SUPERCONDUCTING ALLOYS IN A STRONG ALTERNATING FIELD

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The dynamic properties of the parameter Δ for superconducting alloys are investigated in the vicinity of the transition temperature T_c. It is shown that the transition to the adiabatic pattern occurs only at anomalously low frequencies. The corresponding characteristic frequency Ω_0 for a bulk superconductor is of the order of Δ^3/T_c^2, and for small-size superconductors it depends on inelastic processes (electron-electron and electron-phonon interactions). The expression for the current density goes over to the static formula for frequencies $\omega < \Omega_1 \sim \Delta^2/T_c$. The dependence of Δ on the amplitude of the applied alternating field is nonunique at the superconductor boundary for intermediate frequencies $\Omega_0 < \omega < \Omega_1$. This corresponds to instability of the regime in a certain amplitude range.

In[1] the authors set up a general scheme which made it possible to investigate the properties of superconductors in strong, nonstationary electromagnetic fields. In the case of superconductors with a large concentration of paramagnetic impurities we obtained closed equations which were a natural generalization of the scheme of the Ginzburg-Landau theory.[2] In the same paper[1] we noted that the problem of the properties of superconductors in a strong variable field is amenable to a general formulation above all in the vicinity of the critical temperature. In this region one can separate the Joule losses from the basic effects connected with the action of the magnetic field on the "superconducting" electrons. The condition that the temperature change due to Joule heating during a period be small compared to the chosen scale of temperature near T_c is[1)]

$$(H/H_c)^2 \ll T_c/(T_c - T).$$

It is hence seen that alternating fields comparable with the critical fields are in fact admissible in this temperature range. Another important point is connected with the necessity of conducting away the Joule heat in an actual experimental situation; this requires adequate heat conduction. The vicinity of T_c is in this respect also the most convenient region.

Below we shall investigate on the basis of the microscopic theory of Bardeen, Cooper, and Schrieffer[3] superconducting alloys (with the usual impurities) in an alternating field and we shall show that even close to T_c and down to very low frequencies one cannot, generally speaking, write simple differential equations including as a special case the static Ginzburg-Landau scheme. Nevertheless, for a large number of phenomena one can reduce the problem to some simple scheme containing time-averaged values of the energy gap and of the magnetic fields.

1. A SMALL PARTICLE WITH MIRROR WALLS

As an example which will subsequently greatly facilitate the understanding of the general situation we shall consider the problem of a small particle of pure superconductor with mirror boundary conditions. The properties of such particles in a static magnetic field have been calculated by Larkin.[4] The convenience of the model consists in the fact that in this case the one-electron functions can be classified by the projection of the orbital angular momentum on the magnetic field direction, as a result of which the equations for the Green's functions have exact solutions. Going over from Fourier components in Eq. (5) of[1] to the time representation, we obtain

$$\Delta(t) = \frac{|g|}{2} \sum_{\xi,\mu} \Big\{ \int_{-\infty}^{t} dt_2 [F_\xi^A(t_2, t) - F_\xi^R(t, t_2)] \frac{T}{\operatorname{sh} \pi T(t - t_2)}$$
$$- \iint_{-\infty}^{t} dt_1\, dt_2 \frac{T}{\operatorname{sh} \pi T(t_1 - t_2)} [(G_\xi^{+R}(t, t_2) G_\xi^{-A}(t_1, t)$$
$$+ F_\xi^R(t, t_2) F_\xi^{+A}(t_1, t))(\Delta(t_1) - \Delta(t_2)) + (G_\xi^{-R}(t, t_2) F_\xi^A(t_1, t)$$
$$+ F_\xi^R(t, t_2) G_\xi^{+A}(t_1, t))(\mu H(t_1) - \mu H(t_2))] \Big\}, \qquad (1)$$

where the Green's functions satisfy the equations[2)]

$$\Big[i\frac{\partial}{\partial t} - \xi + \mu H(t) \Big] G_\xi(t, t') - \Delta(t) F_\xi^+(t, t') = \delta(t - t'),$$
$$\Big[i\frac{\partial}{\partial t} + \xi + \mu H(t) \Big] F_\xi^+(t, t') - \Delta^*(t) G_\xi(t, t') = 0. \qquad (2)$$

In (1) and (2) we have chosen the vector potential $\mathbf{A}(\mathbf{r}, t) = [\mathbf{H}(t) \times \mathbf{r}]/2$ assuming that the currents appearing in the particle are sufficiently small; the field $\mathbf{H}$ is directed along the symmetry axis of the particle (a sphere or cylinder). The summation sign in (1) denotes summation over ξ and over all momentum projections. The above choice of the vector potential allows one to consider Δ to be a real quantity.

The first term in the curly brackets of (1) represents the regular part which can near T_c be expanded in powers of Δ/T. Although the other, irregular part vanishes on first sight if Δ and H do not depend on the time, it contains in fact a singularity which does not

[1)]Compare with [1], Eq. (1), which contains a typographical error.

[2)]The functions F and F^+ differ in their sign from those defined in [1].

always allow one to carry out this limiting procedure. We now proceed to investigate it.

We note that by replacing all the quantities

$$G, F \to \exp\left\{ i\mu \int_{t_2}^{t} H(\tau)\,d\tau \right\} \bar{G}, \bar{F}$$

one can eliminate from Eqs. (2) the field:

$$\bar{G}_\xi(t,t') = \frac{1}{\Delta(t)}\left(i\frac{\partial}{\partial t} + \xi\right)\bar{F}_\xi^+(t,t'),$$

$$\left[\left(i\frac{\partial}{\partial t} - \xi\right)\frac{1}{\Delta(t)}\left(i\frac{\partial}{\partial t} + \xi\right) - \Delta\right]\bar{F}_\xi^+(t,t') = \delta(t-t'). \qquad (2')$$

The second equation can be readily solved by a quasiclassical method if the change in Δ occurs during a time large compared with Δ^{-1}. We present the result:

$$\bar{F}_\xi^{+R}(t,t') = \frac{i\Delta(t')}{\sigma_+(t') - \sigma_-(t')}\left\{\exp\left(i\int_{t'}^{t}\sigma_+ d\tau\right) - \exp\left(i\int_{t'}^{t}\sigma_- d\tau\right)\right\}, \qquad (3)$$

$$\bar{G}^R(t,t') = \frac{i\Delta(t')}{\Delta(t)[\sigma_+(t') - \sigma_-(t')]}\left\{(\xi - \sigma_+(t))\exp\left(i\int_{t'}^{t}\sigma_+ d\tau\right) - (\xi - \sigma_-(t))\exp\left(i\int_{t'}^{t}\sigma_- d\tau\right)\right\},$$

where

$$\sigma_\pm = \pm\varepsilon + \frac{i}{2}\left[\frac{\dot\varepsilon}{\varepsilon} - \frac{\dot\Delta}{\Delta}\left(1 \mp \frac{\xi}{\varepsilon}\right)\right], \qquad \varepsilon = \sqrt{\xi^2 + \Delta^2(t)}. \qquad (3')$$

The functions $\bar{F}$ and $\bar{G}^+$ are obtained from (3) by the replacement $\xi \to -\xi$.

Let us go back to Eq. (1). Replacing all G and F by $\bar{F}$ and $\bar{G}$ leads to the appearance in (1) of factors

$$\exp\left\{i\mu\int_{t}^{t_2} H(\tau)\,d\tau\right\}.$$

In this connection we draw attention to the fact that for $T \approx T_c$ the function $\sinh \pi T(t - t')$ plays the role of a δ function. However, whereas in the regular term the instant t_2 is now close to t (locality in time), the irregular term remains integral.

Further calculations in Eq. (1) with the use of formulas (3) and (3′) are elementary. Omitting intermediate steps and retaining only the leading terms in Δ/T, we obtain the following result:

$$-\frac{7\zeta(3)\overline{\mu^2}\Delta(t)}{4\pi^2T_c^2}\left[H^2(t) - \int_{-\infty}^{t}\frac{d}{d\tau}(H^2)\,d\tau\right] - \frac{\Delta(t)}{2T}\int_0^\infty d\xi\int_{-\infty}^{t}d\tau\,\frac{\Delta(t)\,\dot{\Delta}(\tau)}{\varepsilon(t)\,\varepsilon(\tau)}, \qquad (4)$$

where $\overline{\mu^2}$ is the average of μ^2 which depends on the shape of the sample (see[4]). This result is most readily grasped if one assumes that $\Delta = \Delta_0 + \Delta_1$ where Δ_0 is the equilibrium value of the gap and Δ_1 is its change under the action of a weak magnetic field. The equation obtained for Δ_1

$$-\frac{7\zeta(3)\overline{\mu^2}\Delta_0}{4\pi^2T_c^2}\left[H^2(t) - \int_{-\infty}\frac{d}{d\tau}(H^2)\,d\tau\right] - \frac{\pi\Delta_0}{4T}\int_{-\infty}^{t}\dot{\Delta}_1 d\tau = 0, \qquad (4')$$

taken seriously, yields $\Delta_1 = 0$, i.e., the gap does not change when the field is switched on. The same result is also obtained from (4) for arbitrary fields. It can be shown (and we shall not dwell on this) that this is also correct for arbitrary temperature. The reason for this, at first sight paradoxical, result is the absence from our model of any uniform relaxation mechanism. Inclusion of such a mechanism (for instance, interaction with phonons) will lead to the circumstance that terms integral in time in (4′) will have to be written with account of the uniform relaxation time τ_0:

$$\int_{-\infty}^{t}\frac{d}{d\tau}(H^2)\,d\tau \to \int_{-\infty}^{t}\frac{d}{d\tau}(H^2)\,e^{-(t-\tau)/\tau_0}\,d\tau, \qquad \int_{-\infty}^{t}\dot{\Delta}_1 d\tau \to \int_{-\infty}^{t}\dot{\Delta}_1 e^{-(t-\tau)/\tau_0}\,d\tau.$$

As a result of such a replacement Eqs. (4′) and (4) become comprehensible.

The problem of the uniform relaxation of the gap was investigated in part in the work of Woo and Abrahams.[5] It is not essential for us to develop here the microscopic theory with allowance for scattering effects. This could only be done with certain model assumptions. The only thing of importance for us is the fact that in metals the quantity τ_0^{-1} is extremely small: $\tau_0^{-1} \sim T_c^3/\Theta_0^2$ for electron-phonon interaction and $\tau_0^{-1} \sim T_c^2/\epsilon_F$ for electron-electron interactions. It can be shown that with account of the time relaxation one obtains instead of (4) for the corrections to Δ the following expression which we write in Fourier components:

$$-\left[\frac{7\zeta(3)\Delta_0^2}{4\pi^2T_c^2} + \frac{\pi\Delta_0}{2T}\frac{\omega_0}{\omega_0 + i/\tau_0}\right]\Delta_{1,\omega_0} + \frac{\overline{\mu^2}\Delta_0 7\zeta(3)}{4\pi^2T_c^2}(H^2)_{\omega_0}\left(1 - \frac{\omega_0}{\omega_0 + i/\tau_0}\right) = 0. \qquad (5)$$

It is hence seen that the transition to the static case takes place only for very low frequencies $\omega_0\tau_0 \ll \Delta/T$, a fact which is connected with the anomalously large coefficient in the irregular term of (5) preceding Δ_1. As we shall see below, nonuniform relaxation takes place considerably more rapidly. Therefore, in describing nonstationary properties of bulk samples the above-mentioned scattering processes need not be taken into account.

2. THE CHANGE OF Δ FOR SUPERCONDUCTING ALLOYS IN AN EXTERNAL FIELD

Let us go over to the case of superconducting alloys near T_c and calculate the change of Δ under the influence of a weak magnetic field. The formulas obtained make it also possible to draw rather general conclusions for the case of strong fields. For simplicity, we shall restrict ourselves to the situation in which all the quantities depend only on a single coordinate z. For an alloy with a given impurity distribution we choose a complete set of one-electron functions $\{\psi_n\}$ and expand the Green's functions in ψ_n. Let ξ_n denote the energy of the n-th state counted from the Fermi level. ω_1 and ω_2 will denote the frequencies of the alternating field; $\omega_0 = \omega_1 + \omega_2$ is then the frequency corresponding to the Fourier component of Δ_1. On the Matsubara frequency axis we obtain

$$\Delta_{1,\omega_0}(\mathbf{r}) = -|g|T\sum_{\omega}\sum_{m,n}\delta(\mathbf{r}-\mathbf{r}')_{n,m}F^{(1)}_{mn}, \qquad (6)$$

where $\delta(\mathbf{r} - \mathbf{r}')_{nm}$ is the matrix element of the δ function, and

$$F^{(1)}_{mn} = \frac{\Delta_0^2 + (\omega - \xi_n)(\omega - \omega_0 + \xi_m)}{(\omega^2 - E_n^2)((\omega - \omega_0)^2 - E_m^2)}(\Delta_{1,\omega_0})_{mn}$$
$$+ \Delta_0\frac{(\omega - \omega_1 + \xi_k)(\omega - \omega_0 + \xi_m) + (\omega - \xi_n)(2\omega - \omega_1 - \omega_0 + \xi_k + \xi_m)}{(\omega^2 - E_n^2)((\omega - \omega_1)^2 - E_k^2)((\omega - \omega_0)^2 - E_m^2)} \times \left(\frac{e}{mc}\right)^2(\hat{\mathbf{p}}\mathbf{A}_{\omega_1})_{nk}(\hat{\mathbf{p}}\mathbf{A}_{\omega_2})_{km}. \qquad (6')$$

Here $A\omega$ are the components of the vector potential, and $E_n = (\xi_n^2 + \Delta_0^2)^{1/2}$. The matrix elements are calculated in the Appendix.

Making use of a method developed in[1], we continue Eq. (6) to the real frequency axis. Dividing the obtained result into a regular part which simplifies near T_c and into an irregular part, we obtain after some calculations the following equation for determining Δ_{1,ω_0}:

$$\frac{i\pi}{8}\omega_0\Delta_{1,\omega_0} + \frac{\pi}{24}\frac{lv}{T_c}\left[\frac{\partial^2\Delta_{1,\omega_0}}{\partial z^2} - \frac{4e^2}{c^2}\Delta_0(A^2)_{\omega_0}\right] - \frac{7\zeta(3)\Delta_0^2}{4\pi^2 T_c^2}\Delta_{1,\omega_0} + \frac{i\pi}{2mp_0}\int_{-\infty}^{\infty} d\omega \hat{A}^{(1)}_{\omega,\omega_0} = 0, \tag{7}$$

where the irregular part is in turn obtained from the equation

$$\left[i(\xi_\omega{}^R + \xi_{\omega''}{}^A) + \frac{lv}{3}\nabla^2\right]\hat{A}^{(1)}_{\omega,\omega_0} = \frac{mp_0}{2\pi}\left\{\left[1 + \frac{\omega\omega'' + \Delta_0^2}{\xi_\omega{}^R\xi_{\omega''}{}^A}\right]\left(\operatorname{th}\frac{\omega}{2T} - \operatorname{th}\frac{\omega''}{2T}\right)\Delta_{1,\omega_0} - \frac{ie^2lv}{6}\Delta_0\int\frac{d\omega_1}{2\pi}A_{\omega_1}A_{\omega_0-\omega_1}\left[1 + \frac{\omega\omega'' + \omega'\omega'' + \omega\omega' + \Delta_0^2}{\xi_\omega{}^R\xi_{\omega''}{}^A}\right] \times\left[\frac{1}{\xi_{\omega'}{}^R}\left(\operatorname{th}\frac{\omega'}{2T} - \operatorname{th}\frac{\omega''}{2T}\right) + \frac{1}{\xi_{\omega'}{}^A}\left(\operatorname{th}\frac{\omega}{2T} - \operatorname{th}\frac{\omega'}{2T}\right)\right]\right\} \tag{8}$$

Here for brevity $\omega' = \omega - \omega_1$, $\omega'' = \omega - \omega_0$, and ξ_ω^R and ξ_ω^A are the values of the root $(\omega_0^2 - \Delta_0^2)^{1/2}$ determined with the cuts $(-\infty, -\Delta_0)$ and $(\Delta_0, +\infty)$ taken on approaching in the complex ω plane the real axis from above and from below. Expressions (7) and (8) are symmetrical with respect to the replacement $\omega \to \omega_0 - \omega$. The derivation of (7) and (8) imposes a boundary condition at $z = 0$: $\partial A^{(1)}/\partial z = 0$. Because of this, both Δ_1 and $\hat{A}^{(1)}$ can be symmetrically continued in z and one can also go over to Fourier components in the z coordinate. It is not our purpose here to obtain the coordinate dependence of Δ_1 and to calculate the correction to the field penetration depth, and we shall therefore restrict ourselves to a qualitative analysis of the obtained formulas. Estimating the contribution from various regions of integration to the irregular terms with Δ_1 in (7), we obtain

$$\int_{-\infty}^{\infty} d\omega\left[1 + \frac{\omega\omega'' + \Delta_0^2}{\xi_\omega{}^R\xi_{\omega''}{}^A}\right]\frac{\operatorname{th}(\omega/2T) - \operatorname{th}(\omega''/2T)}{i(\xi_\omega{}^R + \xi_{\omega''}{}^A) - lvk^2/3} \approx -\frac{2\omega_0}{T}\int_{\Delta_0}^{\infty}\frac{\Delta_0^2\operatorname{ch}^{-2}(\omega/2T)\,d\omega}{\omega\sqrt{\omega^2 - \Delta_0^2}\left(i\omega_0 - \sqrt{\omega^2 - \Delta_0^2}\,lvk^2/3\omega\right)}$$

The latter expression is transformed into the form

$$-\frac{2\omega_0}{T}\int_0^{\infty} d\xi\frac{\Delta_0^2}{\xi^2 + \Delta_0^2}\frac{\operatorname{ch}^{-2}(\varepsilon/2T)}{i\omega_0 - (v(\xi)/v_F)Dk^2}, \tag{9}$$

where $D = lv/3$ and $v(\xi) = v_F\xi/\epsilon$. The numerator in (9) is of the form of a diffuse nucleus with an energy-dependent coefficient of diffusion.

In the expression for the irregular field term the main contribution in integrating over the frequencies is due to regions $|\omega \pm \Delta_0| \sim \omega_1$ and $(Dk^2)^2/\Delta_0$. Taking this into account, we find

$$\int d\omega\left[1 + \frac{\omega\omega'' + \omega'\omega'' + \omega\omega' + \Delta_0^2}{\xi_\omega{}^R\xi_{\omega''}{}^A}\right]\frac{\operatorname{th}(\omega'/2T) - \operatorname{th}(\omega''/2T)}{\xi_{\omega'}{}^R[i(\xi_\omega{}^R + \xi_{\omega''}{}^A) - Dk^2]} \approx \frac{\omega_2}{2T}\int_{-\infty}^{\infty}\frac{du}{\sqrt{u}^R\sqrt{u - \omega_0}^A[i\sqrt{u}^R + i\sqrt{u - \omega_0}^A - Dk^2/\sqrt{2\Delta_0}]} \times\left(\frac{1}{\sqrt{u - \omega_1{}^R}} + \frac{1}{\sqrt{u - \omega_2{}^A}}\right), \tag{10}$$

where $\sqrt{u} = i\sqrt{|u|}$ for $u < 0$. The second term in (8) is obtained from (10) by the replacement $\omega_1 \longleftrightarrow \omega_2$.

First of all, let us note that as follows from (10) the irregular field terms in (7) do not exceed the regular terms in order of magnitude. At the same time, the irregular terms which contain Δ_1 have, as in (4) and (5), a completely different structure. We cite that part of (7) which contains Δ_{1,ω_0}:

$$\left[\frac{i\pi}{8T_c}\omega_0 - \frac{7\zeta(3)\Delta_0^2}{4\pi^2T_c^2} - \frac{\pi}{8T_c}Dk^2\right]\Delta_{1,\omega_0}(k) - \frac{i\omega_0}{2T_c}\int_0^{\infty} d\xi\frac{\Delta_0^2}{\xi^2 + \Delta_0^2}\frac{\operatorname{ch}^{-2}(\varepsilon/2T)}{i\omega_0 - (v(\xi)/v_F)Dk^2}\Delta_{1,\omega_0}(k). \tag{11}$$

Disregarding the logarithmically slow dependences we shall write the last term in (11) schematically in the form

$$\frac{\Delta_0}{T}\frac{\omega_0}{i\omega_0 - Dk^2} \tag{12}$$

The nonuniqueness of the limiting procedure $\omega_0 \to 0$ and $k \to 0$ is eliminated if one takes into account the uniform relaxation time, as was done in the preceding Section. Then one should write instead of (12)

$$\frac{\Delta_0}{T}\frac{\omega_0}{i\omega_0 - Dk^2 - 1/\tau_0} \tag{13}$$

It is hence seen that the transition to an adiabatic situation for $T_c - T \ll T_c$ occurs for $\Delta_{1,\omega_0}(k)$ for very low frequencies, a fact which is connected with the anomalously large coefficient of Δ_0/T in (13). For values of the Ginzburg-Landau coefficient $\kappa \sim 1$ (when $Dk^2 \sim \Delta_0^2/T_c$) the adiabaticity criterion is the condition

$$\omega_0 \ll \Omega_0 \sim \Delta_0^3/T_c^2. \tag{14}$$

When κ differs appreciably from unity, the problem requires more detailed investigation. Let us first consider the Pippard case ($\kappa \ll 1$). The spatial behavior of Δ in the Ginzburg-Landau theory is characterized by two scales: a) $k \sim 1/\delta$ ($DK^2 \sim \Delta_0^2/\kappa^2 T_c$) and b) $k \sim \kappa/\delta$ ($Dk^2 \sim \Delta_0^2/T_c$). The second case leads to the adiabacity criterion (14). In case a) we obtain instead of this

$$\frac{1}{\varkappa^2}\frac{\Delta_0^2}{T_c^2} \gg \frac{\Delta_0}{T_c}\frac{\omega_0}{\omega_0 + \Delta_0^2/\varkappa^2T_c}, \quad \omega_0 \ll \Omega_0' \sim \frac{1}{\varkappa^4}\frac{\Delta_0^3}{T_c^2}. \tag{15}$$

Although for this scale the adiabaticity is retained up to higher values of the frequency, the change in Δ corresponding to this scale is small ($\sim\kappa$).

In the Landau case ($\kappa \gg 1$) one requires for complete adiabaticity very low frequencies:

$$\omega_0 \ll \frac{1}{\varkappa^2}\frac{\Delta_0^3}{T_c^2}.$$

Thus for $\kappa \gtrsim 1$ [and also for $\kappa < 1$ for the main scale b)] for $\omega_0 > \Omega_0$ of (14) the variation of the gap is nonadiabatic. Moreover, on account of the large coefficient in (13) the high-frequency component of Δ_1 is small compared with the correction to the time-averaged value ($\omega_0 \to 0$). We emphasize that all the enumerated results refer to the region in the vicinity of T_c when $\Delta/T \ll 1$.

3. A SMALL PARTICLE IN AN ALTERNATING FIELD (ALLOYS WITH $l \ll \xi_0$)

In Sec. 1 we considered the case of a pure superconductor of small dimensions. Let us now consider the case of superconducting alloys. We shall neglect the dependence of Δ on the coordinates which corresponds formally to $k = 0$ in all formulas of the preceding Section. Let us note that if the alloy is a Pippard alloy then this requires the condition $d \ll \delta/\kappa$ while the penetration depth of the field can be smaller than d. In the case of a type-II superconductor the limitation $d \ll \delta/\kappa$ is stronger and signifies that there are no vortices within the particle. In order to be able to make use of local spatial relations, we assume of course that the mean free path satisfies the condition $l \ll d$ and $l \ll \delta$. Neglecting all derivatives with respect to the coordinates in (7) and (8) one can simplify all expressions considerably. This makes it possible to carry out calculations with an accuracy up to higher-order terms in ω/Δ_0. Omitting the calculations, we present the final result:

$$-\left[\frac{7\zeta(3)\Delta_0^2}{4\pi^2 T_c^2}+\frac{\pi}{4}\frac{\Delta_0}{T_c}\frac{\omega_0}{\omega_0+i/\tau_0}\right]\Delta_{1,\omega_0}$$
$$=\left(\frac{2e}{c}\right)^2\frac{\pi D}{8}\frac{\Delta_0}{T_c}\int\frac{d\omega_1}{2\pi}\overline{A_{\omega_1}A_{\omega_0-\omega_1}}\left\{1-\frac{\omega_0}{2(\omega_0+i/\tau_0)}\right.$$
$$\left.-\frac{i}{2\pi\Delta_0\omega_0}\left[\omega_1^2\ln\frac{8\Delta_0}{\omega_1}+(\omega_0-\omega_1)^2\ln\frac{8\Delta_0}{\omega_0-\omega_1}\right]\right\}. \quad (16)$$

Here $\overline{A_{\omega_1}A_{\omega_0-\omega_1}}$ denotes averaging of the field over the volume of the sample:

$$\overline{A_{\omega_1}A_{\omega_0-\omega_1}}=\frac{1}{V}\int d^3\mathbf{r}\,A_{\omega_1}(\mathbf{r})A_{\omega_0-\omega_1}(\mathbf{r}).$$

Disregarding the terms in the square brackets on the right-hand side, we note that unlike in the case considered in Sec. 1, the field term does not cancel for $\omega_0\tau_0 \gg 1$. Nevertheless, the part of Δ_1 which varies with time, although it differs from zero, is small compared with the correction to the static part of Δ.

Let us now go over to an investigation of the nature of the terms quadratic in the frequency in the right-hand side of (16). If $\omega_0 \sim \omega_1$, then these terms are small. However, for the static (time-averaged) part of $\Delta_1(\omega_0 \to 0)$ they provide us with the same disagreeable features as above in Sec. 1. Let us attempt first of all to explain the origin of these terms. In the time representation they have the following structure:

$$\frac{e^2vl}{c^2T}\ln\frac{\Delta}{\omega_1}\int_{-\infty}^{t}(\widetilde{\dot{A}})^2=\frac{e^2vl}{T}\int_{-\infty}^{t}\widetilde{E^2}\ln\frac{\Delta}{\omega_1}d\tau.$$

This form points to their dissipative nature. In order to show this definitively we note that in BCS theory[3] the equation determining the gap is of the form

$$\Delta=|g|\frac{mp_0}{2\pi^2}\int_0^{\tilde{\omega}}d\xi\frac{\Delta}{\sqrt{\xi^2+\Delta^2}}[1-2n(\varepsilon)]. \quad (17)$$

Let $\epsilon = \epsilon_0 + \delta\epsilon$; then in the right-hand side there appears the term

$$-\Delta\int\frac{d\xi}{\sqrt{\xi^2+\Delta^2}}2\frac{\partial n}{\partial\varepsilon}\delta\varepsilon\sim\frac{\Delta}{T}\int\frac{d\xi}{\sqrt{\xi^2+\Delta^2}}\mathrm{ch}^{-2}\frac{\varepsilon}{2T}\delta\varepsilon. \quad (18)$$

The energy change $\delta\epsilon$ consists of two parts $\delta\epsilon = \delta\epsilon_1 + \delta\epsilon_2$ where $\delta\epsilon_1$ is the change of the spectrum under the influence of a magnetic field and $\delta\epsilon_2$ corresponds to "heating"

$$\delta\varepsilon_2=\int_{-\infty}^{t}e\widetilde{Ev_d}d\tau. \quad (18')$$

Here the tilde denotes averaging over the period and v_d is the drift velocity which depends on ξ:

$$\dot{v}=eE\frac{\partial v(\xi)}{\partial p}-\frac{v_d}{\tau(\xi)}=0,\quad v_d=eE\tau(\xi)\frac{\partial v(\xi)}{\partial p}.$$

From the condition $\tau(\xi)v(\xi) = l$ we find [compare with (9)]

$$v_d=eElv/\xi\quad(\xi\ll\Delta). \quad (19)$$

Collecting (19) and (18) and substituting this in (18′), we obtain terms of just the same logarithmic structure as in (16). Thus for the quasiparticles in a superconductor the time between the collisions increases on approaching the threshold; in this connection the energy taken up by the electric field increases. It should be noted that this effect provides a small correction to the conductivity to which the principal contribution is due to electrons with an energy $\sim T$:

$$\sigma_{\text{eff}}=\sigma\left(1+\frac{\Delta}{2T}\ln\frac{\Delta}{\omega}\right).$$

Thus we see that the terms of second order in the frequency in (16) are really connected with Joule heating. With heat being conducted away and sufficiently high heat conduction these terms are apparently small.

4. DISCUSSION AND GENERALIZATION OF THE OBTAINED RESULTS

If we now reconsider the derivation of the formulas cited above, we note that the majority of the results depends essentially on the fact that the density of states has a singularity at an energy equal to Δ. This circumstance does not allow one to set up a simple scheme for describing nonstationary phenomena in superconductors. The equations will in this case have to be of a nonlocal nature with respect to time.

However, at the end of Sec. 2 it was shown that in the region of nonadiabaticity for $\omega_0 \gg \Omega_0 \sim \Delta_0^3/T_c^2$ the oscillating part of the gap is small compared with the static part. At the same time the change of the static part of the gap is only determined by the regular part of Eq. (5) averaged over the time, which corresponds to averaging of the square of the field in the Ginzburg-Landau equations. There exist two frequency ranges:

$$\begin{aligned}&\text{I. } \Omega_0\ll\omega_0\ll\Delta_0^2/T_c;\\&\text{II. } \Delta_0^2/T_c\ll\omega_0\ll\Delta_0.\end{aligned} \quad (20)$$

In the first range one can neglect the term $i\pi\omega_0/8$ in Eq. (11) for the oscillating part of the gap. As regards the field, it is determined in this range by the London equations. In the second range the smallness of the oscillating part of Δ is insured by the term $i\pi\omega_0/8$ in conjunction with the irregular term in (11). The field distribution is obtained from Maxwell's equations where in the expression for the current density

$$j=\sigma E-\frac{Ne^2\tau_{tr}}{mc}\frac{\pi}{2T_c}|\Delta|^2A=-\frac{\sigma}{c}\left(\frac{\partial A}{\partial t}+\frac{\pi|\Delta|^2}{2T_c}A\right)$$

one can neglect the term with Δ^2, i.e., the field is determined by the skin effect in the normal metal.

The high-frequency case [region II in (20)] coincides completely with that already considered before[6]; we shall therefore restrict ourselves to an investigation of the intermediate frequency range [range I in (20)]. Introducing the frequency scale $\Omega_1 = \pi\Delta_0^2/2T_c$, the dimensionless time parameter $t' = \Omega_1 t$, and going over to dimensionless variables in accordance with[2], we obtain the equations

$$(\Delta^2 - 1)\Delta - \frac{1}{\varkappa^2}\frac{\partial^2\Delta}{\partial z^2} + \overline{A^2}(z)\Delta = 0,$$
$$\frac{\partial^2 A}{\partial z^2} = A + \Delta^2 A. \qquad (21)$$

(Let us note that we always have in mind the one-dimensional case when Δ can be considered to be a real quantity.) The time-averaged square of the field enters in the first of these equations.

Equation (21) describes the destruction of superconductivity by an alternating field.

Above we estimated the variation of $|\Delta|$ with the field. Generally speaking, the phase of the gap appears in spatial problems. The question of when one can write down the three-dimensional averaging of the equation is connected with the problem of the motion of a vortex in an electromagnetic field. So long as the vortex displacements are small, the changes of phase will also be small. Therefore, it seems to us that one can utilize the averaged Ginzburg-Landau equations at least in the high-frequency range II of (20). However, this question is necessary in our further investigation.

Let us now consider the problem of the destruction of superconductivity by a high-frequency field $h(t) = h_1 \sin \omega t$ for the frequency range I of (20). An important point here is the dependence of the field distribution on Δ. If one neglects in the second equation of (21) the time derivative of the field (i.e., the σE term in the current), then the equation for the gap reduces by the replacement $h_0^2 \to h_1^2/2$ to the static equation investigated in detail by Ginzburg.[7] Figure 5 in Ginzburg's article now depicts the dependence of the gap on the boundary of the semispace $\Delta(0)$ on $h_1/\sqrt{2}$. This curve has a hysteresis character which corresponds in the static case to a first-order phase transition to the normal state. In our case we should in the equation for the current for small values of Δ take into account $\dot{A}$ which will lead to a transition to the branch where $\Delta^2 \sim \omega$. For small κ one can obtain in explicit form an equation for the hysteresis curve by using the matching method described by the authors[6] (an assumption conceptually close to this was also used by Kemoklidze[8]):

$$h_1^2 = \frac{2}{\varkappa}\frac{1-\Delta^2(0)}{\Delta(0)}\sqrt{\omega^2 + \Delta^4(0)}\,[\Delta^2(0) + \sqrt{\omega^2 + \Delta^4(0)}]^{1/2}.$$

For thin films with dimensions $d \ll 1/\kappa$ the hysteresis curves were obtained in the static case by Ginzburg.[7] Here too in the case of an alternating field there occurs a collapse of the regime to the branch on which $\Delta^2 \sim \omega$. Thus the collapse of the regime in a strong high-frequency field is a characteristic peculiarity of the intermediate frequency range I of (20). For very thin films whose thickness is comparable with the penetration depth of the field the collapse disappears. This corresponds to the region of the second-order phase transition in the static case.

Concerning the unavoidability of collapses for type-II superconductors we merely note that in accordance with[7] the "superheating" field for $\kappa \gg 1$ is $h_1 = 1$, i.e., for field amplitudes greater than $\sqrt{2}H_{st}$ there is no stable plane regime.

APPENDIX

We shall present here an assumption by means of which one can average over the positions of the impurities in formulas (6) and (6′) in a comparatively simple manner. First of all we note that the expression for the Fourier component of $\Delta_{1,\omega_0}(k)$ contains products of the matrix elements

$$\sum (\Delta_1)_{mn}(e^{-ikr})_{nm}, \qquad \sum (\hat{p}A_{\omega_1})_{mn}(\hat{p}A_{\omega_2})_{nk}(e^{-ikr})_{km},$$

summed over all quantum numbers except for the energies. It is precisely these quantities which underlie the averaging over the impurity positions. Let us consider for simplicity the first of these quantities which we shall write in the form

$$\sum (\Delta_1)_{12}(e^{-ikr})_{21}\delta(\xi_1 - \xi_m)\delta(\xi_2 - \xi_n) \equiv f(\xi_n, \xi_m).$$

Here the summation is over all quantum numbers, including the energies ξ_1 and ξ_2. Using the relation

$$\delta(\xi_1 - \xi_m) = \frac{1}{2\pi i}\left(\frac{1}{\xi_1 - \xi_m - i\delta} - \frac{1}{\xi_1 - \xi_m + i\delta}\right),$$

we represent $f(\xi_n, \xi_m)$ in the form of combinations of four terms of the type

$$\sum \frac{(\Delta_1)_{12}(e^{-ikr})_{21}}{(\xi_m - \xi_1)(\xi_n - \xi_2)}$$

which differ from one another in the circuiting of the poles. However, each of the quantities is nothing else but the Fourier component over the coordinates of

$$\int G^{\pm}_{\xi_m}(\mathbf{r}, \mathbf{r}_1)\Delta_1(\mathbf{r}_1)G^{\pm}_{\xi_n}(\mathbf{r}_1, \mathbf{r})d^3r_1, \qquad (A.1)$$

where $G^{\pm}_{\xi}(\mathbf{r}, \mathbf{r}_1)$ is a retarded or advanced Green's function of the electrons of the normal metal in which ξ plays the role of a frequency argument. The averaging of such an expression over the impurity positions is readily carried out by means of the well-known diagram technique (see, for example,[9]). The result is a sum of ladder diagrams. At the same time a nonzero contribution is only made by those expressions (A.1) which contain one retarded or advanced Green's function.

The second matrix element which contains the electromagnetic field is transformed analogously and its calculation reduces to averaging of a product of three Green's functions of the normal metal over the impurity positions. In the ladder approximation the problem reduces to pairwise averaging of Green's functions. The calculations become particularly simple in the case of isotropic scattering in which case one has only to sum over one ladder of diagrams. The summation of the other two ladders makes a contribution proportional to div **A** which plays no role in the solution of the plane problem considered by us, since one can choose a gauge with div **A** = 0. As a result of simple calculations one obtains the following expression for $\Delta_{1,\omega_0}(k)$ which is valid for sufficiently large impurity concentrations when $kl \ll 1$:

$$\Delta_{1,\omega_0}(k) = |g| \frac{mp_0}{2\pi^2} \int d\xi_1 d\xi_2 \frac{1}{\pi} \frac{Dk^2}{(\xi_1-\xi_2)^2+(Dk^2)^2} \Big\{ Q_1(\xi_1\xi_2)\Delta_{1,\omega_0}(k)$$

$$+ \frac{D}{\pi}\left(\frac{e}{c}\right)^2 \int \frac{d\omega_1}{2\pi} (\mathbf{A}_{\omega_1}\mathbf{A}_{\omega_0-\omega_1})_k \int d\xi_3 Q_2(\xi_1\xi_3\xi_2) \Big\},$$

$$Q_1(\xi_1\xi_2) = T\sum_\omega \frac{(\omega_0-\omega)\omega+\xi_1\xi_2-\Delta_0^2}{(\omega^2-E_1^2)((\omega_0-\omega)^2-E_2^2)}, \qquad (A.2)$$

$$Q_2(\xi_1\xi_3\xi_2) = T\sum_\omega \Delta_0 \frac{\omega(\omega-\omega_0)+\omega(\omega-\omega_1)+(\omega-\omega_0)(\omega-\omega_1)-\xi_1\xi_2+\xi_1\xi_3+\xi_2\xi_3+\Delta_0^2}{(\omega^2-E_1^2)((\omega-\omega_1)^2-E_3^2)((\omega-\omega_0)^2-E_2^2)}.$$

Here $D = lv/3$ is the diffusion coefficient of the electrons, and l is the mean free path. After analytic continuation and integration over ξ_1, ξ_2, and ξ_3 (A.2) leads to formulas (7) and (8). We shall also note that since the derivation was for the Fourier components we have assumed that the terms linear in k vanish on account of averaging over the angles. Near the boundary of a superconductor this becomes incorrect. As a result there appear the boundary conditions indicated in the text.

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