# *QUANTUM OSCILLATIONS OF THERMODYNAMIC QUANTITIES OF METALLIC FILMS IN WEAK MAGNETIC FIELDS*

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Oscillations of thermodynamic quantities with variation of the magnetic field are studied in the case of specular reflection of the electrons from the film boundary. For weak magnetic fields, when the radius of the electron orbit is much greater than the film thickness, the oscillations permit one to observe the quantum size effect without varying the film thickness. Formulas are obtained for the oscillating part of the thermodynamic potential Ω, the magnetic moment, the magnetic susceptibility and the specific heat; the formulas are valid for arbitrary orientations of the magnetic field H in the film plane. It is shown that the values of the oscillation periods permit one to determine the main curvature radii, the normal and main curvature directions at one of the two points of intersection of the Fermi surface with the extremal chord perpendicular to the film, providing the analogous quantities at the second point are known. The distance between quantum energy levels of a conduction electrons with an extremal chord near the Fermi level can be derived from the temperature dependence of the oscillation amplitude. The condition for the existence of logarithmic singularities of the state density due to quantization of electron energy in the film is derived. The contribution of logarithmic singularities to the oscillations of the magnetic susceptibility and specific heat is analyzed.

## INTRODUCTION

IN investigation of the oscillations of thermodynamic quantities in a magnetic field yields valuable information concerning the structure of the electronic energy spectrum of metal[1]. Additional possibilities arise in the study of magnetic oscillation effects in thin single-crystal films.

The magnetic properties of metal films in a parallel field were considered by Kosevich and I. Lifshitz[2]. They obtained a general formula for the oscillating part of the magnetic moment. It follows from its analysis that when the field decreases to a critical value $H_L$, at which the radius of the electron orbit is of the order of the film thickness L, the period of the oscillations changes noticeably and the amplitude of the oscillations in the film decreases sharply compared with the oscillations in a bulky sample, a fact that can be used to determine the shape of the Fermi surface[3].

In this paper we investigate the region of weak fields, $H \ll H_L$. In this case, without making any additional assumptions concerning the structure of the energy spectrum, it is possible to obtain formulas that describe in greater detail the thermodynamic properties of films in a magnetic field. It is shown that with change of the magnetic field H, the thermodynamic quantities oscillate with an oscillation amplitude on the order of smooth part, and with a period ΔH whose order of magnitude is $\Delta H \sim H_L \mathscr{L}^{-1/2}$, where $\mathscr{L}$ is the number of the atomic layers in the film. A magnetic field of intensity H = ΔH shifts the energy levels of the electron in the film by an amount equal to the distance Δε between the levels. The oscillations under consideration are due to the quantization of the electron energy in the film, and the role of the magnetic field is limited in this case to shifting the system of discrete energy levels that exist in the film.

Formulas were obtained for the period and amplitude of the oscillations, describing the dependence of these quantities on the angle of rotation of the magnetic field intensity H and the plane of the film. This makes it possible to derive, from the experimental investigations of the oscillations in a given region of fields $H \ll H_L$ very detailed information concerning the shape of the Fermi surface in the vicinity of the extremal chord subtending the Fermi surface parallel to the normal to the film. In addition, it may turn out that an investigation of the oscillations will prove the existence of logarithmic singularities of the density of states which, as previously shown by the author[4], arise in some cases upon quantization of the conduction-electron energy in the film. We analyze in detail the condition for the existence of logarithmic singularities of the density of state.

The oscillations are significant in the temperature region $T \lesssim \Delta\epsilon$ at which quantum size effects are usually observed in experiment[5,6].

## QUANSICLASSICAL ENERGY LEVELS

To calculate the oscillations of the thermodynamic quantities it is sufficient to find the energy levels in the quasiclassical approximation.

The formulas for the quasiclassical energy quantization of an electron with an arbitrary dispersion law in a metal film placed in a parallel magnetic field were obtained by Kosevich and I. Lifshitz[2] and can be written in the form

$$\rho(\varepsilon, p_x, p_y; eHL/c) = 2\pi\hbar n/L, \quad n = 1, 2, 3, \dots, \tag{1}$$

where

$$\rho(\varepsilon, p_x, p_y; \Delta p_y) = S(\varepsilon, p_x, p_y; \Delta p_y)/\Delta p_y. \tag{1'}$$

and we put $\Delta p_y = eHL/c$; $S(\epsilon, p_x, p_y; \Delta p_y)$ is the area of the intersection of the equal-energy surface $\mathscr{E}(p) = \epsilon$ with the plane $p_x$ = const, bounded by the straight lines $p_0$ and $p_y + \Delta p_y$ (see Fig. 1). The quantum energy levels $\epsilon_n(p_x, p_y; H, L)$ are determined by formulas (1) and (1′), and when account is taken of the spin they take the form

$$\varepsilon_{ns} = \varepsilon_n(p_x, p_y; H, L) + (-1)^s \cdot {}^1/_2 \mu_0 H, \quad s = 1, 2. \qquad (2)$$

Calculating S under the condition $\Delta p_y \ll |R_i|$, where $R_i$ is the curvature radius in the i-th point (i = 1, 2), which is the intersection point of the straight line $p_x$ = const, $p_y$ = const with the equal energy surface $\mathscr{E}(\mathbf{p}) = \epsilon$, we get from (1′)

$$\rho(\varepsilon, p_x, p_y;\ \Delta p_y) = d(\varepsilon, p_x, p_y) - \frac{1}{2}\Delta p_y\left(\frac{v_{y1}}{v_{z2}} + \frac{v_{y2}}{|v_{z2}|}\right) - \frac{1}{6}\left(\Delta p_y\right)^2\left[\frac{1}{R_1}\left(1 + \frac{v_{y1}^2}{v_{z1}^2}\right)^{3/2} + \frac{1}{R_2}\left(1 + \frac{v_{y2}^2}{v_{z2}^2}\right)^{3/2}\right]. \qquad (3)$$

In the case of electrons, $v_{yi}$ and $v_{zi}$ are the projections of the velocity $v_i = \partial\mathscr{E}/\partial p_i$ at the i-th point; in the case of holes, $v_{yi}$ and $v_{zi}$ are the projections of the velocity with the opposite sign, i.e., $v_i = -\partial\mathscr{E}/\partial p_i$. Since $v_{z1}$ and $v_{z2}$ have different signs, we can put $v_{z1} > 0$ and $v_{z2} < 0$. The signs of the curvature radii $R_i$ are determined relative to the normal $v_{\perp i}$ ($v_{\perp i} = jv_{yi} + kv_{zi}$), such that when $R_i < 0$ the center of curvature is on the normal in the direction of $v_{\perp i}$ from the tangent. The quantity $d(\epsilon, p_x, p_y) = |p_{z1} - p_{z2}|$ is the length of the chord intersecting the equal-energy surface parallel to the normal to the film.

Formulas (1) and (3) determine implicitly the energy levels $\epsilon_n$ in the weak-field region

$$H \ll H_L, \quad H_L = c|R|/eL. \qquad (4)$$

At H = 0 there follow from (1) and (3), naturally, formulas that determine the quantum energy levels of the electron with an arbitrary dispersion law in the film[7].

The main contribution to the oscillations is made by energy levels $\epsilon_n(H)$ at the extremal points, at which the function $\rho(\epsilon, p_x, p_y; eHL/c)$ has an extremum at fixed values of $\epsilon$ and eHL/c. In the case of an isolated extremal point ($\partial\rho/\partial p_x = 0$, $\partial\rho/\partial p_y = 0$) the vectors $v_{\perp 1}$ and $-v_{\perp 2}$ are parallel and $v_{y1}/v_{z1} = -v_{y2}/v_{z2} \equiv v_y/v_z$, so that for the extremal value of $\rho_e(\epsilon, H)$ we get from (1) and (3)

$$\rho_e(\varepsilon, H) = d_e(\varepsilon) - \frac{1}{24}\left(\frac{eHL}{c}\right)^2\left(1 + \frac{v_y^2}{v_z^2}\right)^{3/2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{2\pi\hbar n}{L}, \qquad (5)$$

where $d_e(\epsilon)$ is the extremal length of the chord.

The distance $\Delta\epsilon$ between neighboring quantum levels $\epsilon_n(H)$ is determined from (5), and is given in the vicinity of the fields (4) by

$$\Delta\varepsilon = 2\pi\hbar/L|d_e'(\varepsilon)| = 2\pi\hbar/L(v_{z1}^{-1} + |v_{z2}|^{-1}), \qquad (6)$$

i.e., it coincides with the distance between the levels at H = 0. Thus, the magnetic field shifts the system of levels in the film, without changing the distances between them, by an amount

$$\Delta\varepsilon_H = \frac{1}{24}(d_e'(\varepsilon))^{-1}\left(\frac{eHL}{c}\right)^2\left(1 + \frac{v_y^2}{v_z^2}\right)^{3/2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \qquad (6')$$

The oscillations occur, with increasing field, as soon as the level shift $\Delta\epsilon_H$ becomes larger than the distance $\Delta\epsilon$ between them.

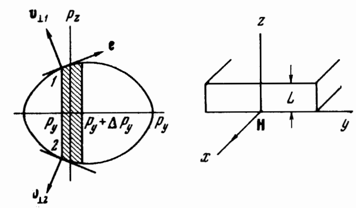


FIG. 1

## CALCULATION OF THE THERMODYNAMIC POTENTIAL Ω

The calculation of the thermodynamic quantities is best started with the determination of the thermodynamic potential Ω from the known formula of statistical physics

$$\Omega = -\frac{VT}{L(2\pi\hbar)^2}\sum_{s=1}^{2}\sum_{n=1}^{\infty}\iint dp_x dp_y \ln\left[1 + \exp\frac{\zeta - \varepsilon_{ns}(p_x, p_y;\ H, L)}{T}\right], \qquad (7)$$

which can be represented in the form

$$\Omega = \tilde{\Omega} + \Omega_{osc}, \qquad (7')$$

where $\tilde{\Omega}$ is a smooth function, by summing (7) by the Poisson formula. The thermodynamic potential is determined here as a function of the external field intensity H, since the difference between H and the magnetic induction B is insignificant in the considered field region $H \ll H_L$, and the inhomogeneity[1)] of the magnetic moment can be neglected (corresponding estimates are made in the analysis of the oscillations of the magnetic susceptibility).

As a result of calculations similar to those of Lifshitz and Kosevich[10], we obtain for the oscillating part of the potential the following expression:

$$\Omega_{osc} = -\frac{2VT}{(2\pi\hbar)^3}\sum_{s=1}^{2}\sum_{k=1}^{\infty}\int d\varepsilon \ln\left(1 + \exp\frac{\zeta_s - \varepsilon}{T}\right) \times \iint dp_x dp_y \frac{\partial\rho}{\partial\varepsilon}\cos\left(k\frac{L\rho}{\hbar}\right), \qquad (8)$$

where $\rho$ is determined by formula (1′),

$$\zeta_s = \zeta - (-1)^s \cdot {}^1/_2 \mu_0 H, \qquad (8')$$

V—volume of film, and $2\pi\hbar$—Planck's constant.

Inasmuch as $L\rho/\hbar \sim \mathscr{L} \gg 1$, the integrals in (8) can be calculated asymptotically. In the case of an isolated extremal point, we obtain as a result

$$\Omega_{osc} = \frac{V}{2\pi^2 L^3}\sum_{s=1}^{2}\left|\frac{\partial\rho}{\partial\zeta_s}\right|^{-1}|J|^{-1/2}\sum_{k=1}^{\infty}\frac{1}{k^3}\Psi\left(2\pi^2 k\frac{T}{\Delta\varepsilon}\right) \times\cos\left[k\frac{L}{\hbar}\rho_e(\zeta_s, H) \pm \frac{\pi}{4} \pm \frac{\pi}{4}\right], \qquad (9)$$

1) The thermodynamic potentials are determined as functions of the magnetic induction **B**, since the field acting on the charges is the magnetic induction, as was first pointed out by Shoenberg [8]. This can lead to effects due to inhomogeneity of the magnetic moment [9].

where the function $\rho_e(\zeta_s, H)$ is determined by formula (5), $\Psi(z) = z/\sinh z$, and $\Delta\epsilon$ is determined by formula (6) at $\epsilon = \zeta_s$,

$$J = [(\partial^2\rho_e/\partial p_x^2)(\partial^2\rho_e/\partial p_y^2) - (\partial^2\rho_e/\partial p_x\partial p_y)^2]_{\zeta_s}. \tag{10}$$

In the argument of the cosine, identical signs are chosen in the case $J > 0$, the plus sign in the case when $\partial^2\rho_e/\partial p_x^2 > 0$ and the minus sign when $\partial^2\rho_e/\partial p_x^2 < 0$, and opposite signs in the case $J < 0$. It follows from (9) and (5) that when the magnetic field changes, the thermodynamic quantities oscillate with a period equal to

$$\Delta(H^2) = \frac{48\pi\hbar c^2}{e^2L^3}\left(1+\frac{v_y^2}{v_z^2}\right)^{-3/2} \times \left|\frac{1}{R_1}+\frac{1}{R_2}\right|^{-1}. \tag{11}$$

The considered oscillations will take place in the field region

$$\Delta H \sim H \ll H_L, \tag{12}$$

where $\Delta H = \sqrt{\Delta(H^2)}$.

In formula (9) we can put $\zeta_s = \zeta$, i.e., we can neglect the spin splitting, since estimates have shown that in the investigated region of fields (12) the spin paramagnetism makes no contribution to the oscillations of the thermodynamic quantities. Oscillations due to spin paramagnetism in films occur in stronger fields (see[11]). In addition, in this region of fields we can neglect the shift of the chemical potential $\zeta$ with changing magnetic field, so that in (9) the value of $\zeta$ is taken at $H = 0$.

## DEPENDENCE OF THE OSCILLATIONS ON THE MAGNETIC FIELD DIRECTION

When the magnetization vector of the magnetic field H is rotated in the plane of the film, the period of the oscillations $\Delta(H^2)$ changes appreciably. Let $(\vartheta, \varphi, \psi_1)$ and $(\vartheta, \varphi, \psi_2)$ be the Euler angles which specify, with respect to a certain basis (i, j, k) reference frames $(\boldsymbol{\nu}_1, \boldsymbol{\tau}_1, \mathbf{v}_1)$ and $(\boldsymbol{\nu}_2, \boldsymbol{\tau}_2, \mathbf{v}_2)$, which in turn determine the principal directions and the outward normal at the points of intersection of the Fermi surface by the chord parallel to the normal k to the film. By virtue of the extremal character of the chord, $\mathbf{v}_1$ is parallel to $-\mathbf{v}_2$, so that $\vartheta_1 = \vartheta_2 \equiv \vartheta$ and $\varphi_1 = \varphi_2 \equiv \varphi$, and by definition $v_{z1} > 0$, $v_{z2} < 0$, and $\cos\vartheta > 0$ (see Fig. 2).

Using the well known differential-geometry relations between the radius of curvature $R_i$ of an arbitrary section with the principal curvature radii $R_{\nu i}$ and $R_{\tau i}$ at the i-th point of the surface, we obtain from (9) and (5) after simple transformations

$$\Omega_{osc} = \frac{V}{\pi^2L^3}\left(\frac{1}{v_{z1}}+\frac{1}{|v_{z2}|}\right)^{-1}|J|^{-1/2}\sum_{k=1}^{\infty}\frac{1}{k^3}\Psi\left(2\pi^2k\frac{T}{\Delta\varepsilon}\right) \times\cos\left\{k\frac{L}{\hbar}d_e \pm\frac{\pi}{4}\pm\frac{\pi}{4} - k\frac{H^2}{24}\frac{e^2L^3}{\hbar c^2} \times[\gamma_0-\gamma_1\cos 2(\alpha-\varphi)-\gamma_2\sin 2(\alpha-\varphi)]\right\}, \tag{13}$$

where $d_e$ is the length of the extremal chord

$$\gamma_0 = \frac{1+\cos^2\vartheta}{2\cos^3\vartheta}(\varkappa_1+\varkappa_2) - \frac{\sin^2\vartheta}{2\cos^3\vartheta}(\delta_1\cos 2\psi_1+\delta_2\cos 2\psi_2),$$

$$\gamma_1 = \frac{\sin^2\vartheta}{2\cos^3\vartheta}(\varkappa_1+\varkappa_2) - \frac{1+\cos^2\vartheta}{2\cos^3\vartheta}(\delta_1\cos 2\psi_1+\delta_2\cos 2\psi_2),$$

$$\gamma_2 = -\frac{1}{\cos^2\vartheta}(\delta_1\sin 2\psi_1+\delta_2\sin 2\psi_2). \tag{14}$$

Here

$$\varkappa_i = \frac{1}{2}\left(\frac{1}{R_{\nu i}}+\frac{1}{R_{\tau i}}\right),\quad \delta_i = \frac{1}{2}\left(\frac{1}{R_{\nu i}}-\frac{1}{R_{\tau i}}\right),\quad i = 1, 2; \tag{15}$$

$\kappa_i$ is the average curvature, and the unit vectors $\boldsymbol{\nu}_i$ and $\boldsymbol{\tau}_i$ are chosen such that $\delta_i \geq 0$.

For J we obtain from (10)

$$J = \frac{1}{\cos^4\vartheta}[(\varkappa_1+\varkappa_2)^2-(\delta_1+\delta_2)^2+4\delta_1\delta_2\sin^2(\psi_1-\psi_2)], \tag{16}$$

where $(\psi_1 - \psi_2)$ is the angle between the principal directions $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$.

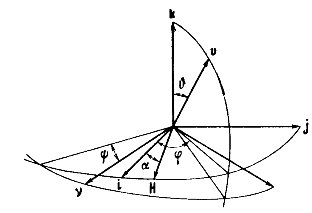


FIG. 2

It follows from (13) that the dependence of the period of the oscillations on the angle $\alpha$ that characterizes the rotation of the vector H in the plane of the film is determined by the formula

$$\Delta(H^2) = 48\pi\frac{\hbar c^2}{e^2L^3}|\gamma_0-\gamma_1\cos 2(\alpha-\varphi)-\gamma_2\sin 2(\alpha-\varphi)|^{-1}, \tag{17}$$

## OSCILLATIONS OF THE MAGNETIC MOMENT, OF THE MAGNETIC SUSCEPTIBILITY, AND OF THE SPECIFIC HEAT

Differentiating $\Omega_{osc}$ with respect to the field, we obtain the oscillating part of the magnetic moment $M_{osc} = -\partial\Omega_{osc}/\partial H$. The component of the moment along the magnetic field is $M_{\parallel osc} = -\partial\Omega_{osc}/\partial H$, while the component perpendicular to H and to the normal k to the film is $M_{\perp osc} = -H^{-1}\partial\Omega_{osc}/\partial\alpha$. Hence, differentiating (13), we obtain

$$M_{\parallel osc} = (\gamma_0-\gamma_1\cos 2(\alpha-\varphi)-\gamma_2\sin 2(\alpha-\varphi))G(H,\alpha,T),$$

$$M_{\perp osc} = (\gamma_1\sin 2(\alpha-\varphi)-\gamma_2\cos 2(\alpha-\varphi))G(H,\alpha,T), \tag{18}$$

where

$$G(H,\alpha,T) = -\frac{Ve^2H}{12\pi^2\hbar c^2}\left(\frac{1}{v_{z1}}+\frac{1}{|v_{z2}|}\right)^{-1}|J|^{-1/2}\sum_{k=1}^{\infty}\frac{1}{k^2}\Psi\left(2\pi^2k\frac{T}{\Delta\varepsilon}\right) \times\sin\left\{k\frac{L}{\hbar}d_e\pm\frac{\pi}{4}\pm\frac{\pi}{4}-k\frac{H^2}{24}\frac{e^2L^3}{\hbar c^2} \times[\gamma_0-\gamma_1\cos 2(\alpha-\varphi)-\gamma_2\sin 2(\alpha-\varphi)]\right\} \tag{18'}$$

Comparing in order of magnitude the amplitudes of the oscillations (18) and (18′) with the smooth part M (for estimates we can use the well known expression[10] for the smooth part of the magnetic moment of a bulky sample), we get[2)]

$$|M_{osc}| \sim |\bar{M}|. \tag{19}$$

2)The same follows from the exact formulas [12] for the magnetic moment of a gas of free electrons in a layer.

We determine analogously the oscillating part of the magnetic susceptibility $\chi_{osc} = \partial M_{\|osc}/\partial H$.

We shall show that at T = 0 the susceptibility $\chi_{osc}$ has logarithmic singularities. As a result of differentiation of (18′) and the summation of the series at T = 0 we obtain

$$\chi_{osc} = \tilde{\chi}_{osc} - V\left(\frac{1}{v_{z1}} + \frac{1}{|v_{z2}|}\right)^{-1} \times |J|^{-1/2}\left[\frac{H}{12\pi}\frac{e^2L^{3/2}}{\hbar c^2}(\gamma_0 - \gamma_1\cos 2(\alpha-\varphi) - \gamma_2 \sin 2(\alpha-\varphi))\right]^2 \times \cos\left(\pm\frac{\pi}{4}\pm\frac{\pi}{4}\right)\ln\left|2\sin\left[\frac{Ld_e}{2\hbar} - \frac{H^2e^2L^3}{48\hbar c^2}(\gamma_0 - \gamma_1\cos 2(\alpha-\varphi) - \gamma_2\sin 2(\alpha-\varphi))\right]\right|, \quad (20)$$

where $\tilde{\chi}_{osc}$ is a certain bounded oscillating function whose explicit form will not be written out. At $J > 0$, identical signs are chosen, so that $\cos(\pm\pi/4 \pm \pi/4) = 0$; at $J < 0$ we have $\cos(\pm\pi/4 \pm\pi/4) = 1$.

Thus, in the case when

$$J < 0 \quad (21)$$

the magnetic susceptibility at T = 0 has logarithmic singularities (see Fig. 3a) at field values $H = H_n$, where

$$H_n^2 = \frac{48\pi\hbar c^2}{e^2L^3}\frac{n + \{Ld_e/2\pi\hbar\}}{\gamma_0 - \gamma_1\cos 2(\alpha-\varphi) - \gamma_2\sin 2(\alpha-\varphi)}, \quad n = 0, \pm 1, \pm 2, \ldots \quad (22)$$

The sign of n is chosen to satisfy the condition $H_n^2 > 0$, and $\{x\}$ is the fractional part of the number x.

Calculating the amplitude of the oscillations $\chi_{osc}$ at the points $H = H_n$ for temperatures $T \ll \Delta\epsilon$ different from zero, we obtain

$$\chi_{osc}(H_n, \alpha, T) = -V\left(\frac{1}{v_{z1}} + \frac{1}{|v_{z2}|}\right)^{-1}|J|^{-1/2}\left[\frac{H_n}{12\pi}\frac{e^2L^{3/2}}{\hbar c^2} \times (\gamma_0 - \gamma_1\cos 2(\alpha-\varphi) - \gamma_2\sin 2(\alpha-\varphi))\right]^2 \ln\left(\frac{T}{\Delta\varepsilon}\right). \quad (23)$$

At sufficiently low temperatures $T \ll \Delta\epsilon$, the dependence of $\chi_{osc}(H, \alpha, T)$ on the magnetic field at $H \neq H_n$ is determined by formula (20), and at $H = H_n$ by formula (23). The amplitude of the oscillations increases in proportion to the square of the field intensity (see Fig. 3b).

According to formulas (18) and (18′), in the considered region of fields $H \ll H_L$ the value of the magnetic moment is much smaller than the period of the oscillations $\Delta H$. In addition, $|M| \ll H$, so that in our case the difference between the magnetic induction **B** and the intensity **H** is insignificant, and the inhomogeneity of the magnetic moment can be neglected (see[9]). It must also be noted that as $T \to 0$ the magnetic susceptibility $\chi(H_n, \alpha T) \to +\infty$ in accordance with formula (23), since the coefficient of the logarithmic singularity is negative. To obtain values of $\chi$ of the order of unity (let alone larger values) it is necessary to have exceedingly low temperatures, which can be determined from the estimate of the magnetic susceptibility per unit volume $\chi \sim 10^{-3}\, v_F c^{-1}\ln(\Delta\epsilon/T)$, where $v_F$ is the electron velocity. In addition, the logarithmic singularities are smeared out as a result of the scattering of the electrons by the impurity[13]. Nonetheless, the oscillations considered here are appreciable compared with the smooth part of the magnetic susceptibility.

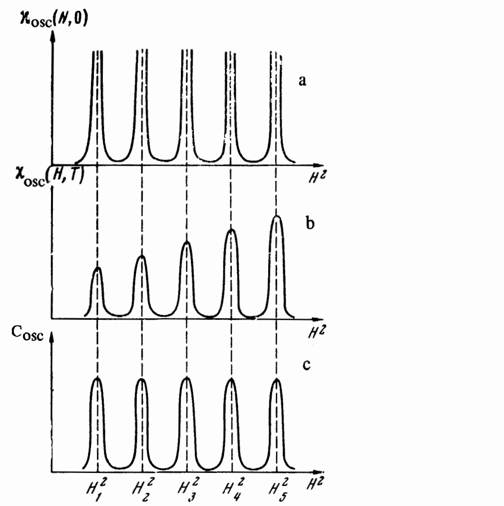


FIG. 3. Oscillations of the magnetic susceptibility $\chi_{osc}$ at T = 0 (a), of $\chi_{osc}$ at $0 < T \ll \Delta\epsilon$ (b), and of the specific heat at $T \ll \Delta\epsilon$ (c); $\Delta\epsilon$ — distance between quantum energy levels.

The oscillating part of the specific heat $C_{osc} = -T\partial^2\Omega_{osc}/\partial T^2$ is determined from (13):

$$C_{osc} = -T\frac{V}{\hbar^2 L}\left(\frac{1}{v_{z1}} + \frac{1}{|v_{z2}|}\right)|J|^{-1/2}\sum_{k=1}^{\infty}\frac{1}{k}\psi''\left(2\pi^2 k\frac{T}{\Delta\varepsilon}\right) \times \cos\left\{k\frac{L}{\hbar}d_e \pm \frac{\pi}{4} \pm \frac{\pi}{4} - k\frac{H^2}{24}\frac{e^2L^3}{\hbar c^2}[\gamma_0 - \gamma_1\cos 2(\alpha-\varphi) - \gamma_2\sin 2(\alpha-\varphi)]\right\}, \quad (24)$$

where

$$\Psi''(z) = \frac{z(1+\mathrm{ch}^2 z)}{\mathrm{sh}^3 z} - \frac{2\,\mathrm{ch}\, z}{\mathrm{sh}^2 z}. \quad (24')$$

At $J < 0$ and at sufficiently low temperatures $T \ll \Delta\epsilon$, summing the series in (24), we obtain

$$C_{osc} = \begin{cases} -\dfrac{TV}{3\hbar^2 L}\left(\dfrac{1}{v_{z1}} + \dfrac{1}{|v_{z2}|}\right)|J|^{-1/2}\cdot\ln\left|2\sin\left[\dfrac{Ld_e}{2\hbar} - \dfrac{H^2}{48}\dfrac{e^2L^3}{\hbar c^2} \times (\gamma_0 - \gamma_1\cos 2(\alpha-\varphi) - \gamma_2\sin 2(\alpha-\varphi))\right]\right|, & H \neq H_n, \\ -\dfrac{TV}{3\hbar^2 L}\left(\dfrac{1}{v_{z1}} + \dfrac{1}{|v_{z2}|}\right)|J|^{-1/2}\ln\left(\dfrac{T}{\Delta\varepsilon}\right), & H = H_n. \end{cases} \quad (25)$$

The amplitude of the oscillations does not depend on the magnetic field (see Fig. 3c). At T = 0, the quantity $C_{osc}/T$, in analogy with the susceptibility $\chi_{osc}$, goes logarithmically to infinity at the points $H = H_n$, this being due to the presence of logarithmic singularities in the density of states.

## LOGARITHMIC SINGULARITIES OF THE DENSITY OF STATES

According to the results obtained in[4], the state density $\nu(\epsilon)$ has logarithmic singularities if the corresponding isochords have hyperbolic points (Fig. 1 of[4]). Isochords are curves in the $(p_x, p_y)$ planes, at the points of which the given equal-energy surface has chords of equal length parallel to the $p_z$ axis. (The z axis is normal to the film.) Certain particular cases of the shape of the equal-energy surface, for which $\nu(\epsilon)$ has logarithmic singularities, were noted in[4]. We present below a more detailed investigation of the topology of the equal-energy surfaces, the shapes of which determine the presence of logarithmic singularities in the density of states.

In the vicinity of the hyperbolic point, the isochords satisfy the equation

$$p_y^2/2m_2 - p_x^2/2m_1 = \varphi(\varepsilon, n),$$

where the $p_x$ axis is the axis of the neck, so that $m_2 > 0$ and $m_1 > 1$.

Calculating $m_i$ and $\varphi$ near the extremal chord $d_e(\epsilon)$, we obtain

$$m_1^{-1} = \sqrt{\gamma_1^2 + \gamma_2^2} - \gamma_0, \quad m_2^{-1} = \sqrt{\gamma_1^2 + \gamma_2^2} + \gamma_0,$$
$$\varphi(\varepsilon, n) = d_e(\varepsilon) - 2\pi\hbar n / L,$$

where the $\gamma_i$ are determined by formulas (14), and $(m_1 m_2)^{-1} = -J$.

Substituting the obtained expressions in formulas (7) or (8) of[4], we obtain the corresponding expression for the density of states:

$$\nu(\varepsilon) = \nu_0(\varepsilon) - \frac{V|J|^{-1/2}}{L(\pi\hbar)^2}\left(\frac{1}{v_{z1}} + \frac{1}{|v_{z2}|}\right)\ln\left|2\sin\frac{Ld_e(\varepsilon)}{2\hbar}\right|, \quad (26)$$

where $\nu_0(\epsilon)$ is a certain smooth function. The same follows from formula (25) at H = 0, using the relation $C = \pi^2\nu T/3$.

At the points $\epsilon = \epsilon_n$, where the quantum energy levels $\epsilon_n$ are determined by the equation

$$d_e(\varepsilon_n) = 2\pi\hbar n / L, \quad n = 1, 2, 3, \ldots, \quad (26')$$

the state density $\nu(\epsilon)$ goes logarithmically to infinity.

From the inequality $m_1 m_2 > 0$, which is characteristic of the hyperbolic point, follows the condition $J < 0$, which with allowance for (16) can be written in the form

$$(\varkappa_1 + \varkappa_2)^2 + 4\delta_1\delta_2 \sin^2(\psi_1 - \psi_2) < (\delta_1 + \delta_2)^2. \quad (21')$$

Let us analyze in greater detail the condition (21′) with respect to the type of points on the equal energy surface that is intersected at these points by the extremal chord, taking into account the fact that $\delta_1 > 0$ and $\delta_2 > 0$ by definition. We denote by $K_i = (R_{\nu i} R_{\tau i})^{-1}$ the Gaussian curvature at the i-th point, i = 1, 2.

## Elliptic Points $K_1 > 0$, $K_2 > 0$

If $\kappa_1\kappa_2 > 0$ (see Fig. 4a), where $\kappa_i$ is the average curvature at the i-th point, the inequality $J > 0$ is always satisfied. On the other hand, if $\kappa_1\kappa_2 < 0$ (see Fig. 4b), then J can be either positive or negative (the specific case J = 0 is not considered in this paper), depending on the satisfaction of the conditions

$$J > 0 \quad \text{for} \quad |\varkappa_1 + \varkappa_2| > \delta_1 + \delta_2, \quad (27)$$

$$J < 0 \quad \text{for} \quad |\varkappa_1 + \varkappa_2| < \delta_1 - \delta_2. \quad (27')$$

In the intermediate case

$$|\delta_1 - \delta_2| < |\varkappa_1 + \varkappa_2| < \delta_1 + \delta_2, \quad (27'')$$

$J > 0$ if $\sin^2(\psi_1 - \psi_2) > s$ and $J < 0$ if $\sin^2(\psi_1 - \psi_2) < s$, where

$$s = [(\delta_1 + \delta_2)^2 - (\varkappa_1 + \varkappa_2)^2] / 4\delta_1\delta_2.$$

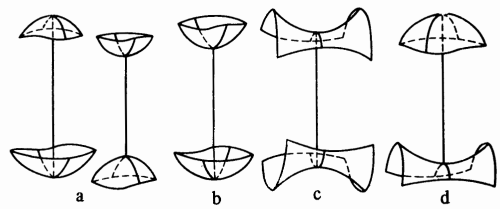


FIG. 4

In the case (27″), the mutual orientation of the principal directions is important.

## Saddle Points: $K_1 < 0$, $K_2 < 0$.

In this case (see Fig. 4c), condition (27) is not satisfied. There can be realized either the condition (27′), when $J < 0$, or the condition (27″), and then, depending on the mutual orientation of the principal directions, $J < 0$ or $J > 0$.

## Elliptic and Saddle Points: $K_1 > 0$, $K_2 < 0$

The conditions (27), (27′), and (27″) can be realized (see Fig. 4d), and accordingly $J < 0$ or $J > 0$.

Thus, with the exception of the case of elliptic points, in which the average curvatures have identical signs (Fig. 4a), in all the remaining cases, at appropriate values of the principal curvature radii and of the mutual orientation of the principal directions of the curvature, the condition $J < 0$ is satisfied, and consequently the density of states has logarithmic singularities. This condition can be formulated in the form of the following criterion: in order for the density of states to have logarithmic singularities, it is necessary that, upon displacement from the extremal point, the corresponding changes of the length of the chord along certain directions have opposite signs. This follows directly from the fact that the quantity $\varphi(\epsilon, n)$, which characterizes the change of the length of the chord in the vicinity of the hyperbolic point, can assume both positive and negative values.

## CONCLUSION

The relative large magnitude of the oscillations investigated in the present paper facilitates their experimental study. We present several estimates for fields and temperatures in which the oscillations are appreciable. For films of thickness $L \sim 10^{-4}$ cm we have $H_L \sim 10^3$ Oe, $\Delta H \sim 10$ Oe, and $\Delta\epsilon \sim 1°$K. Similarly, for $L \sim 10^{-5}$ cm we get $H_L \sim 10^4$ Oe, $\Delta H \sim 10^2$ Oe, and $\Delta\epsilon \sim 10°$K.

There exists a certain optimal film thickness for a given temperature region in which the effect under consideration takes place. If the film is too thick, then the value of $\Delta\epsilon$ will be accordingly small and lower temperatures are needed. To the contrary, if the film is too thick, then the region of fields in which the oscillations are observed is appropriately decreased, since $\Delta H$ increases more rapidly with decreasing film thickness than $H_L$ ($\Delta H \sim L^{-3/2}$, $H_L \sim L^{-1}$).

By determining experimentally the periods of the oscillations in the given regions of fields at different orientations of the intensity in the plane of the film, we can, using the formulas presented in the paper, find the principal curvature radii, the normal, and the principal curvature directions in one of the points where the Fermi surface is intersected by an extremal chord parallel to the normal to the film, provided analogous quantities at the second point are known. The length of the extremal chord, as is well known[14], can be determined from experiments on quantum size effects. Thus, using differently oriented single-crystal films, we can obtain valuable information concerning the shape of the Fermi surface.

An interesting question connected with the clarification of the existence of logarithmic singularities of the state density might be solved by experimentally investigating the oscillations at sufficiently low temperatures, $T \ll \Delta\epsilon$. As seen from the foregoing analysis, depending on the topology of the Fermi surface in the vicinity of the extremal chords, the electron state density has logarithmic singularities due to the quantization of the conduction-electron energy in the film.

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