EFFECT OF AN ELECTROMAGNETIC FIELD ON DECAYS OF ELEMENTARY PARTICLES

V. I. RITUS

P. N. Lebedev Physics Institute, USSR Academy of Sciences

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Probabilities are obtained for the decays of the muon, $\mu \to e\nu\tilde{\nu}$, the pion, $\pi^\pm \to \pi^0 e^\pm \nu$, $\pi \to \mu\nu$, $e\nu$, and for neutrino emission by an electron, $e \to e\nu\tilde{\nu}$, in the presence of strong variable or constant electromagnetic fields. The characteristic properties of these processes are considered and parameters are found which determine the dependence of the decay probability on the field for various ratios between the particle masses, in particular for small mass differences which increase the sensitivity of the probability to the field. Depending on the form of the decay interaction between the particles, an external electromagnetic field may either increase or decrease the decay rate. It is shown that the probability, which depends on the external field $F_{\mu\nu}$ by means of the parameter $\chi = \sqrt{(eF_{\mu\nu}p_\nu)^2}/m^3$, has an essential singularity at the point $\chi = 0$.

1. INTRODUCTION

In the present article the effect of an electromagnetic field on the decays of elementary particles is considered. The decays are described by the well-known V-A theory of weak interactions to first order in perturbation theory. The external electromagnetic field is approximated by the field of a plane, monochromatic wave or by a constant crossed field.[1] The interaction of charged particles with such a field is taken into account exactly by using the Volkov solutions. Principal attention is given to decays into three particles: $\mu \to e\nu\tilde{\nu}$, $\pi^\pm \to \pi^0 e^\pm \nu$. The decay $\pi \to \mu\nu$, $e\nu$ which was investigated earlier by Nikishov and the author[2] is considered once again in connection with the utilization of a more effective method of investigation, enabling one to obtain certain new results, and a mistake in the asymptotic expression (9) of article[2] is corrected.

The effect of the field of a plane, electromagnetic wave on the decay of an elementary particle leads to the result that the total decay probability begins to depend on the field by means of the two invariants[1)]

$$x = eB / m\omega, \qquad \chi = \sqrt{(eF_{\mu\nu}p_\nu)^2} / m^3, \tag{1}$$

where B and ω are the amplitude of the field intensity of the wave and its frequency, $F_{\mu\nu}$ is the electromagnetic field tensor, p_μ and m are the momentum and mass of the decaying particle. For $x \ll 1$ one can expand the decay probability in powers of the parameter x^2. The term $\propto x^2$ corresponds to decay in vacuum, the term $\propto x^2$ corresponds to decay with absorption of a single photon from the wave, etc. For $x \gtrsim 1$ the decay amplitude depends on the field in a substantially nonlinear way. For $x \gg 1$ the probability essentially coincides with the decay probability in a constant, crossed field. This case is very important since the decays of ultrarelativistic particles in an arbitrary, constant electromagnetic field[1] reduce to it. Therefore, in this article most attention will be given to decays in a constant, crossed field.

From general considerations it is clear that if the masses of all particles are of the same order, then the field will substantially change the decay probability for $\chi \sim 1$, that is, for particle energy $p_0 \sim (B_0/B)m$, where $B_0 = m^2/e$ is the characteristic intensity of the field. For the known decaying particles, B_0 is extremely large. However, if there are small mass differences, then the situation may be changed. Thus, for the decay of a charged particle into another charged particle and a neutral system (corresponding to masses m, m′, and m_0) the parameter which determines the probability will be $\tilde{\chi} = \chi\delta^{-2}$ if the quantity $\delta = (m - m_0)/m \ll 1$, or $\tilde{\chi} = \delta^{-2}\epsilon^{-3}$ if, in addition, $\epsilon = [2(m - m_0 - m')/(m - m_0)]^{1/2} \ll 1$, and the characteristic field intensity turns out to be $\tilde{B}_0 = m^2\delta^2/e$ or $\tilde{B}_0 = m^2\delta^2\epsilon^3/e$. For example, for the decay $\pi^\pm \to \pi^0 e^\pm \nu$ the characteristic field intensity $\tilde{B}_0 = m^2\delta^2/e = 10^{15}$ Oe is a thousand times smaller than B_0, but still very large in order to be able to obtain $\tilde{\chi} \approx Bp_0/\tilde{B}_0 m \sim 1$ for presently-available fields B and energies p_0. However, it would apparently be possible to achieve this for nuclear decays where the level differences may be very small.

The analytic properties of the probability as a function of the parameter χ are also considered in this article, i.e., in the presence of an external field or charge, and it is shown that the point $\chi = 0$ is an essential singularity.

2. MUON DECAY $\mu \to e + \nu + \tilde{\nu}$

The matrix element for the decay $\mu \to e\nu\tilde{\nu}$ in the field of a plane, electromagnetic wave has the usual form

$$M = 2^{-1/2} G \int d^4x (\bar{e}_p \gamma_\mu (1 + \gamma_5) \nu_{l_2}) (\overline{\nu_{l_1}} \gamma_\mu (1 + \gamma_5) \mu_p), \tag{2}$$

if μ, e, ν are understood as the wave functions of muon, electron, and neutrino in the presence of the field. We recall[1] the form of the wave function for a charged particle with spin 1/2 in the field of a wave with potential $A_\mu = A_\mu(kx)$, $k^2 = Ak = 0$:

$$\mu_{pr}(x) = \left[1 + \frac{e\hat{k}\hat{A}}{2kp}\right] u_{pr} \exp\left[i \int_0^{kx} \left(\frac{epA(\varphi)}{kp} - \frac{e^2A^2(\varphi)}{2kp} \right) d\varphi + ipx \right]. \tag{3}$$

[1)]Units with $\hbar = c = 1$, $e^2/4\pi = 1/137$ are used, and the following notation: $p_\mu = (\mathbf{p}, ip_0)$, $kp = \mathbf{k} \cdot \mathbf{p} - k_0 p_0$, $\hat{p} = \gamma p = \boldsymbol{\gamma} \cdot \mathbf{p} + i\gamma_4 p_0$, where the γ_μ are Hermitian matrices.

In what follows we will consider a linearly-polarized monochromatic wave $A_\mu = a_\mu \cos(kx)$. Using in (2) the functions (3) for muon and electron and performing calculations similar to these which are described in detail in[1,2], we arrive at the following expression for the square of the matrix element, averaged over the spin projections of the muon, summed over the spin projections of the electron, and integrated over the momenta l_1 and l_2 of the emitted neutrinos:

$$dW = \int \frac{1}{2} \sum_{\text{polarizations}} \frac{|M|^2}{VT} \frac{d^3l_1 d^3l_2 d^3q'}{(2\pi)^9} = -\frac{G^2 n}{6\pi q_0 q_0'} \sum_{s>s_0} \Big\{ \Big[l^4 + \frac{1}{2}(m^2+m'^2) l^2 - \frac{1}{2}(m^2 - m'^2)^2 \Big] A_0^2 + \Big[\frac{(2l^2 - m^2 - m'^2)(kl)^2}{2(kq)(kq')} + 2l^2 \Big] \times e^2 a^2 (A_1^2 - A_0 A_2) \Big\} \frac{d^3 q'}{(2\pi)^3}. \quad (4)$$

The integration over the neutrino momenta, which substantially simplifies the calculation and the expression for the probability dW, is carried out with the aid of the well-known formula of Lenard.[3] Here we present more general formulas for the integration over the momenta of particles with nonvanishing masses, which will be required in Sec. 5:

$$\int \delta(l - l_1 - l_2) \frac{d^3l_1 d^3l_2}{l_{10} l_{20}} = -\frac{2\pi[(l^2 + m_1^2 + m_2^2)^2 - 4m_1^2 m_2^2]^{1/2}}{l^2}, \quad (5a)$$

$$\int l_{1\alpha} \delta(l - l_1 - l_2) \frac{d^3l_1 d^3l_2}{l_{10} l_{20}} = \frac{\pi(m_1^2 - m_2^2 - l^2)[(l^2 + m_1^2 + m_2^2)^2 - 4m_1^2 m_2^2]^{1/2}}{l^4} l_\alpha, \quad (5b)$$

$$\int l_{1\alpha} l_{2\beta} \delta(l - l_1 - l_2) \frac{d^3l_1 d^3l_2}{l_{10} l_{20}} = -\frac{\pi[(l^2 + m_1^2 + m_2^2)^2 - 4m_1^2 m_2^2]^{1/2}}{6 l^6} \Big\{ 2\big[l^2(l^2 - m_1^2 - m_2^2) - 2(m_1^2 - m_2^2)^2 \big] l_\alpha l_\beta + [(l^2 + m_1^2 + m_2^2)^2 - 4m_1^2 m_2^2] l^2 \delta_{\alpha\beta} \Big\}. \quad (5c)$$

For $m_1 = m_2 = 0$, the last of these formulas goes over into Lenard's formula. In expression (4) $l = sk + q - q'$, q and q′ are the quasimomenta of the muon and electron, m and m′ are their masses; s is the number of photons absorbed from the wave (or emitted into the wave if $s < 0$), $s_0 = (m^2 - m'^2)/2kq$ is the minimal possible value for s, determined from the condition for a minimum of the square $E_s^2 = -(sk + q)^2$ of the total energy of the system; for decays $s_0 < 0$, and $-s_0$ represents the largest number of photons which may be emitted into the wave during the decay; n is the average number density of muons. The functions $A_n(s, \alpha, \beta)$, n = 0, 1, 2, which are considered in detail in[1], describe the motion of charged, spinor particles in a field and depend on s and on the invariant variables

$$\alpha = e\Big(\frac{ap}{kp} - \frac{ap'}{kp'}\Big), \quad \beta = \frac{e^2 a^2}{8}\Big(\frac{1}{kp} - \frac{1}{kp'}\Big). \quad (6)$$

The differential decay probability dW is represented by an infinite sum of terms, each of which describes the decay associated with absorption from the wave (or emission into the wave) of a definite number of photons with momentum k_μ and polarization $e_\mu = a_\mu/a$.

The total decay probability is obtained by integrating dW over the electron quasimomenta. Usually such an integration is carried out with respect to the direction and magnitude of the vector q′ in the center-of-mass system. In the case under consideration, however, a center-of-mass system exists for each s, and therefore it is more convenient to use as independent variables the angle φ between the planes (**k**, **q**′) and (**k**, **a**) in the system where **k** and **p** are directed towards each other, and the invariant variables $u = kl/kq'$ and $\lambda = -l^2/m^2$. Then for reactions with the conservation law $sk + q = q' + l_1 + l_2$ we have

$$W = \sum_{s>s_0} \int \frac{d^3q'}{q_0'} w = \sum_{s>s_0} \int_0^{2\pi} d\varphi \int_{u_1}^{u_2} \frac{du}{(1+u)^2} \int_{\lambda_1}^{\lambda_2} d\lambda \frac{m^2}{2} w, \quad (7)$$

where

$$u_{2,1} = \frac{E_s^2 - m_0^2 - m_*'^2 \pm [(E_s^2 - m_0^2 - m_*'^2)^2 - 4m_0^2 m_*'^2]^{1/2}}{2m_*'^2}$$

$$\lambda_1 = \frac{m_0^2}{m^2}, \quad \lambda_2 = \frac{E_s^2}{m^2}\frac{u}{1+u} - \frac{m_*'^2}{m^2} u; \quad (8)$$

$m_0 = m_1 + m_2$ is the total mass of the neutral particles, which is equal to zero in our case. The function w is determined from Eq. (4) and is given by

$$w = \frac{G^2 m^4 n}{48\pi^4 q_0} \Big\{ \Big[\frac{1}{2}(1-\mu)^2 + \frac{1}{2}(1+\mu)\lambda - \lambda^2 \Big] A_0^2 + \Big[(2\lambda + 1 + \mu)\frac{u^2}{2(1+u)} + 2\lambda \Big] x^2 (A_1^2 - A_0 A_2) \Big\}, \quad (9)$$

where $\mu = m'^2/m^2$. The arguments of the functions A_n are expressed in terms of φ, u, and λ in the following way:

$$\alpha = z\cos\varphi, \quad z = \frac{x^2}{\chi}\sqrt{(1+u)(\lambda_2 - \lambda)}, \quad \beta = \frac{x^3 u}{8\chi}. \quad (10)$$

For $x \ll 1$ one can expand the functions A_n in terms of the parameter x. Then for s = 0 we have $A_0^2 = 1$, $A_1^2 - A_0A_2 = -1/2$ to the lowest approximation, and the probability for free decay of a muon (see[4]) follows from Eqs. (7) and (9). For $s = \pm 1$ we have $A_0^2 = \alpha^2/4$, $A_1^2 - A_0A_2 = 1/4$, and the probability for muon decay with the absorption or emission of a single photon (see[5]) follows from expressions (7) and (9), and so forth.

One can also write formula (7) in the form

$$W = \int_0^{2\pi} d\varphi \int_0^\infty \frac{du}{(1+u)^2} \int_{\lambda_1}^\infty d\lambda \sum_{s>s_{min}} \frac{m^2}{2} w, \quad (11)$$

where

$$s_{min} = \frac{x^3 u}{4\chi}\Big[1 + \frac{2}{x^2}\Big(1 - \Delta\frac{1+u}{u} + \lambda\frac{1+u}{u^2}\Big)\Big], \quad \Delta = 1 - \mu, \quad (12)$$

if the summation over s is carried out first. For $x \gg 1$ the effective values of the variables s and φ will be $s \sim x^3$, $\varphi \sim x^{-1}$. In this case one can replace the sum over s by an integral and change from variables s and φ to variables ρ and τ, whose effective values are of the order of unity (see[1]):

$$\rho = \frac{\alpha}{8\beta} = \frac{eF_{\mu\nu} q_\mu' q_\nu}{x m^4 \chi} = \frac{1}{x}\Big[\Big(1 + \frac{x^2}{2}\Big)\Big(1 - \frac{u_1}{u}\Big)\Big(\frac{u_2}{u} - 1\Big)\Big]^{1/2} \cos\varphi,$$

$$\tau = -\frac{eF_{\mu\nu}^* q_\mu' q_\nu}{m^4 \chi} = \Big[\Big(1 + \frac{x^2}{2}\Big)\Big(1 - \frac{u_1}{u}\Big)\Big(\frac{u_2}{u} - 1\Big)\Big]^{1/2} \sin\varphi,$$

$$s = \frac{x^3 u}{2\chi}\Big(\rho^2 + \frac{\tau^2}{x^2}\Big) + s_{min}, \qquad \mathrm{tg}\,\varphi = \frac{\tau}{x\rho}. \quad (13)$$

From (13) it is seen that if $(s - s_{min})^{1/2}$ and φ are regarded as polar coordinates on a plane, then $(x^3u/2\chi)^{1/2}\rho$ and $(x^3u/2\chi)^{1/2}\tau/x$ will be Cartesian coordinates. Therefore, for $x \gg 1$ instead of expression (11) one can write

$$W=\int_0^\infty \frac{du}{(1+u)^2}\int_{\lambda_1}^\infty d\lambda\int_{-\infty}^\infty d\tau\int_{-\infty}^\infty d\rho\,\frac{m^2x^2u}{2\chi}w. \tag{14}$$

We note that formula (14) is obtained from (7) or (11) by only replacing the summation over s by an integration.

In article[1] it was shown that for $x \gg 1$ the functions A_0^2 and $A_1^2 - A_0A_2$ entering into w in the interval $-1 \le \rho \equiv \cos\psi \le 1$ are determined by the asymptotic expressions (B.13) and (B.21), and for $|\rho| > 1$ they decrease exponentially like $\exp(-Cx^3)$. Therefore, in Eq. (14) one can restrict oneself to the interval $|\rho| \le 1$. Then we obtain

$$W(\chi)=\frac{1}{\pi}\int_0^\pi d\psi\,F(\chi\sin\psi), \tag{15}$$

where

$$F(\chi)=\frac{G^2m^6c}{24\pi^4}\int_0^\infty\frac{du}{(1+u)^2}\int_0^\infty d\lambda\int_{-\infty}^\infty d\tau\left(\frac{u}{2\chi}\right)^{1/3}$$
$$\times\left\{\left[\frac{(1-\mu)^2}{2}+\frac{1+\mu}{2}\lambda-\lambda^2\right]\Phi^2(y)+\left[(2\lambda+1+\mu)\frac{u^2}{2(1+u)}+2\lambda\right]\right.$$
$$\left.\times\left(\frac{2\chi}{u}\right)^{2/3}(y\Phi^2+\Phi'^2)\right\}. \tag{16}$$

Here $\Phi(y)$ is the Airy function,

$$y=\left(\frac{u}{2\chi}\right)^{2/3}\left(1+\tau^2-\Delta\frac{1+u}{u}+\lambda\frac{1+u}{u^2}\right), \tag{17}$$

c is an invariant which is equal to the ratio of the number density of muons to their energy. We note that cmF^{-1} is the muon's lifetime in its rest system. In expression (16) the terms which are rapidly oscillating for $x \gg 1$ are emitted, and in this case it represents the decay probability in a constant, crossed field. Of course, one can immediately obtain this probability from the initial matrix element (2) if the functions in a crossed field (see[2]) are used as the wave functions.

Now let us analyze the decay probability in a crossed field in detail. First of all we note that the integrand in Eq. (16) is the differential decay probability with respect to the variables u, λ, and τ. Carrying out the integration with respect to τ in (16) just as it is done in Sec. 4 of article[6], we obtain

$$F(\chi)=\frac{G^2m^6c}{48\pi^3\sqrt{\pi}}\int_0^\infty\frac{du}{(1+u)^2}\int_0^\infty d\lambda\left\{\left[\frac{1}{2}(1-\mu)^2+\frac{1}{2}(1+\mu)\lambda-\lambda^2\right]\right.$$
$$\left.\times\,\Phi_1(t)-\left(\frac{\chi}{u}\right)^{2/3}\left[\frac{2\lambda+1+\mu}{1+u}u^2+4\lambda\right]\Phi'(t)\right\}, \tag{18}$$

where

$$\Phi_1(t)=\int_t^\infty dx\,\Phi(x),\qquad t=\left(\frac{u}{\chi}\right)^{2/3}\left(1-\Delta\frac{1+u}{u}+\lambda\frac{1+u}{u^2}\right). \tag{19}$$

The rather simple dependence of the differential probability on λ allows one to carry out an integration with respect to this variable in expression (18). Then for the distribution of probability with respect to u we obtain

$$\frac{dF(\chi,u)}{du}=\frac{G^2m^6c}{48\pi^3\sqrt{\pi}}\frac{1}{(1+u)^2}\left\{A\Phi_1(z)+B\Phi'(z)+C\Phi(z)\right\}, \tag{20}$$

where

$$A=\frac{1}{3}(\mu u-v)^3+\frac{1+\mu}{4}(\mu u-v)^2$$
$$-\frac{(1-\mu)^2}{2}(\mu u-v)+4\chi^2v^2\left(\frac{1}{3}uv+1\right),$$

$$B=\left(\frac{\chi}{u}\right)^{2/3}uv\left[\frac{1}{3}(\mu u-v)^2+\frac{1+\mu}{4}(\mu u-v)-\frac{(1-\mu)^2}{2}\right],$$
$$C=\left(\frac{\chi}{u}\right)^{4/3}u^2v^2\left[\frac{1}{3}(\mu u-v)^2+\frac{5(1+\mu)}{4}\right],$$

$v = u(1+u)^{-1}$, and z is the minimal value of t as a function of λ:

$$z=\left(\frac{u}{\chi}\right)^{2/3}\left(1-\Delta\frac{1+u}{u}\right). \tag{21}$$

As $u \to 0$ the distribution dF/du tends to zero, undergoing damped oscillations with an infinitely-increasing frequency. As $u \to \infty$ it falls off exponentially.

Integrating this distribution over u (here using integration by parts in the first and second terms) and changing from u to the variable z, which is a monotonic function of u, we obtain

$$F(\chi)=\frac{G^2m^6c}{48\pi^3\sqrt{\pi}}\int_{-\infty}^\infty dz\,\Phi(z)h(z), \tag{22}$$

where h(z) depends on z only through u:

$$h(z)=-\frac{v^4}{12}+\frac{1-3\mu}{12}v^3+\frac{1-3\mu}{4}v^2-\frac{\mu^2(27-5\mu)}{12}v$$
$$+3\mu^2\ln(1+u)-\frac{\mu^2(9+5\mu)}{12}u+\frac{\mu^3}{6}u^2-\frac{4}{3}\chi^2\left\{\frac{v^4}{4}-\frac{2v^3}{3}\right.$$
$$+\frac{v^2}{2}+v-\ln(1+u)-\frac{3v^4}{4(2\mu u+1-\mu)}\left[\mu u-v+\frac{15(1+\mu)}{4}\right]$$
$$+\frac{9v^3}{4(2\mu u+1-\mu)^2}\left[\left(\frac{(\mu u-v)^2}{3}+\frac{(1+\mu)(\mu u-v)}{4}-\frac{(1-\mu)^2}{2}\right)\right.$$
$$\left.\left.\times\left(5-9v+\frac{3(1-\mu)}{2\mu u+1-\mu}\right)+3\left(\frac{2}{3}(\mu u-v)+\frac{1+\mu}{4}\right)(\mu u-v+v^2)\right]\right\}, \tag{23}$$

where $v = u(1+u)^{-1}$, and u and z are related by Eq. (21). The integrand in (22) is no longer the probability distribution with respect to u; therefore expression (22) should be understood simply as an integral representation for $F(\chi)$.

One can obtain an asymptotic representation for F as $\chi \to 0$ from expression (22) if h is written down as a function of z in the form

$$h(z)=h(0)+z\tilde{h}(z),\qquad \tilde{h}(z)=\frac{h(z)-h(0)}{z} \tag{24}$$

and this expression is substituted in (22). The integral of the first term is equal to $\pi^{1/2}h(0)$, and the integral of the second, after using the equation $z\Phi(z) = \Phi''(z)$ and double integration by parts, reduces to the original but with replacement of the function h(z) by the function $h_1(z) = \tilde{h}''(z)$. Applying the same procedure to it again, we obtain

$$\int_{-\infty}^\infty dz\,\Phi(z)h(z)=\pi^{1/2}h(0)+\int_{-\infty}^\infty dz\,\Phi(z)h_1(z)$$
$$=\pi^{1/2}\left[h(0)+\sum_{n=1}^\infty h_n(0)\right],$$
$$h_n(z)=\left[\frac{h_{n-1}(z)-h_{n-1}(0)}{z}\right]''. \tag{25}$$

Since $h_n(0) = h^{(3n)}(0)/(3n)!!!$ where, by definition, $(3n)!!! = 3\cdot 6\cdot 9\cdot\ldots 3n$, we finally have

$$F(\chi)=\frac{G^2m^6c}{48\pi^3}\sum_{n=0}^\infty\frac{h^{(3n)}(0)}{(3n)!!!}. \tag{26}$$

This representation is an asymptotic expansion of $F(\chi)$ in a series in powers of χ^2. Actually, as is evident from (23), the function h is a bilinear combination $f + \chi^2 g$ of the functions f and g which depend only on u, and u depends on z and χ only through the combination $z\chi^{2/3}$ (see Eq. (21)). Therefore $h^{(3n)}(0) \sim a_n\chi^{2n} + b_n\chi^{2n+2}$. Evaluation of the derivatives $h^{(3n)}(0)$ reduces to a calculation of the derivatives of h with respect to u at the point $u = u_0 = \mu^{-1}(1-\mu)$ and the derivatives of u with respect to z at the point z = 0. The latter is contained, in particular, in the following expansions of the function u = u(z), which we cite here for reference ($\epsilon = (\chi/u_0)^{2/3}\mu^{-1}$):

$$u = u_0\varphi(\varepsilon z),$$

$$\varphi(x) = \begin{cases} 1 + x + \dfrac{x^2}{3} - \dfrac{x^4}{81} + \dfrac{x^5}{243} - \dfrac{4x^7}{6561} + \dfrac{5x^8}{19683} + \ldots, & x \to 0, \\ x^{3/2} + \dfrac{3}{2} - \dfrac{3}{8x^{3/2}} + \dfrac{1}{2x^3} - \dfrac{105}{128x^{9/2}} + \ldots, & x \to +\infty, \\ -x^{-3} - 3x^{-6} - 12x^{-9} - 55x^{-12} - 273x^{-15} - \ldots, & x \to -\infty. \end{cases} \tag{27}$$

Using the representation (26) and limiting ourselves to a calculation of the first three terms, we obtain the expansion

$$F(\chi) = \frac{G^2m^6c}{192\pi^3}(c_0 + c_2\chi^2 + c_4\chi^4 + \ldots),$$
$$c_0 = 1 - 8\mu + 12\mu^2\ln\mu^{-1} + 8\mu^3 - \mu^4,$$
$$c_2 = {}^8/_9(6\ln\mu^{-1} - 2 - 9\mu^2 + 20\mu^3 - 9\mu^4),$$
$$c_4 = {}^8/_9(1-\mu)(17 - 73\mu - 145\mu^2 + 315\mu^3). \tag{28}$$

The first term $G^2m^6cc_0/192\pi^3$ is the decay probability in vacuum, and the remaining terms are corrections due to the effect of the electromagnetic field. The coefficient $c_2(\mu)$ is positive for all values of μ lying in the interval $0 < \mu < 1$, i.e., the decay rate increases when the field is switched on. In a hypothetical case when μ is close to unity, i.e., for $\Delta = 1 - \mu \to 0$, the ratios $c_2/c_0 \to 40/\Delta^2$, $c_4/c_0 \to 760/3\Delta^4$, ..., so that the expansion parameter for the series (28) is actually $(\chi/\Delta)^2$. This means that the effect of a field on decays is enhanced with a decrease of the mass difference between the initial and final particles. For the actual decay of a muon, $\mu = (207)^{-2}$, i.e., μ is very small, and one can use the limiting values of the coefficients c_n as $\mu \to 0$. In this connection it is necessary to understand that the series (28) is asymptotic with respect to the parameter χ, that is, a summation of its first few terms will approximate the function $F(\chi)$ well only for sufficiently small values of χ; in any case the last term taken must be small in comparison with unity.

We note that as $\mu \to 0$ the coefficient c_2 diverges logarithmically. This is associated with the infrared divergence of $F(\chi)$ with regard to the electron mass. As we shall see below, $F(\chi)$ diverges logarithmically as $\mu \to 0$. The leading coefficients c_n may diverge as $\mu \to 0$, and because the series (28) is asymptotic with respect to the parameter χ but not with respect to μ, and therefore in it not μ but χ must be regarded as a very small parameter.

Since μ is very small for muon decay, it makes sense to consider the asymptotic value of $F(\chi)$ as $\mu \to 0$. For this purpose let us return to formula (22). The function h(z) depends on the parameter μ directly and through the relation between u and z. Therefore, as $\mu \to 0$ it is necessary to establish the form of h(z) in that region of z or u which gives the principal contribution to the integral.

As $\mu \to 0$ the quantity ϵ in (27) tends to infinity. Values of $z \sim -1$ will be effective in the interval $-\infty < z < 0$; therefore as $\mu \to 0$ one can replace u by its asymptotic value as $\epsilon z \to -\infty$, that is, $u \approx -u_0(\epsilon z)^{-3}$. The contribution from the region $-\epsilon^{-1} \lesssim z \le 0$, where this asymptotic value is wrong, tends to zero as $\mu \to 0$: in this region the integrand $\sim a + b\ln(1+u)$, $a \sim b \sim 1$, whereas the length of the region $\sim \epsilon^{-1} \sim \mu^{1/3} \to 0$. In such a case for $z < 0$ one finds

$$h(z) \approx -\frac{v^4}{12} + \frac{v^3}{12} + \frac{v^2}{4}$$
$$+\chi^2\left[15v^6 - 24v^5 - \frac{11}{6}v^4 + \frac{116}{9}v^3 - \frac{2}{3}v^2 - \frac{4}{3}v + \frac{4}{3}\ln(1+u)\right]$$

where $v = u(1+u)^{-1}$, $u = -\chi^{-2}z^{-3}$.

In the interval $0 \le z < \infty$ as $\mu \to 0$ values of $z \sim 1$ will be effective and, therefore, one can replace u by its asymptotic value as $\epsilon z \to +\infty$, that is, $u \approx u_0(\epsilon z)^{3/2}$ (see (27)), after which one goes to the limit $\mu \to 0$. Then for h(z) in the interval we obtain

$$h(z) \approx \frac{1}{4} + \chi^2\left[\frac{1}{6}z^3 + \frac{4}{3}\left(\ln\frac{\chi z^{3/2}}{\mu^{3/2}} - \frac{13}{12}\right)\right].$$

Integration over z with the aid of the formulas

$$\int_0^\infty dz\Phi(z)z^\nu = \frac{3^{2\nu/3}}{2\sqrt{3\pi}}\Gamma\left(\frac{\nu+1}{3}\right)\Gamma\left(\frac{\nu+2}{3}\right), \quad \int_0^\infty dz\Phi(z)\ln z$$
$$= -\frac{2\sqrt{\pi}}{9}\left(C + \frac{1}{2}\ln 3\right), \quad \int_{-\infty}^0 dz\,\Phi(z)\ln(-z) = -\frac{4\sqrt{\pi}}{9}\left(C + \frac{1}{2}\ln 3\right), \tag{29}$$

in which C = 0.577 ... is Euler's constant, now gives

$$F(\chi) = \frac{G^2m^6c}{576\pi^3}\left\{1 + \frac{16}{3}\chi^2\left(\ln\frac{\chi}{\mu^{3/2}} - C - \frac{1}{2}\ln 3 - \frac{5}{6}\right)\right.$$
$$+\frac{12}{\sqrt{\pi}}\int_{-\infty} dz\,\Phi(z)\left[-\frac{v^4}{12} + \frac{v^3}{12} + \frac{v^2}{4} + \chi^2\left(15v^6 - 24v^5\right.\right.$$
$$\left.\left.\left. - \frac{11}{6}v^4 + \frac{116}{9}v^3 - \frac{2}{3}v^2 - \frac{4}{3}v + \frac{4}{3}\ln(1+u)\right)\right]\right\}. \tag{30}$$

In the last integral $v = u(1+u)^{-1}$ and $u = -\chi^{-2}z^{-3}$.

We note that by the same method which led to formula (26), one can obtain

$$\int_{-\infty}^0 dz\,\Phi(z)f(z)$$
$$= \sum_{n=0}\left[\frac{2\sqrt{\pi}}{3}\frac{f^{(3n)}(0)}{(3n)!!!} + \Phi'(0)\frac{f^{(3n+1)}(0)}{(3n+1)!!!} - \Phi(0)\frac{f^{(3n+2)}(0)}{(3n+2)!!!}\right]. \tag{31}$$

Using this formula and the value of the last integral in (29), it is not difficult to find the following expansion for $F(\chi)$:

$$F(\chi) = \frac{G^2m^6c}{192\pi^3}\left[1 + \frac{16}{3}\chi^2\left(\ln\frac{1}{\chi\mu^{1/2}} + C + \frac{1}{2}\ln 3 + \frac{1}{6}\right) + \ldots\right], \tag{32}$$

which is valid for $\mu^{1/2} \ll \chi \ll 1$. On the other hand, for $\chi \gg 1$ it follows from Eq. (30) that

$$F(\chi) = \frac{G^2m^6c}{108\pi^3}\chi^2\left(\ln\frac{\chi}{\mu^{3/2}} - C - \frac{1}{2}\ln 3 - \frac{5}{6}\right). \tag{33}$$

We recall that formula (30) and its special cases (32) and (33) are asymptotic expansions for $\mu \to 0$. Therefore μ is set equal to zero where it is possible to do so, and where it is impossible to do so, μ is assumed to be a very small parameter.

We note that formula (33) is valid for $\chi \gg 1$ not only for $\mu \to 0$ but also for arbitrary values of μ, and one can obtain it directly from the general expression (22).

3. NEUTRINO EMISSION, $e \to e + \nu + \tilde{\nu}$, BY AN ELECTRON IN THE PRESENCE OF AN ELECTROMAGNETIC FIELD

In the special case $\mu = 1$ the different expressions obtained above for the decay of a muon describe neutrino emission by an electron moving in an electromagnetic field. Thus, formulas (7) and (9) for $\mu = 1$ determine the probability for neutrino emission by an electron in the field of a plane, linearly-polarized electromagnetic wave of frequency ω, and formulas (16), (18), (20), and (22) determine the probability in a constant, crossed field or (for ultrarelativistic electrons) in an arbitrary constant field. Neutrino emission by an ultrarelativistic electron in a magnetic field was considered by Baĭer and Katkov,[7] who obtained an expression for the probability in the form of a triple integral, corresponding to our formula (16). Here we shall analyze more compact expressions for the differential and total probabilities of neutrino emission by an electron in a crossed field, which follow from Eqs. (20) and (22) if μ is set equal to unity.

First of all let us write down the differential distribution of the probability with respect to u, which follows from Eq. (20):

$$\frac{dF(\chi,u)}{du}=\frac{G^2m^6c}{48\pi^3\sqrt{\pi}}v^4\left\{\left[\frac{uv}{3}+\frac{1}{2}+4z^{-3}\left(\frac{uv}{3}+1\right)\right]\Phi_1(z)\right.$$
$$\left.+z^{-1}\left(\frac{uv}{3}+\frac{1}{2}\right)\Phi'(z)+z^{-2}\left(\frac{uv}{3}+\frac{5}{2}\right)\Phi(z)\right\},$$
$$z=(u/\chi)^{2/3},\qquad v=u/(1+u). \tag{34}$$

For small values of u this distribution falls off like u^2, but for large values of u it falls off exponentially, in contrast to the corresponding distribution of electromagnetic radiation by an electron in a field (see Eq. (46) in[6]), which increases like $u^{-2/3}$ for small values of u.

From (22) we have the following result for the total probability of neutrino emission:

$$F(\chi)=\frac{G^2m^6c}{48\pi^3\sqrt{\pi}}\int_0^\infty dz\,\Phi(z)h(z),$$
$$h(z)=-\frac{v^4}{12}-\frac{v^3}{6}-\frac{v^2}{2}-\frac{11v}{6}+3\ln(1+u)-\frac{7u}{6}+\frac{u^2}{6}$$
$$+\frac{4}{3}\chi^2\left(\frac{45}{16}v^6-\frac{99}{16}v^5+\frac{109}{32}v^4+\frac{37}{96}v^3-\frac{1}{2}v^2-v+\ln(1+u)\right) \tag{35}$$

Here $v = u(1+u)^{-1}$, $u = \chi z^{3/2}$.

Values of $u \sim \chi$ are effective in the integral appearing in (35). Therefore for small or large values of χ one can simplify the function h, after which one performs the integration over z. Then we obtain

$$F(\chi)=\frac{119G^2m^6c}{288\sqrt{3}\pi^3}\chi^5,\quad \chi\ll 1, \tag{36}$$

$$F(\chi)=\frac{G^2m^6c}{108\pi^3}\chi^2\left(\ln\chi-C-\frac{1}{2}\ln 3-\frac{5}{6}\right),\quad \chi\gg 1. \tag{37}$$

The last expression differs somewhat from the formula found in[7]. The total expansion in powers of χ may be obtained by the method set forth in[1] in connection with an investigation of radiation by an electron. For this purpose in (35) we shall use the integral representations

$$v^n\equiv\left(\frac{u}{1+u}\right)^n=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dk\frac{\Gamma(-k+n)\Gamma(k)}{\Gamma(n)}u^k,$$
$$\ln(1+u)=-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dk\,\Gamma(-k)\Gamma(k)u^k,$$

in which the contour of integration goes around the point k = 0 on the right. Then, after integration over z with the aid of the first formula in (29), we obtain one more integral representation for $F(\chi)$:

$$F(\chi)=-\frac{G^2m^6c\chi^2}{3456\sqrt{3}\pi^4}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dk\frac{(k-1)(k-2)}{k+2}(21k^3+49k^2+56k$$
$$+48)\Gamma(k)\Gamma(-k)\Gamma\left(\frac{k}{2}+\frac{1}{3}\right)\Gamma\left(\frac{k}{2}+\frac{2}{3}\right)(3\chi)^k. \tag{38}$$

For small values of χ, one can close the contour of the integration over k on the right, reducing the integral to a sum of the residues at the poles k = 3, 4, 5, ..., as a consequence of which we obtain an asymptotic series in powers of χ (determined by the same formula (38) if in it one replaces the integral by a sum over k = 3, 4, 5, ..., and $\Gamma(k)\Gamma(-k)/2\pi i \to (-1)^k$), the first term of which is given by (36). The asymptotic nature of this series is obvious from the fact that its coefficients are proportional to the product of two Γ functions of positive argument. Thus, the probability $F(\chi)$ is nonanalytic at the point $\chi = 0$.

For large values of χ one can close the contour of integration on the left, reducing the integral to a sum of residues at the poles $k = -n, -2n-(2/3), -2n-(4/3)$, where n = 0, 1, 2, ... In this connection it should be kept in mind that the poles k = 0, −2 are of second order, and therefore the corresponding residues contain $\ln\chi$. For $\chi \gg 1$ the principal contribution is determined by the residue at the pole k = 0, given in (37). We note that the resulting series converges well since its coefficients are inversely proportional to the product of two Γ functions of positive argument.

4. PION DECAYS: $\pi \to \mu + \nu$, $\pi \to e + \nu$

The probabilities for these decays in the field of a plane wave and in a constant, crossed field were obtained in article[2]. Here we consider more compact expressions for the differential and total probabilities for the decay of a pion in a crossed field, and we shall correct a mistake made in the asymptotic formula (9) of article[2].

The probability for pion decay in a crossed field has the following form (compare with Eq. (8) in[2]):

$$F(\chi)=\frac{G^2f^2m^2m'^2c}{2\pi^2}\int_0^\infty\frac{du}{(1+u)^2}\int_0^\infty d\tau\left(\frac{u}{2\chi}\right)^{1/3}\left\{\Delta\Phi^2(y)\right.$$
$$\left.+u\left(\frac{2\chi}{u}\right)^{2/3}[y\Phi^2(y)+\Phi'^2(y)]\right\}, \tag{39}$$

where

$$y=\left(\frac{u}{2\chi}\right)^{2/3}\left[1+\tau^2-\Delta\frac{1+u}{u}\right],$$

u and τ are the same physical variables as in Section 2, and the integrand is the differential distribution of the decay probability with respect to u and τ. Integrating with respect to τ by the method described in Sec. 4 of[6], we obtain the following expression for the probability distribution in u:

$$\frac{dF(\chi,u)}{du}=\frac{G^2f^2m^2m'^2c}{8\pi\sqrt{\pi}}\frac{1}{(1+u)^2}\left[\Delta\Phi_1(z)-2u\left(\frac{\chi}{u}\right)^{2/3}\Phi'(z)\right],$$
$$z=\left(\frac{u}{\chi}\right)^{2/3}\left(1-\Delta\frac{1+u}{u}\right). \tag{40}$$

For small values of u this distribution tends to a constant value, undergoing damped oscillations with infinitely-increasing frequency, but for large values of u it falls off exponentially.

If (40) is integrated by parts with respect to u, then we obtain a compact integral representation for the total decay probability:

$$F(\chi)=\frac{G^2f^2m^2m'^2c}{8\pi\sqrt{\pi}}\int_{-\infty}^{\infty}dz\,\Phi(z)h(z),$$
$$h(z)=\Delta\frac{u}{1+u}+6\chi^2\left[\frac{u/(1+u)}{2\mu u+1-\mu}\right]^2\left[2-6\frac{u}{1+u}+\frac{3(1-\mu)}{2\mu u+1-\mu}\right]. \tag{41}$$

The parameter z is related to u in the same way as in (40) or in (22). Therefore, one can obtain an expansion in powers of χ^2 if a formula of type (26) is used. Evaluation at the point z = 0 of the zero, third, and sixth derivatives of the function h with respect to z gives the following expansion, correct to terms of order $\sim\chi^4$:

$$F(\chi)=\frac{G^2f^2m^2m'^2\Delta^2c}{8\pi}\cdot$$
$$\times\left[1+\frac{6-16\Delta+6\Delta^2}{3\Delta^2}\chi^2+\frac{8(18-40\Delta+15\Delta^2)}{3\Delta^2}\chi^4+\ldots\right]. \tag{42}$$

This expansion differs from the corresponding formula (9) of article[2] by the additional term proportional to χ^4 and by the absence of the term proportional to $\chi^{5/3}$. The error which led to the appearance of such a term in[2] was associated with the fact that the contribution of the second "singular" term (A.4) in[2] was not taken into account; this term, which is equal to $f''(0)(3/4)^{2/3}\Gamma(2/3)(2\chi)^{4/3}/24$, must be added to Eqs. (A.5) and (A.6). Now, when the simpler expression (41) is obtained for $F(\chi)$, from the general form of the dependence $u = u(\chi z^{3/2})$ and from formula (26) it is immediately seen that $F(\chi)$ is expansible for small values of χ only in powers of χ^2, and no terms with fractional powers of χ should appear.

The coefficient of the χ^2 term, equal to $c_2 = (6-16\Delta+6\Delta^2)/3\Delta^2$, characterizes the sensitivity of the decay probability to the field. During a change of Δ from zero to unity, $c_2(\Delta)$ decreases from $+\infty$ to $-4/3$, tending to zero for $\Delta = (4-\sqrt{7})/3 \approx 0.451$ and reaching a minimum equal to $-14/9$ for $\Delta = 3/4$. Since $\Delta = 0.427$ and $\Delta = 1$, respectively, for the decays $\pi \to \mu + \nu$ and $\pi \to e + \nu$, switching-on an electromagnetic field accelerates decay into a muon and retards decay into an electron. This is confirmed by the behavior of the corresponding curves for $F(\chi)$ shown in Fig. 2 of[2], which were found numerically.

5. PION DECAY: $\pi^\pm \to \pi^0 + e^\pm + \nu$

The matrix element for the decay $\pi^\pm \to \pi^0 e^\pm \nu$ is written in the usual form

$$M=\frac{1}{\sqrt{2}}G\int J_\alpha{}^\pi J_\alpha{}^{e^+}d^4x,$$

where J_α^π and J_α^e are the pion and electron charged currents in the field of an electromagnetic wave:

$$\begin{gathered}J_\alpha{}^\pi=2^{1/2}[\pi_+{}^*(\nabla-ieA)_\alpha\pi_0-(\nabla-ieA)_\alpha{}^*\pi_+{}^*\pi_0\\ +\pi_0{}^*(\nabla-ieA)_\alpha\pi_--(\nabla-ieA)_\alpha{}^*\pi_0{}^*\pi_-],\\ J_\alpha{}^e=\bar{\nu}\gamma_\alpha(1+\gamma_5)e.\end{gathered} \tag{43}$$

The wave functions in the field of the wave are used for the charged particles. An evaluation of the matrix element M and its square is carried out just as in the usual theory with plane waves, where the relation $(s-2\beta)A_0 - \alpha A_1 + 4\beta A_2 = 0$ turns out to be useful. As a result of the cumbersome calculations we obtain

$$\sum_{\text{polarizations}}\frac{|M|^2}{VT}=\sum_{s=-\infty}^{\infty}\frac{G^2}{2q_0q_0'l_{10}l_{20}}\Big\{[8(ll_2)(l_1p')-4m'^2(ll_2)$$
$$+(4m_1^2-m'^2)(l_2p')]A_0^2+[(8ll_2-4m_1^2+m'^2)[(kl_1)(ap')-(kp')(al_1)]$$
$$-(4m_1^2-m'^2)[(kp')(ap)-(kp)(ap')]]\frac{eA_0A_1}{kp'}$$
$$-[8(ll_2)(kl_1)+(4m_1^2-m'^2)kl_2]\frac{e^2a^2A_1^2}{2kp'}\Big\}(2\pi)^4\delta(sk+q-q'-l_1-l_2). \tag{44}$$

Here p, p′ are momenta; q, q′ are quasimomenta; m, m′ are the masses of the $\pi^\pm$ meson and $e^\pm$ electron; l_1 and l_2 denote the momenta of the π^0 meson and neutrino; $l = l_1 + l_2 = sk + q - q'$; m_1 is the mass of the π^0 meson; the remaining notation is the same as in Sec. 2. Let us integrate this expression over $d^3l_1d^3l_2(2\pi)^{-6}$ with the aid of formulas (5a)–(5c). Then for the differential decay probability we obtain

$$dW=\int\sum\frac{|M|^2}{VT}\frac{d^3l_1d^3l_2d^3q'}{(2\pi)^9}=\sum_{s>s_0}\frac{G^2n}{16\pi q_0q_0'}\frac{(l^2+m_1^2)^2}{l^4}$$
$$\times\Big\{[-4l^4-l^2(4m^2+3m'^2)+m'^2(m^2-m'^2)]A_0^2$$
$$-e^2a^2\frac{(4l^2-m'^2)kl}{kp'}(A_1^2-A_0A_2)\Big\}\frac{d^3q'}{(2\pi)^3}, \tag{45}$$

where $s_0 = [m_*^2 - (m'_* + m_1)^2]/2kq$ is the minimal possible value for s, and n is the average number density of $\pi^\pm$ mesons.

For $x \ll 1$ one can expand the functions A_n in powers of the parameter x, just as in Section 2. Then the term s = 0 gives the probability for the free decay of a pion (see[8]); the term s = ±1 gives the probability for pion decay with the absorption or emission of one photon (see[9]) and so forth. For $x \gg 1$, having utilized the general formalism set forth in the Section on muon decay, we obtain for the decay probability, integrated over $d^3q'(2\pi)^{-3}$, expression (15) in which $F(\chi)$—the decay probability in a crossed field—is given by

$$F(\chi)=\frac{G^2m^6c}{64\pi^4}\int_0^\infty\frac{du}{(1+u)^2}\int_{\lambda_1}^\infty d\lambda\int_{-\infty}^\infty d\tau\left(\frac{u}{2\chi}\right)^{1/3}\frac{(\lambda-\lambda_1)^2}{\lambda^2}\Big\{[\mu(1-\mu)$$
$$+(4+3\mu)\lambda-4\lambda^2]\Phi^2(y)+u\left(\frac{2\chi}{u}\right)^{2/3}(4\lambda+\mu)(y\Phi^2+\Phi'^2)\Big\}, \tag{46}$$

where y is the same as in (17), and $\mu = m'^2/m^2$. An important point here (in contrast to the case of muon decay) is the fact that the lower limit of integration with respect to λ (i.e., with respect to the square of the invariant mass of the neutral particles) is not equal to zero but is equal to $\lambda_1 = m_1^2/m^2$.

Integrating over τ by the method of article [6], we obtain

$$F(\chi) = \frac{G^2m^6c}{128\pi^3\sqrt{\pi}} \int_0^\infty \frac{du}{(1+u)^2} \int_{\lambda_1}^\infty d\lambda \frac{(\lambda-\lambda_1)^2}{\lambda^2} \cdot \times \left\{ [\mu(1-\mu)+(4+3\mu)\lambda - 4\lambda^2]\Phi_1(t) - 2u\left(\frac{\chi}{u}\right)^{2/3}(4\lambda+\mu)\Phi'(t) \right\}, \quad (47)$$

where t is the same as in (19). The integrand of this expression is the distribution of the decay probability with respect to u and λ.

Unfortunately, because $\lambda_1 \neq 0$ the integral with respect to λ cannot be expressed in terms of tabulated functions, and the distribution with respect to u has the form of a single integral which one can bring to the form

$$\frac{dF(\chi,u)}{du} = \frac{G^2m^6c}{128\pi^3\sqrt{\pi}}(1+u)^{-2}\int_{z(u)}^\infty dt \left\{ \left[-\frac{4}{3}(\lambda-\lambda_1)^3 + (4+3\mu)\left(\frac{(\lambda-\lambda_1)^2}{2} - \lambda_1(\lambda-\lambda_1) + \lambda_1^2 \ln\frac{\lambda}{\lambda_1}\right) + \mu(1-\mu)\left(\lambda - \frac{\lambda_1^2}{\lambda} - 2\lambda_1\ln\frac{\lambda}{\lambda_1}\right)\right]\Phi(t) - 2\left(\frac{\chi}{u}\right)^{1/3}\frac{u^3}{1+u}\frac{(\lambda-\lambda_1)^2}{\lambda^2}(4\lambda+\mu)\Phi'(t)\right\}. \quad (48)$$

Here t and λ are related to each other by the linear relation (19), and z(u) is the minimal value of t as a function of λ for fixed u:

$$z(u) = t(u,\lambda_1) = \left(\frac{u}{\chi}\right)^{2/3}\left(1 - \Delta\frac{1+u}{u} + \lambda_1\frac{1+u}{u^2}\right). \quad (49)$$

In order to obtain an integral representation for $F(\chi)$, let us integrate the distribution (48) with respect to u, and let us change the order of integration over u and t. Then, performing an additional integration by parts in the second term of (48), we obtain

$$F(\chi) = \frac{G^2m^6c}{128\pi^3\sqrt{\pi}}\int_{z_0}^\infty dz\,\Phi(z)h(z), \quad (50)$$

where h(z) = f(z) + g(z), and also

$$f(z) = \int_{u_1(z)}^{u_2(z)} \frac{du}{(1+u)^2}\left\{ -\frac{4}{3}\lambda_1^3\xi^3 + \lambda_1^2(4+3\mu)\left[\frac{1}{2}\xi^2 - \xi + \ln(1+\xi)\right] + \lambda_1\mu(1-\mu)\left[1+\xi - \frac{1}{1+\xi} - 2\ln(1+\xi)\right]\right\},$$

$$g(z) = 4\chi^2\int_{u_1(z)}^{u_2(z)} du\frac{u^3}{(1+u)^4}\frac{\xi}{(1+\xi)^3}\left[4+\frac{\mu}{\lambda_1}+6\xi+2\xi^2\right],$$

$$\xi = \left(\frac{\chi}{u}\right)^{2/3}\frac{u^2}{1+u}\frac{z-z(u)}{\lambda_1} \quad (51)$$

Here $u_{1,2}(z)$ are the lower and upper branches of the function which is the inverse to the function z = z(u). The function z(u), which plays an important role, tends to $+\infty$ at the ends of the interval $0 < u < \infty$ like $\lambda_1\chi^{-2/3}u^{-4/3}$ and $\mu(u/\chi)^{2/3}$, respectively, and inside this interval at the point

$$u = u_0 = \frac{4}{\delta(1+K)}, \quad K = \frac{\delta}{|\delta|}(1+8\gamma^2)^{1/2},$$

$$\delta = \frac{1-\mu-\lambda_1}{2\lambda_1}, \quad \gamma = \frac{2\sqrt{\lambda_1\mu}}{1-\mu-\lambda_1} \quad (52)$$

it has a unique minimum

$$z_0 = z(u_0) = -\frac{3\lambda_1(3-K)(1+K)^{1/3}}{4}\left(\frac{\delta^2}{2\chi}\right)^{2/3}, \quad (53)$$

which also appears in the integral representation (50) as the lower limit of integration. For decays, i.e., reactions which occur even in the absence of a field, $m > m_1 + m'$, from here it follows that $\delta > 0$, $0 < \gamma < 1$ and the value of z_0 is negative. In this case the function z(u) has a zero at the points u_1, u_2,

$$u_{1,2} = 1/\delta(1 \pm \sqrt{1-\gamma^2}). \quad (54)$$

For reactions which do not occur in the absence of a field, $m < m_1 + m'$, whence it follows that $\delta > 0$, $\gamma > 1$ or $\delta < 0$, $\gamma < 0$ and z_0 is positive. Thus, the sign of the minimum value of z(u) coincides with the sign of the mass difference between the final and initial systems (we shall dwell on this in more detail in the conclusion).

In connection with an investigation of the integral (50), the behavior of the function h(z) is essential, and consequently so is the behavior of the functions $u_{1,2}(z)$ near the characteristic points $z = z_0$, $z = 0$. Therefore we present an expansion of the functions $u_{1,2}(z)$ close to these points:

$$u_{1,2}(z) = u_0\left\{1 \mp x + \frac{7}{9}\left(1-\frac{3}{7}\alpha\right)x^2 \pm \frac{35}{81}\left(1+\frac{33}{20}\alpha - \frac{9}{14}\alpha^2\right)x^3 + \ldots\right\},$$

$$\text{where } x = \frac{3}{2}\left[\frac{(3-K)(z-z_0)}{K(-z_0)}\right]^{1/2},$$

$$u_{1,2}(z) = u_{1,2}\left\{1 + x + \left(\frac{5}{6} \pm \frac{1}{2\sqrt{1-\gamma^2}}\right)x^2 + 2\left(\frac{1}{2} \pm \frac{1}{2\sqrt{1-\gamma^2}}\right)^2 x^3 + \ldots\right\},$$

$$\text{where } x = \mp\frac{(\chi\delta^{-2})^{2/3}z}{2\lambda_1\sqrt{1-\gamma^2}(1\pm\sqrt{1-\gamma^2})^{1/3}} \quad (55)$$

Here $\alpha = (K-1)/2K$. If $u_{1,2}(z)$ are regarded as functions of $x \sim \sqrt{z - z_0}$, then $u_1(x) = u_2(-x)$. Therefore from (51) it follows that the functions f, g, h are odd functions of this x, which can be expanded in the series

$$f(z) = \sum_{k=7}^\infty f_k\left(\frac{z-z_0}{-z_0}\right)^{k/2}, \quad g(z) = \sum_{k=3}^\infty g_k\left(\frac{z-z_0}{-z_0}\right)^{k/2}, \quad (56)$$

where k is odd, and where the coefficients f_k do not depend on χ, but $g_k \sim \chi^2$.

One can obtain an asymptotic expansion of the decay probability $F(\chi)$ for small values of χ by a method analogous to the one which led to formula (26). Corresponding to the result, utilizing the fact that $z_0 < 0$, will be the formula[2)]

$$\frac{1}{\sqrt{\pi}}\int_{z_0}^\infty dz\,\Phi(z)h(z) = \sum_{n=0}^\infty\left\{\frac{h^{(3n)}(0)}{(3n)!!!} - \left[\frac{t\Phi^2(t)+\Phi'^2(t)}{(-t)^{1/2}} + \frac{3\varphi_{5/2}(z_0)}{4z_0^3}\right]h_{n,-1} + \left[\frac{\Phi(t)\Phi'(t)}{(-z_0)^{3/2}} + \frac{\varphi_{7/2}(z_0)}{4z_0^3}\right](h_{n,-1}+h_{n,1})\right\}, \quad (57)$$

where $t = 2^{-2/3}z_0$,

$$\varphi_s(z_0) = \sum_{k=0}^\infty \frac{\Gamma(s+3k)}{\Gamma(s)\,(3k)!!!\,z_0^{3k}},$$

[2)] The integral representation [10] $\int_0^\infty \frac{dt}{\sqrt{t}}\Phi(t+z_0) = 2^{2/3}\pi^{1/2}\Phi^2(2^{-2/3}z_0)$ is used for the derivation.

and the coefficients $h_{n,\mp 1}$ are related to the coefficients h_k of the function h(z) by the relation

$$h_{n,2m-1}=\frac{(m+{}^1/_2)(m+{}^3/_2)}{z_0{}^{3n}}\sum_{k=0}^{m+2n}R_{2k-1}(n,m)h_{2k-1},$$
$$R_{2k-1}(n,m)=\sum_{m_{n-1}=k_{n-1}}^{m+2}(m_{n-1}+{}^1/_2)(m_{n-1}+{}^3/_2)\ldots$$
$$\ldots\sum_{m_2=k_2}^{m_3+2}(m_2+{}^1/_2)(m_2+{}^3/_2)\sum_{m_1=k_1}^{m_2+2}(m_1+{}^1/_2)(m_1+{}^3/_2),$$
$$k_i=\begin{cases}0, & 0\leqslant k\leqslant 2i\\ k-2i, & k\geqslant 2i\end{cases} \tag{58}$$

The asymptotic expansion (57) turns out to be much more complicated than its special case (26), which corresponds to $z_0=-\infty$. Now, besides the terms involving 3n-th derivatives which, as is evident from the expansions of $u_{1,2}(z)$ near z = 0, are proportional to $(\chi\delta^{-2})^{2n}$, there are still specific terms which contain the Airy function and its derivative. These terms depend on χ only through z_0 and moreover, as is evident from (58), the coefficients $h_{n,\mp 1}\sim z_0^{-3n}\sim(\chi\delta^{-2})^{2n}$, and the known functions of z_0 inside the square brackets (57) behave in the following way as $\chi\to 0$ or $z_0\to-\infty$:

$$\frac{t\Phi^2+\Phi'^2}{(-t)^{1/2}}+\frac{3\varphi^{5/2}(z_0)}{4z_0^3}\to-\sin\left[\frac{2}{3}(-z_0)^{3/2}\right],$$
$$\frac{\Phi\Phi'}{(-z_0)^{3/2}}+\frac{\varphi^{3/2}(z_0)}{4z_0^3}\to-\frac{1}{2(-z_0)^{3/2}}\cos\left[\frac{2}{3}(-z_0)^{3/2}\right], \tag{59}$$

i.e., they oscillate with an infinitely-increasing frequency. Although, as we shall see below, these terms enter into the probability with coefficients $\infty(\chi\delta^{-2})^4$, i.e., the amplitude of the oscillations will be damped as $\chi\delta^{-2}\to 0$, nevertheless one can now assert that the probability $F(\chi)$ is not only nonanalytic at the point $\chi=0$ but it has an essential singularity there.

Another property which is important from a practical point of view is the result that the expansion parameter is not χ but $\chi\delta^{-2}$. Therefore, for $\delta\ll 1$ (for pion decay $\delta=0.034$) the effect of the field on the decay is to significantly increase the decay probability. For $\chi\delta^{-2}\sim 1$ the decay probability changes by a factor of two or three times in comparison with the probability for free decay. One can assert that the characteristic field in the rest system of a $\pi^\pm$ meson is not $B_0=m^2/e$ but $\widetilde{B}_0=(m\delta)^2/e=10^{15}$ Oe. In connection with the motion of a pion through a field $\sim 10^7$ Oe, a field $\widetilde{B}_0$ arises in its rest system for an energy $\sim 10^7$ GeV.

Going on to a concrete evaluation of the quantities entering into the expansion (57), we note that the next, nonvanishing coefficients of the oscillating terms will be $f_{2,\mp 1}$ and $g_{1,\mp 1}$, which are proportional to $(\chi\delta^{-2})^4$. Therefore, carrying out the calculation with this degree of accuracy, it is still necessary for us to evaluate f(0), $f^{(3)}(0)$, $f^{(6)}(0)$ and g(0), $g^{(3)}(0)$. Although the integrals in terms of which these quantities are expressed can be evaluated exactly, the results obtained are cumbersome. Therefore, let us take advantage of the smallness of the parameter δ and write down all coefficients to lowest order in this parameter:

$$F(\chi)=\frac{G^2m^6c}{128\pi^3}\delta^5\left\{c_0+c_2\left(\frac{\chi}{\delta^2}\right)^2+c_4\left(\frac{\chi}{\delta^2}\right)^4+\ldots-\left[\frac{t\Phi^2+\Phi'^2}{(-t)^{1/2}}+\frac{3\varphi^{5/2}(z_0)}{4z_0^3}\right](a_4z_0^{-6}+\ldots)+\left[\frac{\Phi\Phi'}{(-z_0)^{3/2}}+\frac{\varphi^{3/2}(z_0)}{4z_0^3}\right](b_4z_0^{-6}+\ldots)\right\},$$
$$c_0=\frac{32}{15}\left[(2-9\gamma^2-8\gamma^4)(1-\gamma^2)^{1/2}+15\gamma^4\operatorname{Arch}\frac{1}{\gamma}\right],$$
$$c_2=\frac{128}{3}\left[\operatorname{Arch}\frac{1}{\gamma}-\frac{3}{4}(1-\gamma^2)^{1/2}\right],$$
$$c_4=-\frac{4}{3}(1-\gamma^2)^{-3/2}\left[1-\frac{\gamma^2}{2(1-\gamma^2)}+\frac{8\delta}{\gamma^2}\right],$$
$$a_4=-\frac{81}{128}(3-K)^{7/2}(1+K)K^{-1/2}(5-3K),$$
$$b_4=\frac{243}{32}(3-K)^{7/2}(1+K)K^{-1/2}\left\{5\left[1+\frac{9}{20}\frac{3-K}{K}\left(\varkappa_3-\varkappa_2+\frac{1-K}{3}\right)\right]-\frac{3}{2}(3-K)\left[1+\frac{9}{8}\frac{3-K}{K}\left(\varkappa_3-\frac{2}{3}\varkappa_2+\frac{10}{27}\right)\right]\right\}. \tag{60}$$

In the last expression the κ_i are the coefficients of an expansion of the function $u_2(z)/u_0$ in powers of $x={}^3/_2[(3-K)(z-z_0)/K(-z_0)]^{1/2}$ (see formula (55). The subscripts on the coefficients a, b, and c correspond to the powers of the parameter $\chi\delta^{-2}$. If we confine our attention to only those terms which are written down here, then one can replace the oscillating functions inside the square brackets by their lowest approximation according to (59).

The first term in (60) represents the decay probability in vacuum. The second term $\infty c_2(\chi\delta^{-2})^2$ is the main correction to the decay probability due to the effect of the field. It is positive for all physical values of γ, corresponding to a decay. This means that the decay probability increases when the field is switched on.

For the decay of a pion the parameter γ is small: $\gamma=0.11$, and the temptation arises to set it equal to zero everywhere. However, this cannot be done since the probability $F(\chi)$ has an infrared (logarithmic) divergence with respect to the electron mass m′, and consequently with respect to the parameter γ, which is proportional to m′. In the expansion (60) this is expressed by the fact that as $\gamma\to 0$ the coefficient c_2 diverges logarithmically, c_4 diverges quadratically, etc. Although the terms in the coefficients c_4, c_6, ... which diverge more strongly than logarithmically as $\gamma\to 0$ appear in a higher order of smallness with respect to the parameter δ, one should retain them if γ is small. This is also done in the coefficient c_4. The appearance in the expansion (60) of terms which diverge more strongly than logarithmically as $\gamma\to 0$, i.e., more strongly than the probability $F(\chi)$ itself diverges, is associated with the fact that this expansion is asymptotic in the parameter $\chi\delta^{-2}$, but not with respect to the parameter γ.

In conclusion we note that if for the decay not only δ but also $\epsilon=(1-\gamma^2)^{1/2}$ is small, then from expansions of the coefficients c_n in (60) in powers of ϵ and from formula (53) for z_0 it is obvious that in this case the expansion parameter for the decay probability in the presence of a field is $\chi\delta^{-2}\epsilon^{-3}$.

6. GENERAL REMARKS AND DISCUSSION

In this article all particles were assumed to be elementary. In reality, for pions it would be necessary to introduce the appropriate form factors into the interaction (see, for example,[11]); however, their influence would be essential only for $\chi\sim 1$ whereas for the decay $\pi^\pm\to\pi^0e^\pm\nu$, for example, values of $\chi\sim\delta^2\ll 1$ are essential.

If it is assumed that decay corresponds to a penetration of particles through a potential barrier, then one might think that switching-on the electromagnetic field will always increase the decay probability. In the majority of cases under consideration this is actually so, except the decay $\pi \to e\nu$, whose probability decreases when the field is switched on. A decrease of the decay probability associated with the switching-on of the field is observed even in the simplest model for decay of a scalar particle into two scalar particles, which is considered in[12], formula (39). If in this model, for simplicity, the mass of the neutral particle is set equal to zero, then it is not difficult to show that for small values of χ the probability $F(\chi) = (f^2 c\Delta/16\pi)[1 - (4 - 6\Delta)\chi^2/3\Delta + \ldots]$, that is, it is decreased if $\Delta < 2/3$ and it is increased if $\Delta > 2/3$. Thus, it should be assumed that not only the magnitude but also the sign of the correction associated with the external field depend on the form of the interaction between the particles, which is responsible for the decay (i.e., depend on the function h).

The investigation presented here, together with the results of article[6], indicate that the total probability of the decay process in the presence of a field may be represented in the form[3)]

$$F(\chi) = A \int_{z_0}^{\infty} dz\, \Phi(z) h(z) \quad \text{or} \quad F(\chi) = -A \int_{z_0}^{\infty} dz\, \Phi'(z) g(z), \tag{61}$$

where the functions h and g are determined by the interaction between particles and depend on z only through the physical variable u, which is related to z by Eq. (49). The Airy function (or its derivative) describes the motion of charged particles in an external field. The variable z, on which it depends, has a definite physical meaning, to the elucidation of which we now pass.

For processes in a variable field, it follows from the law of conservation, $sk + q = q' + l$, that the number s of photons absorbed from the field is proportional to the square of the "effective" mass transfer of the system, $-(q' + l)^2 + q^2$:

$$s = \frac{(q'+l)^2 - q^2}{2kq} = \frac{x}{2\chi} Q_*;$$
$$Q_* \equiv -\frac{(q'+l)^2 - q^2}{m^2} =$$
$$= \frac{ux^2}{2} + ux^2\rho^2 + u\left(1 + \tau^2 - \Delta\frac{1+u}{u} + \lambda\frac{1+u}{u^2}\right). \tag{62}$$

The relation between the square of the effective mass transfer Q_* and the physical variables ρ, τ, λ, and u is written on the basis of formula (13) for s. The term $ux^2/2$ on the right-hand side is related to the difference between the effective masses and the ordinary masses. Actually, for the square of the mass transfer (which is obtained by replacement of the quasimomenta by the momenta) we have

$$Q(\rho, \tau, \lambda, u) \equiv -\frac{(p'+l)^2 - p^2}{m^2}$$
$$= ux^2\rho^2 + u\left(1 + \tau^2 - \Delta\frac{1+u}{u} + \lambda\frac{1+u}{u^2}\right). \tag{63}$$

3)We note that the first representation (61) may always be transformed into the second representation by an integration by parts, but not vice versa if $g(z_0) = \infty$, as holds for the emission of an electron or for the creation of a pair of protons in a field. [6]

Owing to the absorption (or emission) of momentum and energy by the system from the field, the square of the mass transfer Q may vary within the limits $Q_0 \equiv [(m_0 + m')^2 - m^2]/m^2 \leq Q < \infty$. Since the differential probability in the presence of a constant field does not depend on ρ, it may only depend on the minimal value of the square of the mass transfer as a function of ρ, i.e., on the second term in (63), which we denote by $Q(\tau, \lambda, u)$. It is precisely this quantity which is proportional to the argument y of the Airy functions which enter into the differential probabilities for the decay processes (see Eqs. (16), (39), and (46)). The distributions with respect to λ, u, or u depend on the variables t or z, which are also proportional to the corresponding squares of the mass transfers: $t = (\chi^2 u)^{-1/3} Q(\lambda, u)$, $z = (\chi^2 u)^{-1/3} Q(u)$. Finally, z_0 in the representation (61), defined by formula (53), is proportional to the absolute minimum Q_0 of the square of the mass transfer.

Thanks to the exclusive analytic properties of the Airy function, all probabilities have a unique form both for processes with the evolution of energy (decays with Q_0, $z_0 < 0$) and for processes with the absorption of energy (Q_0, $z_0 > 0$), and by analytic continuation with respect to the particles' masses one can pass from one decay process to another.

From the representation (61) and from the relation $z_0 \sim \chi^{-2/3} Q_0$ it follows that the point $\chi = 0$ is an essential singularity for the probability $F(\chi)$. In fact, if the function h(z) is replaced by unity, then $F(\chi)$ will be proportional to the function

$$\Phi_1(z_0) = \int_{z_0}^{\infty} dz\, \Phi(z) = \begin{cases} 2^{-1} z_0^{-3/4} \exp(-2 z_0^{3/2}/3), & z_0 \to +\infty, \\ \pi^{1/2} - (-z_0)^{-3/4} \cos\left(\tfrac{2}{3}(-z_0)^{3/2} + \pi/4\right), & z_0 \to -\infty, \end{cases}$$

which has an essential singularity at the point $z_0 = \pm\infty$ (or $\chi = 0$). The difference between h(z) and unity does not change this qualitative picture.

The theory under consideration also possesses the property of crossing symmetry. For example, by replacing the momenta $q \to -q$, $l \to -l$ in the squares of the matrix elements, one can pass to the process of creation of two charged particles by a neutral particle. One can represent the probabilities of such processes in the form (61) with the same physical meaning of z and z_0, and so forth.

[1] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. 46, 776 (1964) [Sov. Phys.-JETP 19, 529 (1964)].

[2] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. 46, 1768 (1964) [Sov. Phys.-JETP 19, 1191 (1964)].

[3] A. Lenard, Phys. Rev. 90, 968 (1953).

[4] T. Kinoshita and A. Sirlin, Phys. Rev. 108, 844 (1957). R. E. Behrends, R. J. Finkelstein, and A. Sirlin, Phys. Rev. 101, 866 (1956).

[5] S. M. Berman, Phys. Rev. 112, 267 (1958). V. P. Kuznetsov, Zh. Eksp. Teor. Fiz. 37, 1102 (1959) [Sov. Phys.-JETP 10, 784 (1960)]. T. Kinoshita and A. Sirlin, Phys. Rev. 113, 1652 (1959).

[6] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. 52, 1707 (1967) [Sov. Phys.-JETP 25, 1135 (1967)].

[7] V. N. Baĭer and V. M. Katkov, Dokl. Akad. Nauk SSSR 171, 313 (1966) [Sov. Phys.-Doklady 11, 947 (1967)].

[8] Ya. B. Zel'dovich, Dokl. Akad. Nauk SSSR 97, 421 (1954). E. Feenberg and H. Primakoff, Phil. Mag. 3, 328

(1958).
[9] G. Da Prato and G. Putzolu, Nuovo Cimento **21**, 541 (1961).
[10] D. E. Aspnes, Phys. Rev. **147**, 554 (1966).
[11] L. B. Okun', Slaboe vzaimodeĭstvie élementarnykh chastits (Weak Interactions of Elementary Particles), Fizmatgiz, 1963 (English Transl., Israel Program for Scientific Translations, Jerusalem, 1965). K. Nishijima, Fundamental Particles, W. A. Benjamin, Inc., 1964 (Russ. Transl., Mir, 1965).
[12] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **50**, 255 (1966) [Sov. Phys.-JETP **23**, 168 (1966)].

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