pomeranchuk,[2] for relativistic particles moving in matter, the multiple scattering suppresses the emission of soft quanta beginning with frequencies $\omega < \omega_1$, where

$$\omega_1 = \left(\frac{E_s}{m}\right)^2\left(\frac{E}{m}\right)^2 L_{\text{rad}}^{-1}, \quad E \gg m,$$
$$E_s = 21\,\text{MeV}, \quad L_{\text{rad}}^{-1} = n_0 e^6 Z^2 m^{-2}\ln(183Z^{-1/3}), \tag{1.1}$$

and the condition for the applicability of perturbation theory for frequencies $\omega > \omega_1$ has the form

$$e^2\ln(E/\omega_1) \ll 1. \tag{1.2}$$

Since for $\omega < \omega_1$, the influence of the scattering guarantees the applicabiltiy of perturbation theory with $e^2 \ll 1$, (1.2) is a general condition for the applicability of perturbation theory for all frequencies. Condition (1.2) restricts the density of atoms in the medium from below by the negligibly small value

$$n_{\text{crit}} = 10^{-15}\left(\frac{m}{E}\right)\frac{1}{Z^2\ln(183Z^{-1/3})}$$

It has been pointed out in[2] that the multiple scattering leads to the removal of infrared divergences in the radiative corrections, but the effect of the multiple scattering on the infrared divergences in radiative corrections has so far not been considered in detail. An estimate of the frequency at which the effect of the scattering leads to a deviation of the formulas for the radiative corrections from their form in the vacuum, can be obtained in the following way.

An infrared divergence appears in the radiative corrections when one adds an internal photon line between two external electron lines in the Feynman graph for the process under consideration. Writing the Green's function for the electromagnetic field corresponding to this photon line in the form

$$D_{\mu\nu}(x, x') = \frac{\delta_{\mu\nu}}{(2\pi)^4}\int\frac{d^4k}{k^2 - i\delta}\exp\{ik(x - x')\}$$
$$= -\frac{i}{(2\pi)^3}\int d^3k\frac{\delta_{\mu\nu}}{\sqrt{2\omega(\mathbf{k})}}\exp\{i\mathbf{k}\mathbf{r} - i\omega(k)t\}$$
$$\times\frac{1}{\sqrt{2\omega(\mathbf{k})}}\exp\{-i\mathbf{k}\mathbf{r}' + i\omega(k)t'\},$$

one can interpret the inclusion of the additional photon line as the result of the exchange of a real quantum with arbitrary **k**. Therefore, the estimate of the characteristic frequency at which the effect of the scattering sets in is obtained in the same way as for bremsstrahlung, and leads to the value of ω_1 given by (1.1).

2. The multiple scattering of charged particles in a medium is conveniently taken into account in the calculation of the radiative corrections by using the method developed in[3]. In this method one employs the approximate solution of the wave equation for particles in the summed potential of the atoms of the medium,

$$\psi_p(\mathbf{r}, t) = \text{const}\cdot\exp\left\{i\mathbf{p}\mathbf{r} + \sum_a S_a(\mathbf{r}, \mathbf{p}) - iEt\right\} \tag{2.1}$$

where $S_a(\mathbf{r}, \mathbf{p})$ is determined from the solution of the one-center scattering problem, and the asymptotic behavior of $S_a(\mathbf{r}, \mathbf{p})$ at distances far from the scatterer is connected with the one-center scattering amplitude $f(\mathbf{p}_1 - \mathbf{p}_2)$ by the formula[3]

$$S_a(\mathbf{r}, \mathbf{p}) = \ln\left\{1 + \frac{1}{2\pi^2}\int\frac{d^3l f(\mathbf{l})\exp\{i\mathbf{l}(\mathbf{r} - \mathbf{R}_a)\}}{l^2 + 2\mathbf{p}\mathbf{l} - i\delta}\right\} \tag{2.2}$$

where $\mathbf{R}_a$ is the coordinate of the atom in the medium. The condition for the validity of this approximation is that the one-center scattering amplitude f be small compared with the mean free path of the particle: $fn_0\sigma \ll 1$, and that the mean square angle of the multiple scattering over the considered distances L be small:

$$\langle\theta^2\rangle_L = \left(\frac{E_s}{E}\right)^2\frac{L}{L_{\text{rad}}} \ll 1.$$

In calculating the radiative corrections the matrix element of any process is written in the form

$$M = M_0 + M_2,$$

where M_0 is the matrix element in first nonvanishing order of perturbation theory for the process under consideration, and M_2 is the following perturbation term (in e^2). We assume that M_0 does not contain infrared divergences, so that the radiative correction to the probability, $(M_0M_2^+ + M_0^+M_2)$, contains the divergent matrix element linearly. Hence, the average of the probability over the multiple scattering, i.e., over the coordinates of the atoms $\mathbf{R}_a$, reduces to an average of M_2. In the calculation of the probability we must also take into account that the Green's functions of charged particles in matter are given by

$$G(x, x') = (2\pi)^{-3} \int d^3p \frac{\psi_p(\mathbf{r}, t)\psi_p^*(\mathbf{r}', t')}{v^2 - v_0^2 + i\delta}, \qquad (2.3)$$

where $\psi_p(\mathbf{r}, t)$ is given by (2.1).

In the infrared region the four-momentum of the virtual quantum k_μ is small, and one may use the first nonvanishing approximation in the expansion of M_2 in powers of k_μ. For the average value $\langle M_2 \rangle$ one can write

$$\langle M_2 \rangle \cong -M_0 \frac{ie^2}{\pi^3} \int \frac{d^4k}{k^2 - i\delta} \int\int d^4s_1 d^4s_2 \int\int d^3r d^3r'$$
$$\times \frac{p_{2\mu}}{(k^2 - 2p_2k - 2p_2s_2 - i\delta)} \frac{p_{1\mu}}{(k^2 - 2p_1k - 2p_1s_1 - i\delta)} \cdot \exp\{is_2\mathbf{r}' + is_1\mathbf{r}\}$$
$$\times \exp\{i\varkappa_1 r_z |r_z| \omega + i\varkappa_2 r_z' |r_z'| \omega\} \delta(s_{14}) \delta(s_{24});$$

where we have used the following formula for the average over atomic coordinates:

$$\left\langle \exp \sum_a S_a \right\rangle = \exp \left\langle \sum_a (\exp S_a - 1) \right\rangle,$$

which holds if one assumes that the coordinates of different atoms are independent; we also used the notation

$$\varkappa_{1,2} = \frac{n_0}{2p_{1,2}} \int d^2l_\perp |f(l_\perp)|^2 l_\perp^2.$$

The axes r_z and r_z' in (2.4) are directed along $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. Integrating (2.4), we find for $\omega < E_s^2 E^2 / m^4 L_{rad}$

$$\langle M_0 M_2^+ + M_2 M_0^+ \rangle = -|M_0|^2 \left\{ \sqrt{\frac{\pi}{2}} \frac{e^2}{\sqrt{\varkappa_1 + \varkappa_2}} \int_0^{\omega_r} \frac{d\omega}{\sqrt{\omega}} \right\}. \qquad (2.5)$$

It is seen from this that there is no infrared divergence. In an analogous fashion one can easily show, by taking account of the multiple scattering, that no infrared divergence occurs in the renormalization of the vertices and self-energy graphs.

3. In calculating radiative corrections in quantum electrodynamics one usually considers the so-called "experimentally observable cross section," which is the sum of the cross section for the considered process with radiative corrections and the cross section for the same process with emission of an additional soft quantum. If the frequency of the additional quantum is below the threshold of the detector, these processes are indistinguishable experimentally. The unitarity condition for the S matrix implies that the infrared divergences of the two processes cancel each other in the experimentally observable cross section, but the answer now depends significantly on the threshold frequency ω_e.[1]

The consideration of the process with emission of an additional quantum is analogous to that of the bremsstrahlung process,[3] and leads to

$$d\sigma_1 = d\sigma_0 \frac{e^2}{(2\pi)^3} \int\int d^3r d^3r' d^3s_1 d^3s_2 \int_{\omega_e}^{\omega_s} \frac{d^3k}{2\omega} \left| \frac{\mathbf{p}_1\mathbf{e}}{\mathbf{p}_1\mathbf{k} + \mathbf{p}_1\mathbf{s}_1} - \frac{\mathbf{p}_2\mathbf{e}}{\mathbf{p}_2\mathbf{k} + \mathbf{p}_2\mathbf{s}_2} \right|^2$$
$$\times \exp\{is_1\mathbf{r} + is_2\mathbf{r}'\} \exp\{i\varkappa_1\omega r_z |r_z| + i\varkappa_2\omega r_z' |r_z'|\}. \qquad (3.1)$$

Integrating (3.1) we obtain in the region where multiple scattering is important, i.e., for $\omega < E_s^2 E^2 / m^4 L_{rad}$,

$$d\sigma_j = d\sigma_0 \sqrt{2\pi} \frac{e^2 E_s}{E\sqrt{L_{rad}}} \omega_e^{1/2}, \qquad (3.2)$$

where $d\sigma_0$ is the cross section for the process without radiative corrections but with emission of an additional quantum. It follows from (3.2) that the multiple scattering of charged particles leads to a qualitative change in the dependence of the experimentally observable cross section on the threshold of the detector ω_e. The experimentally observable cross section has the form

$$d\sigma_{exp} = d\sigma_0 \left\{ 1 - e^2 \sqrt{2\pi} \frac{E_s}{E\sqrt{L_{rad}}} \omega_e^{1/2} \right\}. \qquad (3.3)$$

We emphasize that the inequality $\omega < E_s^2 E^2 / m^4 L_{rad}$ used in deriving this formula, imposes a restriction on the matter density which enters in L_{rad}. Therefore one cannot go directly from (3.2) or (3.3) to the vacuum limit.

4. In using the above formulas, one must take account of the effect of the polarization of the medium on the emission of the additional quantum and on the radiative corrections. Formula (3.3) is applicable in the case when the energy of the particle satisfies the condition

$$E > m(m / E_s)^2, \qquad (4.1)$$

and when the threshold of the detector ω_e satisfies

$$\left(\frac{3E^2 L_{rad} \omega_L^4}{E_s^2} \right)^{1/3} < \omega_e < \frac{E_s^2 E^2}{L_{rad} m^4}, \omega_L^2 = \frac{4\pi n_0 e^2}{m}, \qquad 4.2)$$

where n_0 is the number of electrons per unit volume of the medium. The multiple scattering leads to the replacement of $e^2 \ln(\omega_e/m)$ by

$$\pi \omega_e^{1/2} e^2 \sqrt{2\pi} E_s / 2m L_{rad}^{1/2}. \qquad (4.3)$$

in the experimentally observable cross section. If the threshold of the detector satisfies the inequality

$$\omega_{aT}, \omega_L < \omega_e < (3E^2 L_{rad} \omega_L^4 / E_s^2)^{1/3}, \qquad (4.4)$$

then it is not the scattering which is important, but the polarization of the medium, and $e^2 \ln(\omega_e/m)$ must be replaced by

$$E_s^2 m \omega_e^2 / 24\pi^2 L_{rad} n_0 E^2. \qquad (4.5)$$

[1] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya elektrodinamika (Quantum Electrodynamics), Fizmatgiz, 1959.

[2] L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR 92, 535 and 735 (1953).

[3] N. P. Kalashnikov and M. I. Ryazanov, Zh. Eksp. Teor. Fiz. 47, 1055 (1964) [Sov. Phys.-JETP 20, 707 (1965)].

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