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NATURAL GAUGE OF A TETRAD FIELD AND GRAVITATIONAL ENERGY

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The tetrad or γ-matrix gauge is uniquely found on the basis of the method of observation by means of light rays (isotropic geodesics). The gauge is used to calculate the gravitational energy density near the observer in Riemannian normal coordinates. This method excludes any paradoxes of the Bauer type. Averaging over the observation angles yields for arbitrary fields a non-negative value of the gravitational field energy density. The density is zero on the world line of the observer. The case of weak plane gravitational waves is analyzed as an example. A short review of the problem of localizability of gravitational energy is presented.

THE tetrad (orthogonal reference) formalism was first used in relativity theory in the late 20's[1] for description of the fermion field in Riemann space, where this formalism (or the isomorphic generalized Sommerfeld formalism of Dirac matrices) is indispensable. The next fundamental application of tetrads was a detailed analysis of the problem of localizability of the gravitational energy[2]. These two cases, however, are qualitatively different, since in the general covariant Dirac equation the tetrads actually carry no additional information whatever compared with the metric tensor (owing to the invariance of this equation against local tetrad rotations or, in the matrix formalism, against similarity transformations), in contrast to the fact that the expression for the gravitational energy depends essentially on the indicated transformations. In view of this, Møller[3] proposed to subject the tetrad gauge to certain conditions, to which, however, he could not ascribe a deep physical meaning.

By choosing the tetrad gauge it is possible, even in the case of an empty flat space, to "obtain" a nonzero and furthermore diverging gravitational energy, in perfect analogy with the behavior of the Einstein pseudotensor on transforming to spherical coordinates[4]. In his well known theorem, Einstein[5] has shown, in connection with this and with a criticism on the part of Schrödinger[6], that in the analysis of the gravitational energy it is necessary to use asymptotic Cartesian coordinates on a spatial infinity, for island systems (models). The energy of the physical system is considered here either as a whole (integral energy) or, when dealing with the distribution of this energy, it is taken from the point of view of a remote observer (inertial or located in a practically flat space). The same pertains to the pseudotensor of Landau and Lifshitz[7] and Fock[8]. Attempts to obtain a localizable representation of the gravitational energy[2,3,9,10,11] 1) were not successful, and therefore the point of view that the gravitational energy is in principle not localizable (or does not exist at all[14]) gained support. These conclusions are frequently connected with the inhomogeneity of space-time, due to the curvature, in view of which, generally speaking, there may be no Killing vectors characterizing the mobility of space. This, however, concerns only the integral energy-momentum conservation laws, since in the small one can always go over to a tangential plane space; indeed, Noeter's theorem gives differential conservation laws also in the case of gravitation[15,19]. The transition to correct integral laws is impossible in such an approach, in view of the appearance of another aspect of inhomogeneity of space-time, namely the absence of the operation of "covariant integration" in the traditional formalism.

We consider in this article the gravitational energy from the point of view of an observer moving in the immediate vicinity of gravitating bodies. Unlike Møller, we approach the tetrad-field gauge not as the introduction of new physical field components (Møller hoped to obtain in this manner a new unified theory), but as the choice of a reference frame. In the curved space-time, generally speaking, there are no "rigid" reference frames; moreover, it is advantageous here to start not from a reference frame that is extended in space, but only from the observer's word line, with respect to which the physical system is ascribed. When speaking, for example, of the velocity of a body seen by him, the observer actually executes a parallel transfer of the vector of this velocity along an isotropic geodesic of the light ray that joins him with this body. This is equivalent to a parallel transfer of the tetrad from the observer to the body along the isotropic geodesic, with subsequent taking of the tetrad (invariant) components of the velocity vector (or of any other tensor quantity). The initial tetrad itself, naturally, is transported with the observer along its word line in accordance with the Fermi-Walker transfer rules (if the word line is a geodesic, i.e., the motion is inertial, the transfer is the usual parallel one).

Thus, we have a constructive method of tetrad formation (gauge) for a specified initial tetrad in a certain initial world point, based on the physical process of observation with the aid of optical signals. This method can be called also an extension of the reference frames; as a result, the observer's motion, by uniquely inducing a tetrad gauge (at least in the world tube

1) References 9 and 10 were anticipated already by Schrödinger [12], and were subsequently duplicated in a different mathematical form by Stanyukovich [13].

where the isotropic geodesics still do not intersect), leads to establishment of an effective homogeneity of space in this system, for in this case, besides the ordinary parallel transfer, there is induced also a new ("tetrad") transfer, which is independent of the path. This, of course, leads to the appearance of covariant differentiation of a new type ("tetrad differentiation"), in the sense of which there always exists a maximal set of Killing vectors[2]; the operation of covariant integration with respect to a given reference frame is also defined in a trivial manner.

For simplicity, let us consider inertial motion of an observer and use the normal Riemannian coordinates y^μ with origin at that point P on the time-like geodesic word line $\Gamma_{(t)}$ of the observer, to which we shall refer the instant of the description of the physical system. The proper time of the observer s changes along $\Gamma_{(t)}$; assume that there is a variation of the canonical parameter u along the isotropic geodesics that begin on $\Gamma_{(t)}$ and are directed towards the past. We take the world point R with coordinates y^μ; passing through it is an isotropic geodesic $\Gamma_{(i)}$, which intersects $\Gamma_{(t)}$ at the point Q. In order to express the energy-momentum density accurate to terms quadratic in the coordinates (in the expansion for the normal coordinates), it is sufficient to confine oneself to two terms in the expansion of the metric tensor

$$g_{\mu\nu} = \delta_{\mu\nu} + {}^1/_3 R_{\mu\alpha\beta\nu} y^\alpha y^\beta + \ldots. \tag{1}$$

In the normal coordinates, the geodesics passing through the origin are described in the simplest manner: $y^\mu = v^\mu s$. We choose for $\Gamma_{(t)}$ the direction vector $v^\mu = (1,0,0,0,)$. We then obtain for $\Gamma_{(i)}$ starting from the point Q (the instant s), an equation in the form

$$y^\mu = v^\mu s + a^\mu u + {}^1/_3 R^{\mu}_{\cdot\alpha\beta\nu} a^\alpha a^\beta v^\nu s u^2 \tag{2}$$

in view of the fact that

$$\Gamma^\mu_{\alpha\beta} = -{}^1/_3 (R^{\mu}_{\cdot\alpha\beta\nu} + R^{\mu}_{\cdot\beta\alpha\nu}) y^\nu.$$

The directional isotropic vector of the geodesic $\Gamma_{(i)}$ has in this case the components

$$a^\mu = (-1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

where the spherical angles φ and θ characterize the direction of the ray from the point of view of the observer. Since in our case only world lines are geodesics, the field of the γ matrices (equivalently—the tetrad) is established by parallel transfer along these lines,

$$D\gamma^\mu \equiv (\gamma^\mu_{,\alpha} + \gamma^\lambda \Gamma^\mu_{\lambda\alpha}) dy^\alpha = 0.$$

Assume that $\gamma^\mu = \bar{\gamma}^\mu$ at the point P($\bar{\gamma}^\mu$ are the standard constants of the Dirac matrix). Then successive transfer along the broken line PQR yields

$$\gamma_\alpha = \bar{\gamma}^\lambda (\delta_{\alpha\lambda} + {}^1/_6 R_{\mu\alpha\lambda\nu} y^\mu y^\nu + {}^1/_2 R_{\alpha\lambda\mu\nu} y^\mu v^\nu s) \tag{3}$$

(it is important that the terms contained in the parentheses here are not all symmetrical in the indices α and λ). By the same token, we obtained an expansion of the γ matrices in normal coordinates for the general case of an arbitrary gravitational field; it is obvious that a similarity transformation of the initial γ matrices at the point P involves precisely the same transformation of the matrices at all other points, i.e., does not affect the gauge. The same pertains to tetrad vectors and their rotations.

To calculate the energy, we need expressions for the first derivatives of the γ matrices with respect to the coordinates y^μ; they can be readily obtained on the basis of the relation

$$\frac{\partial s}{\partial y^\beta} = \frac{a_\beta}{a_\mu v^\mu} = -a_\beta, \tag{4}$$

which is obtained when the result of the differentiation of (2) with respect to y^β is multiplied by a_μ (the term with the Riemann-Christoffel tensor should be discarded in this case) and account is taken of the orthogonality property

$$a_\mu \frac{\partial}{\partial y^\beta}(a^\mu u) = 0.$$

The sought derivatives are

$$\gamma_{\alpha,\beta} = \bar{\gamma}^\lambda [-{}^1/_3 (R_{\lambda\alpha\beta\nu} + R_{\lambda\beta\alpha\nu}) y^\nu + {}^1/_2 (R_{\lambda\alpha\beta\nu} - R_{\lambda\alpha\mu\nu} v^\mu a_\beta) a^\nu u].$$

We note that the following relation holds

$$\gamma_{\mu;\nu} = \Phi_{\nu\mu\lambda}\gamma^\lambda, \tag{5}$$

where $\Phi_{\mu\nu\lambda}$ is a tensor that is antisymmetrical with respect to the last two indices (the Ricci torsion symbol), with

$$\Phi_{\mu\nu\lambda} = {}^1/_4 \,\mathrm{Sp}\,(\gamma_\lambda \gamma_{\nu;\mu}). \tag{6}$$

It is now easy to obtain the covariant derivatives of the γ matrices[3] and to obtain on the basis of (6)

$$\Phi_{\mu\nu\lambda} = {}^1/_2 (R_{\lambda\nu\mu\rho} - R_{\lambda\nu\omega\rho} v^\omega a_\mu) a^\rho u. \tag{7}$$

In the Noeter theorem, it is possible to start from the requirement that the Lagrange function $\mathfrak{L}$ of the physical system have the properties of a scalar density[9] (this is equivalent to invariance—in the tensor sense—of the action integral for an arbitrary integration region). If we consider the coordinate transformation $x'^\mu = x^\mu + \xi^\mu$, where ξ^μ is an arbitrary infinitesimally small four-vector (strictly speaking, we separate from it an infinitesimally small scalar multiplier, which later is cancelled out), then this requirement takes the form

$$\delta^* \mathfrak{L} + (\mathfrak{L}\xi^\alpha)_{,\alpha} = 0 \tag{8}$$

or after a number of transformations

$$(\mathfrak{U}^\alpha_\sigma \xi^\sigma + \mathfrak{M}^{\alpha\tau}_\sigma \xi^\sigma_{,\tau})_{,\alpha} = 0. \tag{9}$$

We note that the operation

$$\delta^* F = F'(x) - F(x) = \delta F - \frac{\partial F}{\partial x^\alpha}\xi^\alpha \tag{10}$$

coincides, apart from the sign, with the Lie differential; for any quantity A_B (henceforth A_B will stand for the potentials of the fields) we have here

$$\delta A_B = A_B'(x') - A_B(x) = a_B|^\tau_\sigma \xi^\sigma_{,\tau}. \tag{11}$$

[2]It is easy to see that the tetrad vectors themselves are generalized Killing vectors corresponding to translations along the tetrad axis.

[3]In the sense of the usual covariant differentiation; the γ matrices, of course, like the tetrad vectors, are constant with respect to tetrad covariant differentiation.

The identity (9) is trivial by virtue of the satisfaction of the known relations of the Noeter theorem[9] ("strong identity"). We propose here that the Lagrangian does not depend on the second derivatives of the field potentials, so that

$$\mathfrak{M}_{\sigma}^{\alpha\tau} = \frac{\partial \mathfrak{L}}{\partial A_{B,\alpha}} a_B|_{\sigma}^{\tau}, \tag{12}$$

$$\mathfrak{U}_{\sigma}^{\alpha} = \frac{\delta \mathfrak{L}}{\delta A_B} a_B|_{\sigma}^{\alpha} + \mathfrak{L}\delta_{\sigma}^{\alpha} - \frac{\partial \mathfrak{L}}{\partial A_{B,\alpha}} A_{B,\sigma}, \tag{13}$$

$$\mathfrak{t}_{\sigma}^{\alpha} = \frac{\partial \mathfrak{L}}{\partial A_{B,\alpha}} A_{B,\sigma} - \mathfrak{L}\delta_{\sigma}^{\alpha} \tag{14}$$

(the latter quantity is well known as the canonical energy-momentum density in special relativity theory; the former is already closely connected with the spin of the field and is called the "generalized spin density"). It is obvious that the "strong" identity (9) cannot have a deep physical meaning, since it is valid not only for the physical Lagrangian $\mathfrak{L}$, but also for any quantity with properties of a scalar density. For a synthesis of the consequences of invariance and of the dynamic properties of fields, it is necessary to take into account in this Lagrangian the field equations, particularly the Einstein equations

$$\mathfrak{T}_{f\sigma}^{\alpha} + \mathfrak{T}_{g\sigma}^{\alpha} = 0, \quad \mathfrak{T}_{\sigma}^{\alpha} = 2\frac{\delta \mathfrak{L}}{\delta g^{\tau\sigma}} g^{\alpha\tau},$$

from which it follows, by virtue of (13), that

$$\mathfrak{U}_{g\sigma}^{\alpha} = -\mathfrak{T}_{g\sigma}^{\alpha} - \mathfrak{t}_{g\sigma}^{\alpha}. \tag{15}$$

Therefore the sought "weak" differential conservation law is of the form

$$(\mathfrak{T}_{f\sigma}^{\alpha}\xi^{\sigma} + \mathfrak{t}_{g\sigma}^{\alpha}\xi^{\sigma} - \mathfrak{M}_{g\sigma}^{\alpha\tau}\xi^{\sigma}_{,\tau})_{,\alpha} = 0 \tag{16}$$

(we have taken the g-variant of the identity (9), i.e., we have written it for the gravitational Lagrangian). So far we used the conventional reasoning for the traditional analysis of the Noeter theorem in field theory.

Let us turn to an arbitrary four-vector ξ^{μ} in (16) (according to the remark made above, it need no longer be infinitesimally small). We are interested not only in the differential conservation laws but also in the integral ones; by virtue of the known transformational properties of the quantities $\mathfrak{T}_{\sigma}^{\alpha}$, $\mathfrak{t}_{\sigma}^{\alpha}$ and $\mathfrak{M}_{\sigma}^{\alpha\tau}$ [16], and on the basis of the general definition of the covariant derivative for tensors of arbitrary ranks and for their densities (insofar as we know, used for the first time by Trautman[17])

$$A_{B;\mu} = A_{B,\mu} + a_B|_{\sigma}^{\tau}\Gamma_{\tau\mu}^{\sigma}, \tag{17}$$

where the coefficients $a_B|_{\sigma}^{\tau}$ are determined from the relations (11), we can formally replace of ξ^{σ} by γ^{σ} in (16) as the first step in the construction of the covariant integral for the energy momentum. We then obtain the weak conservation law

$$w^{\alpha}_{,\alpha} = 0 \tag{18}$$

of the matrix quantity

$$w^{\alpha} = \mathfrak{T}_{f\sigma}^{\alpha}\gamma^{\sigma} + \mathfrak{t}_{g\sigma}^{\alpha}\gamma^{\sigma} - \mathfrak{M}_{g\sigma}^{\alpha\tau}\gamma^{\sigma}_{,\tau}, \tag{19}$$

which is the true contravariant vector density. In view of the differential law (18), the integral conservation law is also satisfied under appropriate boundary conditions for the scalar matrix quantity

$$\Pi = \int w^{\alpha} dS_{\alpha}. \tag{20}$$

The concluding step in the construction of the covariant energy-momentum integral of the physical system is the simple operation

$$P_{\mu} = {}^1\!/_4\, \mathrm{Sp}\,(\gamma_{\mu}\Pi). \tag{21}$$

On the whole, the employed procedure consists (in simplified language) of a scalar multiplication of a certain operator A_{μ} by γ^{μ}, transferring the obtained scalar matrix to the world point of the observer, and taking there its trace with the γ matrix at this point; an analogous transfer of scalar matrices from all points of a hypersurface, their summation, and the calculation of the trace give the covariant integral. This operation, obviously, depends essentially on the gauge of the γ matrices, and in our treatment on the choice of the reference system. Taking such a unit tetrad transfer on an infinitesimal path, we can easily find the tetrad covariant differential of the vector A_{μ} and the tetrad covariant derivatives of this vector

$$A_{\mu\|\nu} = A_{\mu,\nu} - {}^1\!/_4\, \mathrm{Sp}\,(\gamma^{\lambda}\gamma_{\mu,\nu})A_{\lambda}. \tag{22}$$

Obviously, at a given gauge of the γ matrices, the tetrad transfer does not depend on the path; the tetrad covariant derivatives of the γ matrices themselves and of the tetrad vectors vanish identically.

We see that the "nongravitational" part of the energy momentum is described by the ordinary symmetrical energy-momentum tensor, and in the limit of flat space and Cartesian coordinates, the principle of correspondence with the special relativity theory is satisfied. We turn to the gravitational energy proper, assuming $\mathfrak{T}_{f\sigma}^{\alpha} = 0$. The expressions for $\mathfrak{t}_{g}^{\alpha}$ and $\mathfrak{M}_{gs}^{\alpha\tau}$, with a Lagrangian

$$\mathfrak{L}_g = \frac{\sqrt{-g}}{8\varkappa}\mathrm{Sp}\,(\gamma^{\mu}_{;\nu}\gamma^{\nu}_{;\mu} - \gamma^{\mu}_{;\mu}\gamma^{\nu}_{;\nu}), \tag{23}$$

that differs by a divergence term from the other gravitational Lagrangians used in general relativity theory, were calculated earlier by one of the authors[18] (it is necessary only to correct some signs that were reversed as the result of a misunderstanding):

$$\mathfrak{M}_{g\sigma}^{\alpha\tau} = \frac{\sqrt{-g}}{4\varkappa}\mathrm{Sp}\,(\gamma^{\alpha}_{;\sigma}\gamma^{\tau} + \gamma^{\nu}_{;\nu}\gamma^{\alpha}\delta_{\sigma}^{\tau} - \gamma^{\nu}_{;\nu}\gamma^{\tau}\delta_{\sigma}^{\alpha}); \tag{24}$$

$$\mathfrak{t}_{g\sigma}^{\alpha} = \frac{\sqrt{-g}}{4\varkappa}\mathrm{Sp}\,[\gamma^{\alpha}_{;\nu}\gamma^{\nu}_{,\sigma} - \gamma^{\nu}_{;\nu}\gamma^{\alpha}_{,\sigma} - \gamma^{\mu}_{;\mu}\gamma^{\alpha}\Gamma_{\sigma\nu}^{\nu} - {}^1\!/_2\,\delta_{\sigma}^{\alpha}(\gamma^{\mu}_{;\nu}\gamma^{\nu}_{;\mu} - \gamma^{\mu}_{;\mu}\gamma^{\nu}_{;\nu})]. \tag{25}$$

Substituting these expressions in (19) and taking (5) into account, we obtain for the gravitational part of w^{α}:

$$w_g^{\alpha} = \frac{\sqrt{-g}}{\varkappa}\gamma^{\mu}\,[\Phi_{\nu\cdot\cdot}^{\cdot\alpha\sigma}\Phi_{\mu\cdot\sigma}^{\cdot\cdot\nu} - \Phi_{\mu\cdot\cdot}^{\cdot\alpha\sigma}\Phi_{\nu\cdot\sigma}^{\cdot\cdot\nu} + \Phi_{\nu\cdot\cdot}^{\cdot\tau\alpha}\Phi_{\tau\cdot\mu}^{\cdot\cdot\nu} - \Phi_{\nu\cdot\cdot}^{\cdot\nu\alpha}\Phi_{\tau\cdot\mu}^{\cdot\cdot\tau} + \Phi_{\nu\cdot\cdot}^{\cdot\nu\tau}\Phi_{\tau\cdot\mu}^{\cdot\cdot\alpha} - {}^1\!/_2\,\delta_{\mu}^{\alpha}(\Phi_{\sigma\cdot\cdot}^{\cdot\nu\tau}\Phi_{\nu\cdot\tau}^{\cdot\cdot\sigma} - \Phi_{\nu\cdot\cdot}^{\cdot\nu\tau}\Phi_{\sigma\cdot\tau}^{\cdot\cdot\sigma})]. \tag{26}$$

We consider here only the problem of the energy density; this corresponds, in accordance with (21), to taking the trace $w = (1/4)\,\mathrm{Tr}\,(\bar{\gamma}_0 w^0)$. Simple transformations show that the general form of the gravitational-energy density can be represented by

$$w_g = \frac{\sqrt{-g}}{2\varkappa}(\Phi_{ijk}\Phi_{jik} - \Phi_{iik}\Phi_{jjk} + \Phi_{ij0}\Phi_{ji0} - \Phi_{ii0}\Phi_{jj0}) \tag{27}$$

(the Latin indices run through the spatial values 1, 2, 3 and obey the summation rule). Finally, substituting in

(27) the expression (7) for the Ricci symbol and taking into account the made choice of vectors v^{μ} and a^{μ}, we obtain

$$w_g = \frac{1}{8\varkappa}(R_{0ij\omega}R_{0ij\varepsilon} + R_{kij\omega}R_{kj i\varepsilon} + R_{i00\omega}R_{i00\varepsilon} + 4R_{i00\omega}R_{ij0\varepsilon}a_j - 2R_{ijk\omega}R_{ik0\varepsilon}a_j)\, a^{\omega}a^{\varepsilon}u^2. \quad (28)$$

This representation of the density of the gravitational energy does not yet help readily to analyze its properties, particularly to determine whether it is positive definite. Therefore, using the well known algebraic properties of the Riemann-Christoffel tensor, we reduce it to the form

$$w_g = \frac{1}{8\varkappa}(R_{i00\omega}R_{i00\varepsilon} + R_{0ij\omega}R_{0ij\varepsilon} + {}^1/_2 R_{ijk\omega}R_{ijk\varepsilon} + {}^1/_2 R_{ij0\omega}R_{ij0\varepsilon} + 4R_{i00\omega}R_{ij0\varepsilon}a_j)\, a^{\omega}a^{\varepsilon}u^2, \quad (29)$$

for the first four terms are clearly not negative, and it is obvious here that the last term vanishes when averaged over the observation angles. Expression (29) can, in addition, be transformed in two ways. First, it is possible to simplify the first four terms, using the little-known quadratic identity for the Riemann-Christoffel tensor[19],

$$R_{\alpha\beta\gamma\mu}R^{\alpha\cdot\cdot\gamma\nu} - {}^1/_4 R_{\alpha\beta\gamma\delta}R^{\alpha\cdot\gamma\delta}\delta^{\nu}_{\mu} = 2R_{\mu\alpha}{}^{\beta\nu}{}_{\cdot\cdot}R^{\alpha}_{\beta} + 2R^{\alpha}_{\mu}R^{\nu}_{\alpha} - R_{\alpha\beta}R^{\alpha\beta}\delta^{\nu}_{\mu} - RR^{\nu}_{\mu} + {}^1/_4 R^2\delta^{\nu}_{\mu}. \quad (30)$$

For the case of a pure gravitational field ($R^{\nu}_{\mu} = 0$) we obtain

$$w_g = \frac{1}{8\varkappa}(R_{ijk\omega}R_{ijk\varepsilon} + {}^3/_2 R_{i00\omega}R_{i00\varepsilon} + 4R_{i00\omega}R_{ij0\varepsilon}a_j)\, a^{\omega}a^{\varepsilon}u^2. \quad (31)$$

On the other hand, expression (29) can be reduced to the form

$$w_g = \frac{1}{8\varkappa}[R_{ijk\omega}R_{ijk\varepsilon} - {}^1/_2 R_{k00\omega}R_{k00\varepsilon} + 2(R_{0ij\omega}R_{0ij\varepsilon} - R_{ji0\omega}R_{jk0\varepsilon})a_i a_k]\, a^{\omega}a^{\varepsilon}u^2. \quad (32)$$

In addition, the initial form of (29) can itself be expressed simply in terms of known six-dimensional symbols (see, for example,[20]):

$$w_g = \frac{1}{16\varkappa}(R_{A\mu\omega}R_{A\mu\varepsilon} + 8R_{i00\omega}R_{ij0\varepsilon}a_j)\, a^{\omega}a^{\varepsilon}u^2. \quad (33)$$

Here A is a six-dimensional index subject to summation.

By way of an example, let us consider the case of weak plane gravitational waves. It might seem possible to analyze directly the more general case of locally plane waves (see, for example,[21]). However, in normal Riemannian coordinates the assumption of a locally plane character of the waves leads automatically to the absence of a real gravitational field, i.e., to a flat time-space (this indeed is the shortcoming of Weber's approach). On the other hand, in the case of weak plane gravitational waves it is always possible to go over to normal coordinates, thus depriving these waves of their flat character, but this does not change the values of the Riemann-Christoffel tensor components in the approximation which is of importance here and which is customarily used. Then, if in the initial coordinate system (see, for example,[7]) a weak plane wave propagates in the positive direction of the x^1 axis, there can exist only the following two nonvanishing components of the Riemann-Christoffel tensor (this corresponds to two possible polarizations of the gravitational wave):

$$R_{0202} = R_{1212} = R_{0313} = -R_{0303} = -R_{1313} = -R_{0212} = A;$$
$$R_{0203} = R_{1213} = -R_{0213} = -R_{0312} = B. \quad (34)$$

Recognizing that by virtue of the properties of the vector a^{μ} this leads to the equations

$$(R_{i00\omega}R_{i00\varepsilon} + R_{0ij\omega}R_{0ij\varepsilon})a^{\omega}a^{\varepsilon} = 4(A^2+B^2),$$
$${}^1/_2(R_{ijk\omega}R_{ijk\varepsilon} + R_{ij0\omega}R_{ij0\varepsilon})a^{\omega}a^{\varepsilon} = 4(A^2+B^2),$$

and also

$$R_{i00\omega}R_{ij0\varepsilon} = -(A^2+B^2)(\delta^2_{\omega}\delta^2_{\varepsilon} + \delta^3_{\omega}\delta^3_{\varepsilon})\delta^1_j,$$

we obtain from (29) the value of the energy density of the gravitational wave near the observer:

$$w_g = \frac{1}{\varkappa}(A^2+B^2)\cdot\left[1 - \frac{1}{2}a_1(1-a_1^2)\right]\cdot u^2. \quad (35)$$

Let us compare this result with the well known expression for the density of the gravitational energy of the plane wave[22] 4)

$$t_0{}^0 = \frac{1}{2\varkappa}[(\dot g_{22})^2 + (\dot g_{23})^2]. \quad (36)$$

Going over to coordinates on the light cone (see[23], page 184), we can express the parameter u used above as $u = (x - s)/\sqrt{2}$ (the time axis coincides with the world line of the observer). On the light cone of the past x = −s, so that we have there $u = -s\sqrt{2}$. Then, setting the first derivatives of the matrix tensor at the origin (at the world point of the observer) equal to zero (the origin can always be chosen in this manner), we obtain

$$\dot g_{\mu\nu} \approx \ddot g_{\mu\nu}s = -\frac{1}{\sqrt{2}}\ddot g_{\mu\nu}u$$

as the expression for these first derivatives near the origin, in terms of the second derivative taken at the origin. Thus, for the only physically significant components we obtain $\dot g_{22} = -\dot g_{33} = \sqrt{2}Au$ and $\dot g_{23} = \sqrt{2}Bu$. Substituting these values in the expression for the energy density (36), we get

$$t_0{}^0 = \frac{1}{\varkappa}(A^2+B^2)u^2, \quad (37)$$

which coincides with the first part of the rigorous expression obtained by us for the energy density of a plane gravitational wave near the observer. The second part, which depends on a_1, vanishes upon averaging over the observation angles and can lead to a deviation of the energy density of the wave from the value (37) by at most ±19.2% (for a deviation of ±55° from the wave propagation direction). Of course, such a comparison should be regarded as purely qualitative, all the more since the previous expressions for the gravitational energy did not pretend to define its localization.

The proposed gauge for the tetrad fields, which is conceptually simple, encounters technical difficulties when we seek the form of the γ matrices far from the observer in Riemannian space. These difficulties are not fundamental in character, since they are connected only with finding equations for the isotropic geodesics in algebraic form in specified gravitational fields. Our deduction that the density of the gravitational energy vanishes on the world line of an inertial observer is closely connected with the Einstein equivalence principle; in accordance with the fact that this principle has only a local character, the gravitational energy differs from zero away from the observer's world line. Since

4)See also [7], where the component t^{10} of the Landau-Lifshitz-Fock pseudotensor is calculated; it coincides with (36), since the gravitational waves propagate with fundamental velocity.

w^{α} is a true vector density, and P_{μ} is a true vector, the determination of either the energy-momentum density or the corresponding integral quantities is in our approach strictly covariant in the sense of the transformations of the coordinates for one and the same prescribed observer. On going over from one observer to another (changing the reference system), a change takes place also in the localization of the energy momentum, as would be expected from general physical considerations. The proposed approach excludes in principle the possibility of paradoxes of the Bauer type[4]. It is easy to see, that in the absence of a gravitational field (in flat space-time), the gravitational energy vanishes identically in all of space, regardless of the character of motion of the observer (inertial or non-inertial). Thus, the field of the inertial forces does not have energy and momentum, unlike the gravitational field (the absence of global equivalence principle), and the energy density of the latter in the vicinity of the observer is on the average non-negative.

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[1] V. A. Fock, Z. Physik 57, 261 (1929); ZhRFKhO, (J. Phys. Chem. Soc., Physics Sec.) 62, 133 (1930). V. A. Fock and D. D. Iwanenko, Compt. rend. 188, 1470 (1929).

[2] C. Møller, Mat. Fys. Medd. Dan. Vid. Selsk. 35, No. 3 (1966).

[3] C. Møller, Mat. Fys. Skr. Dan. Vid. Selsk. 1, No. 10 (1961).

[4] H. Bauer, Physik. Z. 19, 163 (1918).

[5] A. Einstein, Collected Works (Russ. Transl.), v. 1, Nauka, 1965, p. 650.

[6] E. Schrödinger, Physik. Z. 19, 4 (1918).

[7] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Nauka, 1967.

[8] V. A. Fock, Teoriya prostranstva, vremeni, i tyagoteniya (Theory of Space, Time, and Gravitation), Fizmatgiz 1961.

[9] N. V. Mitskevich, Ann. Physik 1, 319 (1958).

[10] C. Møller, Ann. of Phys. 4, 347 (1958); Max-Planck-Festschrift, Berlin, 1958, p. 139.

[11] C. Cattaneo, Conservation Laws, in Intern. Conf. on Relativ. Theor. of Gravitat. II, London, 1965.

[12] E. Schrödinger, Proc. Roy. Irish Acad. 54A, 79 (1951).

[13] K. P. Stanyukovich, Gravitatsionnoe pole i élementarnye chastitsy (Gravitational Field and Elementary Particles), Nauka, 1965.

[14] M. F. Shirokov, in: Problemy gravitatsii (Problems of Gravitation), Tbilisi, 1965, p. 18.

[15] E. Noether, Götting. Nachr., 235 (1918).

[16] N. V. Mitskevich, Doklady Bolg. AN 14, 439 (1961).

[17] A. Trautman, Byull. Pol'skoĭ AN, otd. III 4, 665 671 (1956); 5, 721 (1957).

[18] N. V. Mitskevich, Izv. vuzov, seriya fiz. 6, 64 (1964).

[19] N. V. Mitskevich, in: Gravitatsiya i teoriya otnositel'nosti (Gravitation and Relativity Theory), No. 3, Kazan', 1967, p. 129.

[20] A. Z. Petrov, Novye metody v obshchei teorii otnositel'nosti (New Methods in General Relativity Theory), Nauka, 1966.

[21] J. Weber, General Relativity and Gravitational Waves, Wiley, 1961.

[22] A. Einstein, op. cit.[5], p. 631.

[23] D. Bohm. Special Theory of Relativity, Benjamin, 1968.

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114