RADIATIVE TRANSFER OF EXCITATION IN A FINITE VOLUME

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The problem of the transport of resonant radiation in a medium having large optical dimensions and an arbitrary shape is reduced, within the framework of the kinetic equations, to the solution of a singular integral equation. The proposed approach makes it possible to separate the problem of finding the radiation field inside the medium from the problem of the influence of the radiation sources and of the boundaries of the medium. The obtained equation does not contain small parameters and is the analog of the diffusion equation. An asymptotically exact solution is obtained for the problem of the passage of resonant radiation through an optically thick layer, for an arbitrary shape of the absorption line. The influence of the stimulated radiation is taken into account.

1. To describe the interaction of resonant radiation with matter, we shall use the widely employed two-level model and assume that the kinetic equation holds for the photons[1,2]. We shall not discuss the question of the limits of applicability of the kinetic equation, which has been recently the subject of a number of papers (for example[3-5]).

From the kinetic equation for the phonons

$$c\frac{\partial f(\nu,\mathbf{r},\mathbf{e})}{\partial\mathbf{r}} = -k_\nu f(\nu,\mathbf{r},\mathbf{e}) + \varkappa_\nu\frac{n(\mathbf{r})}{4\pi} \quad (1)$$

(here c is the velocity of light, k_ν the absorption coefficient, n(r) the density of the excited atoms, and κ_ν the radiation probability, f(ν, r, e) the density of photons of frequency ν emitted in the direction of the vector e), it is easy to obtain for the density of the excited atoms the equation given in[1,2]:

$$n(\mathbf{r}) = F(\mathbf{r}) + \int\limits_{V_i} K_0(|\mathbf{r}-\mathbf{r}'|)\,n(\mathbf{r}')\,d\mathbf{r}'. \quad (2)$$

The integration here is over the volume occupied by the scattering medium, F(r) is the distribution of the atoms directly excited by the incident light. The kernel $K_0(|\mathbf{r}-\mathbf{r}'|)$, describing the excitation transfer, depends on the absorption line shape and is normalized by the condition

$$\int\limits_{-\infty}^{\infty} K_0(|\mathbf{r}-\mathbf{r}'|)\,d\mathbf{r}' = \lambda \leqslant 1,$$

where λ is the probability that the photon will survive following a single scattering. In the present paper we consider the case $\lambda = 1$, i.e., we neglect nonresonant processes.

2. Let us describe the physical situation occurring in the radiation transfer problem. The distance unit in (2) is chosen to be the photon free path. With this choice, the kernel $K_0(r)$ is a function of the order of unity. If the characteristic dimension of the region is $L \lesssim 1$, then an approximate solution of (2) can be readily obtained by iteration.

We consider the most interesting case of an optically thick medium ($L \gg 1$). In this case the behavior of Eq. (2) depends strongly on the behavior of the kernel $K_0(r)$ at large distances $r \gg 1$. There is a well known method of obtaining solutions of equations of this type for sufficiently rapidly decreasing kernels.

Thus, if the second moment of the function $K_0(r)$ is finite, then Eq. (2) reduces to the diffusion equation, the solution of which has a character that is universal and does not depend on the concrete form of the nucleus. It is then necessary to join this solution to the solution near the boundaries of the region. Thus, in those cases when the diffusion approximation is valid, the problem breaks up into two. In the first problem one solves the standard (diffusion) equation in a region of a given form. This solution is valid far from the boundaries of the regions and from the sources. The second problem is to find the solution in the vicinity of the sources and the surfaces, where it is impossible to confine oneself to the diffusion approximation and it is necessary to solve the entire problem. In such cases, however, the problem always reduces to an analysis of a semi-infinite or infinite medium.

The situation is different for kernels having no second moment, which include just the kernels that appear in the radiation transfer theory. In this case, even far from the boundaries, the solution is not universal and depends on the concrete form of the kernel $K_0(r)$. In addition, the solution depends strongly on the shape of the region. The physical cause of this is the presence (even in a region of large dimensions) of photons whose mean free path is comparable with the dimensions of the region. Moreover, they make the main contribution to the solution. In this case, however, it is possible to breakup the problem for a medium with large optical dimensions into two. The first problem is that of finding the distribution function over the entire region, and the second is that of finding it near the boundaries of the medium and near the sources. The solution of the second problem depends on the concrete structure of the kernel at distances on the order of unity, but does not depend on the dimensions and form of the region. This problem is always reduced to an analysis of an infinite or semi-infinite medium and is solved in a number of papers (for example,[6-9]).

As to the most interesting problem, that of finding the degree of excitation in the entire region, only numerical calculations have been published to date.

We propose in this paper an equation which makes it possible to solve the first of the aforementioned problems, and is the analog of the diffusion equation for slowly decaying nuclei. An asymptotically exact solution of the problem is obtained for a plane-parallel layer of large optical thickness[1)].

3. Let a light source be located inside a medium at a point $\mathbf{r} = \mathbf{r}_0$, located at a distance much larger than unity from the boundaries of the region. Since the principal role in the problem is played by long-range photons, corresponding to large $|\mathbf{r} - \mathbf{r}_0|$, it might seem possible to replace $K_0(|\mathbf{r} - \mathbf{r}'|)$ in (2) by the asymptotic expression

$$K_0(r) \approx ar^{-3-2\gamma}. \quad (3)$$

Here a is a constant[2)], and γ satisfies the inequalities $0 < \gamma \leq 1/2$ for all kernels encountered in the theory of transfer of resonant radiation[6].

However, a direct replacement of $K_0(|\mathbf{r} - \mathbf{r}'|)$ by the asymptotic expression (3) is impossible, since the integral in (2) becomes divergent. To eliminate this divergence, we carry out the subtraction

$$n(\mathbf{r})\left(1 - \int_{V_i} K_0(|\mathbf{r} - \mathbf{r}'|)\,d\mathbf{r}'\right) = \int_{V_i} K_0(|\mathbf{r} - \mathbf{r}'|)\,(n(\mathbf{r}') - n(\mathbf{r}))\,d\mathbf{r}' + F(\mathbf{r}, \mathbf{r}_0), \quad (4)$$

(V_i is the volume occupied by the medium). In this equation it is already possible to go over to the asymptotic expressions

$$n(\mathbf{r}) \int_{V_e} |\mathbf{r} - \mathbf{r}'|^{-3-2\gamma}\,d\mathbf{r}' = \int_{V_i} \frac{n(\mathbf{r}') - n(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|^{3+2\gamma}}\,d\mathbf{r}' + F(\mathbf{r}, \mathbf{r}_0). \quad (5)$$

Here V_e is the region free of scatterers.

Let L be the characteristic dimension of the region occupied by the medium. Carrying out in (5) the scale transformation $\boldsymbol{\rho} = \mathbf{r}/L$, we obtain

$$n(\mathbf{r}) = L^{2\gamma-3}\varphi(\rho),$$

$$\varphi(\rho) \int_{V_e} |\rho - \rho'|^{-3-2\gamma}\,d\rho' - \int_{V_i} \frac{\varphi(\rho') - \varphi(\rho)}{|\rho - \rho'|^{3+2\gamma}}\,d\rho' = \frac{M_0}{a}\delta(\rho - \rho_0), \quad (6)$$

The δ-function in the right side of (6) is the result of the function $F(\mathbf{r}, \mathbf{r}_0)$, which differs from zero when

$$|\rho - \rho_0| \sim 1/L, \quad M_0 = \int_{V_i + V_e} F(\mathbf{r}, \mathbf{r}_0)\,d\mathbf{r}.$$

Equation (6) contains only asymptotic expressions and does not include the mean free path. For the presently considered case, it is the analog of the diffusion equation ($\Delta n = \delta(\mathbf{r} - \mathbf{r}')$). If the region is complicated, then this equation, like the diffusion equation, must apparently be solved numerically. Unlike Eq. (2), our equation does not have small parameters, and all the quantities in it are of the order of unity. In the Appendix we illustrate the use of this equation to obtain asymptotic expressions for the solutions in all of space and in a half-space; these equations will be useful in what follows. Unlike the previously known methods of obtaining these expressions, we shall derive them directly from the equation.

If there are no light sources in the medium, it is necessary to solve the homogeneous equation. The solution of the problem for an arbitrary distribution of the sources of the surface is then given by

$$n(\mathbf{r}) = \int \psi(\mathbf{s})\,G(\mathbf{r}, \mathbf{s})\,ds.$$

Here $\mathbf{s}$ is the coordinate on the surface, $G(\mathbf{r}, \mathbf{s})$ is the surface Green's function, which is also a solution of the homogeneous equation and goes over as $\mathbf{r} \to \mathbf{s}$ into the solution of the problem of a half-space with a point source on its surface. When $|\mathbf{r} - \mathbf{s}| \gg 1$, the latter behaves like $x^\gamma |\mathbf{r} - \mathbf{s}|^{-3}$, where x is the coordinate along the normal to the surface[12].

4. In those cases when the problem has a simple symmetry, it is possible to obtain the solution in explicit form. Let the medium fill a layer of thickness L. The geometry of the region makes it possible to obtain in this case a simpler equation instead of the singular equation. We introduce the coordinate x in such a way that the boundaries of the layer are at the points 0 and L. We put $F(x) = K(x)$ in Eq. (3)[3)]. We denote the solution by $R(L, x)$, where the first argument is the thickness of the layer and the second is the coordinate of the point of interest to us inside the layer. The integral equation for the function $R(L, x)$ has a simple probabilistic meaning. Let us consider the random walk of a particle in a layer of thickness L. Let $K(|x - x'|)$ be the conditional probability of scattering at the point x, if the preceding scattering occurred at the point x′. In this case $R(L, x)$ satisfies the equation

$$R(L, x) = K(x) + \int_0^L K(|x - x'|)\,R(L, x')\,dx' \quad (2')$$

and has the meaning of the probability of scattering, at the point x, of a particle that starts its random walk from the surface $x = 0$.

If we increase the thickness of the layer L by a small amount Δ, then by virtue of the probabilistic meaning of $R(L, x)$ we have

$$R(L + \Delta, x) = R(L, x) + \Delta R(L, L)\,R(L, L - x). \quad (7)$$

The first term in this formula is the probability that a particle which experienced not a single scattering in the layer Δ will fall on the point x; the second term denotes the probability that a particle that stayed in the layer Δ will fall on the point x. Going in (7) to the limit as $\Delta \to 0$, we obtain

$$\frac{\partial R(L, x)}{\partial L} = R(L, L)\,R(L, L - x). \quad (8)$$

5. When $1 \ll x \ll L$, the behavior of the function $R(L, x)$ should coincide with the asymptotic behavior of $R(\infty, x) \approx Ax^{\gamma-1}$ (see the Appendix). In analogy with Sec. 3, it follows therefore that in the region x, $L - x \gg 1$

$$R(L, x) \approx AL^{\gamma-1}\varphi(x/L), \quad (9)$$

where $\varphi(\xi)$ is a function on the order of unity. It is obvious that when $\xi \ll 1$ we have $\varphi(\xi) \approx \xi^{\gamma-1}$.

In order for the right and left sides of (8) to be of the same order, it is necessary to put $R(L, L) = \alpha/L$, where

[1)] For the particular case when the absorption line has a Doppler shape, V.V. Ivanov investigated the asymptotic behavior (with respect to L) of some of the functions encountered in transport theory. He did not consider the behavior of the density of the excited atoms inside the layer [10,11].

[2)] A power-law asymptotic form was chosen only to simplify the exposition. The results can be readily extended to the general case.

[3)] The case of an arbitrary F(x) readily reduces to that considered here. Here $K(x) = \int K_0(|\mathbf{r}|)\,dy\,dz$, $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, and the integration is from $-\infty$ to $+\infty$.

α is a number to be determined. Taking into account the foregoing, we obtain from (9) and (8) an equation for the function $\varphi(\xi)$:

$$\xi\varphi'(\xi) + \alpha\varphi(1-\xi) + (1-\gamma)\varphi(\xi) = 0. \quad (10)$$

Eliminating from (10) the functions of argument $(1-\xi)$ by differentiating and making the substitution $\xi - 1 - \xi$, we obtain a second order differential equation

$$\xi(1-\xi)\varphi''(\xi) + [2-\gamma-(3-2\gamma)\xi]\varphi'(\xi) + [\alpha^2-(1-\gamma)^2]\varphi(\xi) = 0. \quad (11)$$

A solution of this equation, behaving like $\xi^{\gamma-1}$ when $\xi \ll 1$, is $\xi^{\gamma-1}F(\alpha, -\alpha, \gamma, \xi)$, where $F(\alpha, -\alpha, \gamma, \xi)$ is the hypergeometric function. Using the well known properties of the hypergeometric function, we can rewrite the solution in the form

$$\varphi(\xi) = \xi^{\gamma-1}(1-\xi)^{\gamma}F(\gamma-\alpha, \gamma+\alpha, \gamma, \xi). \quad (12)$$

We denote

$$S(x) = \lim_{L\to\infty}\frac{R(L, L-x)}{R(L, L)}.$$

It is obvious that S(x) is a solution of the problem for a half-space with a source at infinity subject to the condition S(0) = 1. Thus, when $L - x \ll L$, we have

$$R(L, L-x) \approx R(L, L)S(x). \quad (13)$$

From (8) we can easily obtain a relation between S(x) and $R(\infty, x)$; the asymptotic behavior of this relation is investigated in the Appendix. Making the substitution $x \to L - x$ in (8), we obtain

$$\frac{\partial R(L, L-x)}{\partial L} + \frac{\partial R(L, L-x)}{\partial x} = R(L, L)R(L, x). \quad (14)$$

Dividing each term of (14) by R(L, L) and going to the limit as $L \to \infty$, we obtain

$$S(x) = 1 + \int_0^x R(\infty, x)\,dx.$$

As shown in the Appendix, $S(x) \sim x^{\gamma}$ and $R(\infty, x) \sim x^{\gamma-1}$. From the connection between S(x) and $R(\infty, x)$ it follows that when $x \gg 1$

$$R(\infty, x) / S(x) = \gamma / x. \quad (15)$$

To determine the number α which enters in (12), let us consider in greater detail the behavior of R(L, x) when $1 \ll x \ll L$. We integrate Eq. (8):

$$R(L, x) = R(\infty, x) - \int_L^\infty R(l, l)R(l, l-x)\,dl.$$

When $x \ll L$ we can use formula (13) for $R(l, l-x)$. Then, with allowance for the equation $R(l, l) = \alpha/l$, we obtain

$$R(L, x) = R(\infty, x) - \frac{\alpha^2}{l}S(x).$$

From a comparison of this formula at $1 \ll x \ll L$ with formula (12) at $\xi \ll 1$, with allowance for (15), we obtain the exact value $\alpha = \gamma$; with this, $F(0, 2\gamma, \gamma, \xi) \equiv 1$, and the solution is written in the form

$$R(L, x) \approx R(\infty, L)\left(\frac{x}{l}\right)^{\gamma-1}\left(1-\frac{x}{l}\right)^{\gamma}. \quad (16)$$

It is easy to verify by direct substitution in the homogeneous singular integral equation (6) that the function $\varphi(\xi) = \xi^{\gamma-1}(1-\xi)^{\gamma}$ is a solution (6).

The expression

$$R(\infty, x)S(L-x) / S(L) \quad (17)$$

inside the layer, i.e., at x, $L - x \gg 1$, coincides with (16) and describes correctly the behavior of the function R(L, x) as $x \to 0$ and $x \to L$. By direct substitution in (8) it is easy to verify that the function (17), accurate to O(1/L), is its solution for all x. This solution is valid also in the case when the asymptotic form of the kernel and of the functions $R(\infty, x)$ and S(x) is not purely a power-law dependence.

Thus, the solution of the problem of the passage of radiation through a thick layer is expressed in terms of the functions $R(\infty, x)$ and S(x), which have been investigated in detail in a number of papers devoted to radiation transfer in a half-space[6-8].

6. With the aid of the solution obtained for a layer we can readily obtain different physical quantities. For example, the value of the radiation flux passing through the layer is

$$j = \int_0^L dx \int_0^\infty d\nu \int d\Omega\, k_\nu \exp\{-k_\nu x/\cos\theta\}\, R(L, L-x).$$

The notation here is the same as in (1), except that dimensionless units of length are used, $d\Omega$ is the solid-angle interval (the integration is over a hemisphere), and θ is the angle between the direction of the emitted radiation and the x axis.

Calculations with $R(L, L-x) = R(\infty, L-x)S(x)/S(L)$ lead to the formula

$$j = 2\pi\frac{\Gamma(1+2\gamma)}{\Gamma^2(1+\gamma)}\frac{1}{S(L)} \sim L^{-\gamma}, \quad (18)$$

where $\Gamma(z)$ is the Euler Gamma function. (We used the asymptotic expression for $R(\infty, x)$ with an exact coefficient[12].)

It is easy to verify that the spectrum of the transmitted radiation has a dip in the center of the line, as well as maxima whose positions are obtained from the approximate equation $k_\nu L \sim \cos\theta$. The spectrum of the reflected radiation in the principal order in L coincides with that possessed by the radiation reflected from a half-space[6].

With the aid of the solution obtained by us it is easy to investigate the question of the bleaching of a layer by high-intensity radiation. As shown in[9], this problem reduces to a linear problem by introducing the variable photon free path, which depends on the radiation density at the point. The new coordinate (X) is connected with the old coordinate (x) by the relation

$$x = X + 2k\int_0^X R(L_{eff}, Y)\,dY, \quad (19)$$

where $R(L_{eff}, X)$ is a solution of equation (2′) for a flat layer, k characterizes the intensity of the radiation from the source, and the effective thickness of the layer L_{eff} is obtained from (19) at x = L and $X = L_{eff}$.

We confine ourselves to the case when $L \gg 1$ and $L_{eff} \gg 1$; it is then possible to use the asymptotic expressions throughout. The quantity L_{eff} is defined by the equation

$$L \approx L_{eff} + 2kS(L_{eff}) \approx L_{eff} + c_1 k L_{eff}^{\gamma}, \quad (20)$$

where c_1 is a constant on the order of unity.

The radiation flux passing through the layer is determined by formula (18) with $L = L_{eff}$. Thus, we can obtain the explicit dependence of the degree of bleaching on the intensity of the incident light (k). To simplify the formulas, we consider the case of strong bleaching

$$k \gg L_{eff}^{1-\gamma} \gg 1.$$

Under this condition we can readily obtain an expression for the transmitted flux j ~ k/L. The dependence of the transmission coefficient on the length is in this case of the diffusion type.

APPENDIX

Let us obtain asymptotic expressions for the solutions in the cases when the medium fills all of space and a half-space.

We consider the homogeneous equation obtained from (6) by integrating with respect to the two transverse coordinates. For a full space we have

$$\int_{-\infty}^{\infty} \frac{n(x)-n(y)}{|x-y|^{1+2\gamma}}\, dy = 0. \tag{A.1}$$

We seek a solution in the form $n(x) \sim x^p$. Introducing $y = xy'$, we obtain an equation for p ($x > 0$):

$$\int_{-\infty}^{\infty} \frac{y'^p - 1}{|1-y'|^{1+2\gamma}}\, dy' = 0. \tag{A.2}$$

Replacing in (A.2) the variable $y' = 1/t$ and adding the obtained expressions, we get

$$\int_{-\infty}^{\infty} \frac{(t^p-1)(t^{2\gamma-1-p}-1)}{|1-t|^{1+2\gamma}}\, dt = 0. \tag{A.3}$$

Since the integrand has a definite sign, Eq. (A.3) has the following two solutions: p = 0 and $p = 2\gamma - 1$. The first of them (n(x) = const) corresponds to the absence of sources, and the second ($n(x) \sim x^{2\gamma-1}$) corresponds to a flat source situated at the point x = 0. (It is then necessary to write $\delta(x)$ in the right side of (A.1).)

In the case of a half-space we obtain in place of (A.1)

$$\int_{0}^{\infty} \frac{n(y)-n(x)}{|x-y|^{1+2\gamma}}\, dy = \frac{x^{-2\gamma}}{2\gamma}\, n(x). \tag{A.4}$$

Substituting $n(x) \sim x^p$, we obtain

$$\int_{0}^{\infty} \frac{y^p - 1}{|1-y|^{1+2\gamma}}\, dy = \frac{1}{2\gamma}. \tag{A.5}$$

Breaking up the integration region into two, (0, 1) and (1, ∞), and making in the second integral the substitution y = 1/z, we obtain

$$\int_0^1 \frac{z^p-1}{(1-z)^{1+2\gamma}}\, dz - \int_0^1 \frac{z^{2\gamma-1}-z^{2\gamma-1-p}}{(1-z)^{1+2\gamma}}\, dz = \frac{1}{2\gamma},$$

or

$$\int_0^1 \frac{z^p-1}{(1-z)^{1+2\gamma}}\, dz - \int_0^1 \frac{z^{2\gamma-1}-1}{(1-z)^{1+2\gamma}}\, dz + \int_0^1 \frac{z^{2\gamma-1-p}-1}{(1-z)^{1+2\gamma}}\, dz = \frac{1}{2\gamma}. \tag{A.6}$$

Equation (A.6) contains three integrals of the same type. To calculate, say, the first of them we note the following. When $\gamma > 0$ this integral cannot be expressed in the form of a difference of two integrals, since each of them diverges. We shall therefore assume $\gamma < 0$, making it possible to break up the integrals and calculate them in explicit form. The obtained expressions are continued analytically to the region $\gamma > 0$. Taking this into account, we get

$$\frac{\Gamma(p+1)\Gamma(-2\gamma)}{\Gamma(p+1-2\gamma)} + \frac{\Gamma(2\gamma-p)\Gamma(-2\gamma)}{\Gamma(-p)} = 0,$$

whence $p = \gamma + n$, where n is an integer. Choosing n from the condition that the initial integral (A.5) be convergent, we get n = 0, −1. The solution $S(x) \sim x^{\gamma}$ corresponds to the absence of sources (source at infinity), and $R(\infty, x) \sim x^{\gamma-1}$ corresponds to a flat source on the boundary of the medium. We see that relation (15) is actually satisfied for these solutions.

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