CONTRIBUTION TO THE THEORY OF LINE WIDTH AND AMPLITUDE FLUCTUATIONS OF A GAS LASER

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Submitted July 25, 1968

Zh. Eksp. Teor. Fiz. 56, 275–289 (January, 1969)

The amplitude fluctuations and line width in a single-mode He-Ne gas laser are calculated for all possible values of the field. The calculations are performed for two field shapes: for a traveling wave (ring laser with unidirectional generation) and a standing wave (linear laser). Part of the calculations pertain to the case of opposite waves in a ring laser. Equations are derived for the field amplitude and phase taking saturation into account. The equations for standing waves differ appreciably from the familiar ones. The calculations also yield expressions for the intensity of the nonequilibrium polarization noise with allowance for dependence on the field strength. The limiting values of the relative amplitude fluctuations are calculated for weak and strong fields. The width of the amplitude fluctuation spectra for traveling and standing waves are different. This fact permits one to explain the experimental results. Correlation of amplitudes of opposite waves in a ring laser observed by Zaĭtsev[3] is investigated. The dependence of the line width on power differs pronouncedly from results obtained previously due to variation of intensity of polarization noise with power. The limiting values of the line width agree with the experimental data.

1. INTRODUCTION

AT the present time there exists a whole series of experimental papers devoted to the investigation of the statistical characteristics of laser emission. Such problems were even investigated in one of the first papers of Javan, Ballik, and Bond.[1] The results of a number of papers are given in the survey by Armstrong and Smith.[2] There are also the later papers of Zaĭtsev,[3] Arecchi, Rodari, and Sona,[4] Zaĭtsev and Stepanov,[5] and Andronova and Zaĭtsev.[6]

There have also been published a large number of theoretical papers on the calculation of the natural line width and amplitude fluctuations in the emission of a gas laser. One of the first papers on these questions is that of Townes and Schawlow.[7] After this appeared the papers of Fleck,[8] Haken,[9] Lamb,[10] Lax,[11] Risken,[12] Bershteĭn, Andronova, and Zaĭtsev,[13] Willis,[14] and Lamb and Scully.[15] The calculations carried out in these papers are valid only for the case of weak fields, when the coupling of the polarization vector with the field can be found by perturbation theory.

We note further, that in almost all of these papers no distinction is drawn between the fluctuations in traveling and standing waves. This distinction exists and leads to a number of novel, interesting effects. It exists because the saturation process in these two cases occurs differently. In addition, the equations for the amplitude and phase in the standing wave regime are dependent even in weak fields, leading to the appearance of an additional term in the expression for the line width, which is proportional to the detuning from the center of the transition line.

The width of the line emitted by a gas laser is due to two factors: equilibrium noise of the free resonator and nonequilibrium fluctuations of the polarization of the working substance (fluctuations due to spontaneous emission of the atoms).

The results of the theoretical papers on line width do not completely agree with each other. But they all, in essence, may be represented in the form of a single general formula, which is given in[9]:

$$\Delta\omega = \frac{(\Delta\omega_p)^2\hbar\omega_0}{4P}\left(\bar{n}+\frac{1}{2}+\frac{1}{2}\frac{\rho_a^{(0)}+\rho_b^{(0)}}{\rho_a^{(0)}-\rho_b^{(0)}}\right). \tag{1.1}$$

Here ω_0 is the frequency of oscillation, $\Delta\omega_r = \omega_0/Q$ is the width of the resonator curve, Q is the quality factor of the resonator, P is the output power, $\bar{n}$ is the average number of photons in the equilibrium state, $\rho_a^{(0)} + \rho_b^{(0)}$ is the sum of the populations of the working levels in the absence of a field, averaged over the Maxwellian distribution, $\rho_a^{(0)} - \rho_b^{(0)}$ is the population difference averaged over the atom velocities. In zero field, it is the same as the threshold value $(\rho_a - \rho_b)_{th}$. The first two terms, $\bar{n} + 1/2$, describe the contribution given by equilibrium fluctuations in the free resonator, and the last is due to nonequilibrium fluctuations of polarization of the working substance.

The result in the form of (1.1) was given in[9]. The majority of papers[7-15] consider either fluctuations in the resonator or fluctuations of polarization. The numerical coefficients in the different papers do not agree.

In correspondence with the initial equations of the theory of the gas laser,[16-18] two parameters, P and $(\rho_a^{(0)} + \rho_b^{(0)})/(\rho_a^{(0)} - \rho_b^{(0)}) \equiv Z$, which are determined by the pump, enter into (1.1). Stating the parameters $\rho_a^{(0)}$ and $\rho_b^{(0)}$ in the initial equations is equivalent to stating P and Z.

Analysis of the experimental data presented in[5] shows that the dependence of line width on power is more complicated than follows from Eq. (1.1). This is because in Eq. (1.1) polarization noise is taken into account only in the zeroth approximation with respect to field. Thus, the problem arises of determining the spectrum of polarization noise with the field dependence taken into account. This is one of the questions considered in this paper. Besides this, we obtain an expression for the limiting value of the line width at the generation threshold. The numerical value of the line width corresponds to the experimental data of[4].

We also calculate the fluctuations of amplitude for all possible values of the output power. The width of the spectrum of amplitude fluctuations for traveling and standing waves turns out to be different. This may explain, in particular the non-correspondence of the experimental data obtained by Andronova and Zaĭtsev[6] with the formula they give, which is valid only for traveling waves or for zero detuning. We also give the results of a calculation of amplitude fluctuations near the generation threshold, where it is already impossible to apply the linearized equations for the fluctuations. In this case it is necessary to use the Fokker-Planck equation to calculate fluctuations and line width. Such calculations were made in[9,11,12,14,15]. These results are extremely interesting, but they do not suffice for determining the line width and relative amplitude fluctuations for small excesses over threshold.

We also investigate the correlation of the amplitudes of opposite waves in a ring laser. The results obtained permit an explanation of the "anti-correlation" (negative correlation) of the opposite waves found experimentally by Zaĭtsev.[3]

2. INITIAL EQUATIONS

For the calculation of fluctuations, we shall start with the equations on which the semi-classical theory of the gas laser is based.[16-18] These are the equations for the elements of the density matrix $\rho_a(\mathbf{r}, v, t)$, ρ_b, ρ_{ab}, ρ_{ba} of the two working levels and the field equation. For our purposes, instead of the equations for the functions ρ_a, ρ_b, ρ_{ab}, and ρ_{ba} it is more convenient to use equations for the functions $D = \rho_a - \rho_b$, $R = \rho_a + \rho_b$ and for the polarization vector $\mathbf{P}(\mathbf{r}, v, t) = en(\mathbf{r}_{ba}\rho_{ab} + \mathbf{r}_{ab}\rho_{ba})$. We may write the equations for these functions, considering we have an isotropic medium, in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + 2\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}\right)D + \frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)R = \frac{2}{\hbar n\omega_{ab}}\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \gamma_{ab}\right)\mathbf{PE} + 2\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}D^{(0)}, \quad (2.1)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + 2\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}\right)R + \frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)D = \frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\frac{2}{\hbar\omega_{ab}n}\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \gamma_{ab}\right)\mathbf{PE} + 2\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}R^{(0)}, \quad (2.2)$$

$$\left[\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \gamma_{ab}\right)^2 + \omega_{ab}^2\right]\mathbf{P} = -2\frac{e^2 n|r_{ab}|^2}{3\hbar}\omega_{ab}D\mathbf{E}. \quad (2.3)$$

Here

$$D^{(0)} = \rho_a^{(0)} - \rho_b^{(0)}, \quad R^{(0)} = \rho_a^{(0)} + \rho_b^{(0)},$$

where n is the concentration of atoms, and $\gamma_a^{-1}, \gamma_b^{-1}, \gamma_{ab}^{-1}$ are the relaxation times of the elements of the density matrix. The other symbols are standard.

We write the field equation in the form

$$\frac{\partial^2 E}{\partial t^2} + \frac{\omega_0}{Q}\frac{\partial E}{\partial t} - c^2\frac{\partial^2 E}{\partial x^2} = -4\pi\frac{\partial^2}{\partial t^2}(\mathbf{eP}) + \omega_0^2(\mathbf{eE})^{(T)}; \quad (2.4)$$

$E^{(T)}$ has its source in thermal noise, and **e** is a unit vector along the field. In the noise calculation, the functions D, R, and **P** have the form

$$D = D^{(i)} + D^{(0)} + D^{(n)},\ R = R^{(i)} + R^{(0)} + R^{(n)},\ \mathbf{P} = \mathbf{P}^{(i)} + \mathbf{P}^{(n)}. \quad (2.5)$$

The superscripts "i" and "n" refer to the induced and noise portions of the corresponding quantities.

The calculation of the induced part of the polarization vector is carried out for traveling and standing waves with saturation taken into account. The shortened equations for the amplitude and phase of the field may be written

$$\frac{dE_0}{dt} = \frac{\omega_0 d}{2}\left\{f(aE_0^2, \mu)[1 + F(aE_0^2, \mu)] - \frac{1}{Qd}\right\}E_0 + \omega_0\xi_1(t), \quad (2.6)$$

$$\frac{d\varphi}{dt} = \frac{\omega_0 d}{2}\mu f(aE_0^2, \mu)[1 - F(aE_0^2, \mu)] + \frac{\omega_0}{E_0}\xi_2(t). \quad (2.7)$$

Here $\mu = (\omega_0 - \omega_{ab})/\gamma_{ab}$ is the detuning, $a = e^2|\mathbf{r}_{ab}|^2(\gamma_a + \gamma_b)/6\hbar^2\gamma_a\gamma_b\gamma_{ab}$ is the saturation parameter, and Q is the quality factor of the resonator. The pump parameter d and functions f and F turn out to be different depending on the operating regime of the laser. In the traveling wave regime, when the field E is given in the form $E(x, t) = E_0\cos(\omega_0 t - k_0 x + \varphi)$, we have

$$f = 1/2\sqrt{1 + aE_0^2}, \quad F = 1, \quad (2.8)$$

$$d = \frac{4\pi^2 e^2 n|r_{ab}|^2}{3\hbar\sqrt{2\pi}k_0 u}D^{(0)}\exp\left\{-\frac{\mu^2\gamma_{ab}^2}{2k_0^2u^2}\right\}, \quad (2.9)$$

$$\xi_{1,2} = -\frac{1}{V}\int[\mathbf{e}(4\pi\mathbf{P}^{(n)} + \mathbf{E}^{(T)})]\sin(\omega_0 t - k_0 x + \varphi_{1,2})dV. \quad (2.10)$$

Here u is the average thermal velocity, V is the resonator volume, $\varphi_1 = \varphi$, and $\varphi_2 = \varphi + \pi/2$.

In the standing wave regime, for a field given in the form

$$E(x, t) = E_0\cos(\omega_0 t + \varphi)\cos k_0 x = {}^1/_2 E_0[\cos(\omega_0 t - k_0 x + \varphi) + \cos(\omega_0 t + k_0 x + \varphi)]$$

we get

$$f = 2^{-1/2}\left\{1 - \mu^2 + \frac{aE_0^2}{4} + \left[(\mu^2 + 1)\left(\mu^2 + 1 + \frac{aE_0^2}{2}\right)\right]^{1/2}\right\}^{-1/2}, \quad (2.11)$$

$$F = \left\{(\mu^2 + 1)\Big/\left(\mu^2 + 1 + \frac{aE_0^2}{2}\right)\right\}^{1/2},$$

$$d = \frac{4\pi^2 e^2 n|r_{ab}|^2}{3\hbar\sqrt{2\pi}k_0 u}D^{(0)}\exp\left\{-\frac{\mu_s^2\gamma_{ab}^2}{2k_0^2u^2}\right\}. \quad (2.12)$$

In the last equation

$$\mu_s^2 = \begin{cases}\mu^2 - 1 + {}^1/_4 f^{-2} & \text{for } E_0 \leqslant E_{cr}\\ 0 & \text{for } E_0 \geqslant E_{cr}\end{cases}, \quad E_{cr} = \left\{\frac{8\mu(\mu + \sqrt{1+\mu^2})}{a}\right\}^{1/2}; \quad (2.13)$$

$$\xi_{1,2}(t) = -\frac{1}{V}\int\sum_{\pm}[\mathbf{e}(\mathbf{E}^{(T)} + 4\pi\mathbf{P}^{(n)})]_{\pm k_0}\sin(\omega_0 t \mp k_0 x + \varphi_{1,2})dV. \quad (2.14)$$

We note that in the zeroth approximation in field $\mu_s = \mu$. As the field increases, μ_s decreases. Equation (2.13) defines the critical value of the field at which μ_s goes to zero.

Below we shall require the stationary solution of Eq. (2.6) in the absence of fluctuations. It is determined from the condition

$$\omega_0 df(a\bar{E}_0^2, \mu)[1+F(a\bar{E}_0^2, \mu)] = \frac{\omega_0}{Q} \equiv \Delta\omega_r. \quad (2.15)$$

The results given in this section were obtained with the assumption of an inhomogeneously broadened line, i.e., under the condition $\gamma_{ab} \ll k_0u$. It can be shown that under this condition the contribution of the second harmonics of the population difference in the expression for the polarization vector is small (of order γ_{ab}/k_0u). This gives us the basis for neglecting the contribution of the second harmonics. An analogous approximation is used also in the following.

3. NOISE INTENSITY

From Eqs. (2.10) and (2.14) follow expressions for the noise intensity for traveling and standing waves, respectively:

$$N_\omega = \langle\xi_1^2\rangle_\omega = \langle\xi_2^2\rangle_\omega = \frac{1}{2V}[\langle(\mathbf{eE}^{(T)})_\perp^2\rangle_{\omega_0, k_0} + (4\pi)^2\langle(\mathbf{eP}^{(n)})^2\rangle_{\omega_0, k_0}], \quad (3.1)$$

$$2N_\omega = \langle\xi_1^2\rangle_\omega = \langle\xi_2^2\rangle_\omega = \frac{1}{2V}\sum_\perp[\langle(\mathbf{eE}^{(T)})^2\rangle_{\omega_0, \pm k_0} + (4\pi)^2\langle(\mathbf{eP}^{(n)})^2\rangle_{\omega_0, \pm k_0}],$$

$$\langle\xi_1\xi_2\rangle_\omega = \langle\xi_2\xi_1\rangle_\omega = 0. \quad (3.2)$$

The thermal part of the noise is determined from the Callen-Wellton equation. In the calculation for one polarization

$$\frac{1}{V}\langle(\mathbf{eE}^{(T)})^2\rangle_{\omega_0, \pm k_0} = \frac{8\pi\hbar\Delta\omega_r}{V\omega_0}\left(\bar{n}+\frac{1}{2}\right), \quad \Delta\omega_r = \frac{\omega_0}{Q}. \quad (3.3)$$

The polarization noise is not in equilibrium, and the determination of its intensity is in general a rather complex problem. Here we proceed as follows. In correspondence with (2.5) we separate the initial equations (2.1) to (2.3) into two systems of equations respectively for the induced and noise parts. For a generation region corresponding to excesses over threshold that are not too small, we may leave out terms containing the noise part of the field **E** in the equations for the functions $D^{(n)}$, $R^{(n)}$, and $\mathbf{P}^{(n)}$, i.e., there remain only terms with the average field. Near threshold terms containing the field may generally be left out, because of the smallness of the total field.

Because of the linearity of the initial equations with respect to the functions P, D, and R, the equations for the functions $D^{(i)} + D^{(0)}$, $R^{(i)} + R^{(0)}$, and $\mathbf{P}^{(i)}$ coincide with the initial equations. The equations for $D^{(n)}$, $R^{(n)}$, and $\mathbf{P}^{(n)}$ also coincide in form with the initial ones, if we set $D^{(0)}$ and $R^{(0)}$ equal to zero in them.

Using the equations for $D^{(n)}$, $R^{(n)}$, and $\mathbf{P}^{(n)}$ it is possible to express the space-time spectral function $\langle\mathbf{P}^2(v)\rangle_{\omega,k}^{(n)}$ in terms of the space-time spectral density of the fluctuations of polarization at the same time

$$\langle\mathbf{P}^2(v)\rangle_k^{(n)} = \frac{1}{2\pi}\int\langle\mathbf{P}^2(v)\rangle_{\omega,k}^{(n)} d\omega.$$

As a result of calculations for the regime of traveling waves we obtain the following expression

$$\langle\mathbf{P}^2(v)\rangle_{\omega_0,k_0}^{(n)} = \frac{2(1+aE_0^2/2)}{\Gamma_1}\langle\mathbf{P}^2(v)\rangle_{k_0}^{(n)},$$

$$\Gamma_1 = \gamma_{ab}\left[\left(\mu-\frac{k_0v}{\gamma_{ab}}\right)^2 + 1 + aE_0^2\right]. \quad (3.4)$$

The corresponding expression for standing waves has the form

$$\sum_\pm\langle\mathbf{P}^2(v)\rangle_{\omega_0,\pm k_0}^{(n)} = \frac{4}{\Gamma_2}\left\{\left[1+\mu^2+\left(\frac{k_0v}{\gamma_{ab}}\right)^2\right]\times\left(1+\frac{aE_0^2}{8}\right)+\frac{aE_0^2}{4}\right\}\langle\mathbf{P}^2(v)\rangle_{k_0}^{(n)}, \quad (3.5)$$

$$\Gamma_2 = \left\{\left[\left(\mu-\frac{k_0v}{\gamma_{ab}}\right)^2+1\right]\left[\left(\mu+\frac{k_0v}{\gamma_{ab}}\right)^2+1\right] + \frac{aE_0^2}{2}\left[\mu^2+\left(\frac{k_0v}{\gamma_{ab}}\right)^2+1\right]\right\}\gamma_{ab}.$$

Thus we see that the problem of determining the spectral density of polarization noise reduces in the final analysis to a determination of the spatial correlation. The equations for these correlation functions contain as sources terms which are determined by the induced part R, i.e., by the average values of the population of the levels. Because of this it is possible to express the function $\langle\mathbf{P}^2(v)\rangle_{k_0}^{(n)}$, and consequently the desired polarization noise intensity, in terms of the average values of the populations ρ_a, ρ_b. However, since the initial equations for the elements of the density matrix are themselves semi-phenomenological, then in the framework of this theory it is most natural to determine the spatial correlation of the elements of the density matrix from the expression

$$\langle\delta\rho_{nm}(v)\delta\overset{*}{\rho}_{n'm'}(v')\rangle_k = \frac{\delta_{nn'}\delta_{mm'}\delta(v-v')}{2n}(\rho_n+\rho_m). \quad (3.6)$$

This determination corresponds to the supposition that the dependence of the spatial correlation on field is only through the field dependence of the function $\rho_n + \rho_m \equiv R^{(i)} + R^{(0)}$. Thus the question of the subsequent determination of the simultaneous correlation function remains open here.

From (3.6) we have

$$\langle\mathbf{P}^2(v)\rangle_k^{(n)} = \frac{e^2n|\mathbf{r}_{ab}|^2}{3}\langle\delta\rho_{ab}^2(v)\rangle_k = \frac{e^2|\mathbf{r}_{ab}|^2n}{6}[R^{(i)}(v)+R^{(0)}(v)]. \quad (3.7)$$

In order to find an explicit expression for the polarization noise, it is necessary to express $R^{(i)}$ via the parameters $D^{(0)}$, aE_0^2 and to integrate over velocity. For traveling waves

$$R^{(i)} = -\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}aE_0^2\frac{\gamma_{ab}}{\Gamma_1}D^{(0)}, \quad (3.8)$$

and for standing waves

$$R^{(i)} = -\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\frac{aE_0^2}{2}\frac{\gamma_{ab}[1+\mu^2+(k_0v/\gamma_{ab})^2]}{\Gamma_2}D^{(0)}. \quad (3.9)$$

Integrating (3.4) and (3.5) over the velocities and substituting the expressions obtained into (3.1) and (3.2) we obtain, using (3.3), an expression for the noise intensity N_ω:

$$N_\omega = \frac{4\pi\hbar\Delta\omega_p}{V\omega_0}\left\{\bar{n}+\frac{1}{2}+\frac{1}{2}f_1(aE_0^2,\mu)\times\left[Z-\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}j_2(aE_0^2,\mu)\right]\right\}, \quad (3.10)$$

where

$$Z = R^{(0)}/D^{(0)} \equiv (\rho_a^{(0)} + \rho_b^{(0)})/(\rho_a^{(0)} - \rho_b^{(0)}).$$

For traveling waves

$$f_1 = 1 + aE_0^2/2, \quad f_2 = \frac{1}{2}\frac{aE_0^2}{1+aE_0^2}. \tag{3.11}$$

For standing waves

$$f_1 = 1 + \frac{aE_0^2}{8} + \frac{aE_0^2}{4(1+\mu^2)\{1+[(\mu^2+1+aE_0^2/2)/(\mu^2+1)]^{1/2}\}} \tag{3.12}$$

$$f_2 = \frac{aE_0^2}{4}\left\{f^2(1+F) + \frac{1}{1+\mu^2+aE_0^2/2}\times\left[1-\left(1+\frac{aE_0^2}{8}\right)f_1^{-1}(1+F)^{-1}\right]\right\}.$$

Here f and F are determined from (2.11). For $aE_0^2 \ll 1$,

$$f_1 = 1 + \frac{aE_0^2}{8}\frac{\mu^2+2}{\mu^2+1}, \quad f_2 = \frac{aE_0^2}{8}\frac{\mu^2+2}{\mu^2+1}; \tag{3.13}$$

for $aE_0^2 \gg 1$,

$$f_1 = aE_0^2/8, \quad f_2 = 1/2. \tag{3.14}$$

In the zeroth approximation with respect to aE_0^2 the expressions for N_ω in the regimes of traveling and standing waves agree. In the first approximation with respect to aE_0^2, a dependence on the detuning μ appears in the formula for the standing wave.

In Eq. (3.10) for the noise intensity, the field dependence appears not only explicitly, but also through the parameters $\Delta\omega_r$ and Z. In order to establish this dependence, it is necessary to concretize the means of varying the field amplitude. The field may be changed by changing either the pumping or the resonator losses. The latter means was used in the experiments of Zaĭtsev and Stepanov.[5] It is not possible to determine the character of the variation of the parameter Z with a variation in pumping in the framework of the initial equations. It is possible only to assume that Z diminishes with increasing aE_0^2. Actually, from the excitation conditions for the laser it follows that in the stationary regime (see Eq. (2.6)) we have $\rho_a^{(0)} - \rho_b^{(0)} = D_{th}^{(0)} f^{-1}(1+F)^{-1}$. Here $D_{th}^{(0)}$ is the threshold value of the population difference. Let us consider two cases:

(1) The sum of the populations is constant, i.e., $\rho_a^{(0)} + \rho_b^{(0)}$ = const. (two-level model). Thus

$$Z = \frac{\text{const}}{D_{th}^{(0)}} f(1+F). \tag{3.15}$$

(2) The population of the lower level is independent of the field, i.e., $\rho_b^{(0)}$ = const. Then

$$Z = 1 + \frac{2\rho_b^{(0)}}{D_{th}^{(0)}} f(1+F). \tag{3.16}$$

It is easy to see that the function f(1 + F) always decreases with increasing field, and consequently so does the parameter Z in both cases.

It is just this second assumption that is made in a number of papers. For example, Willis[14] and Lamb and Scully[15] set $\rho_b^{(0)} = 0$, i.e., Z = 1. This assumption is the most natural only for sufficiently high values of the field, when there is already saturation. However, saturation was not really taken into account in[14,15] and in the other papers.

Since in the experimental papers[3,5,6] the field was varied by varying the resonator losses, we shall consider this particular case in more detail.

The dependence of the field on $\Delta\omega_r$ is determined from Eq. (2.15). Substituting (2.15) into (3.10), we obtain the explicit dependence of noise intensity on field. Analysis of this dependence shows that in weak field the noise intensity decreases with increasing field and is proportional to the field in strong field, i.e.,

$$N_\omega = \frac{4\pi\hbar d}{V}\left\{\bar{n} + \frac{1}{2} + \frac{1}{2}Z - \left(\bar{n} + \frac{1}{2} + \frac{1}{2}\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\right)\alpha\beta\frac{aE_0^2}{4}\right\}, \quad aE_0^2 \ll 1,$$

$$N_\omega = \frac{\pi\hbar d\beta}{2\sqrt{2}\,V}\left(Z - \frac{1}{2}\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\right)\sqrt{\beta aE_0^2}, \quad aE_0^2 \gg 1. \tag{3.17}$$

Here $\alpha = 1$, $\beta = 2$ for traveling waves, $\alpha = (\mu^2+2)/2(\mu^2+1)$, $\beta = 1$ for the standing wave case.

If the power changes due to the pump with a constant value of $\Delta\omega_r$, then in a large field N_ω is determined from

$$N_\omega = \frac{\pi\hbar\Delta\omega_p}{8V\omega_0}\frac{3\gamma_a+\gamma_b}{\gamma_a+\gamma_b}\beta^2 aE_0^2, \quad aE_0^2 \gg 1, \tag{3.18}$$

In concluding this section, we remark that the expression for the noise intensity in zero field agrees with the results of other authors to within a numerical factor.

4. AMPLITUDE FLUCTUATIONS

For excesses over threshold that are not too small (this condition will be made more exact below), Eqs. (2.6) and (2.7) can be linearized and the correlation approximation used to calculate the fluctuations. Setting $E_0 = \bar{E}_0 + \delta E$, $\varphi = \varphi_0 + \delta\varphi$ and expanding over δE, $\delta\varphi$, we obtain

$$\frac{d\delta E}{dt} = \frac{\omega_0 d}{2}\{f'_{aE_0^2/2}(1+F) + fF'_{aE_0^2/2}\}\,a\bar{E}_0^2\,\delta E + \omega_0\xi_1(t), \tag{4.1}$$

$$\frac{d\delta\varphi}{dt} = \frac{\omega_0 d}{2}\mu\,\{f'_{aE_0^2/2}(1-F) - fF_{aE_0^2/2}\}\,a\bar{E}_0^2\frac{\delta E}{\bar{E}_0} + \omega_0\frac{\xi_2(t)}{\bar{E}_0}. \tag{4.2}$$

Here $\bar{E}_0$ is the amplitude of the stationary oscillations without noise.

We introduce the parameter η, which will characterize the excess of pump power over threshold, and define it in the following way:

$$\eta = \frac{4\pi aP}{V\Delta\omega_r} \equiv \gamma P,$$

where P is the power of the oscillation. With this definition, we have $\eta = a\bar{E}_0^2/2$ for the traveling wave $(P = (\bar{E}_0^2/8\pi)V\Delta\omega_r)$, and $\eta = a\bar{E}_0^2/4$ for the standing wave $(P = (\bar{E}_0^2/16\pi)V\Delta\omega_r)$.

It follows from (4.1) that the spectrum of amplitude fluctuations has the form

$$\langle\delta E^2\rangle_\omega = \frac{\omega_0^2\langle\xi_1^2\rangle_{\omega_0}}{\omega^2+(\Delta\omega_a)^2}. \tag{4.3}$$

Here $\Delta\omega_a$ is the width of the spectrum and is equal to

$$\Delta\omega_a = \frac{\Delta\omega_r}{2}\frac{2\eta}{1+2\eta+2(\mu^2+1)(1-\alpha)}[1+8(\mu^2+1)(1-\alpha)f^2/(1+F)]. \tag{4.4}$$

It follows from (4.4) that for the standing wave

$$\Delta\omega_a = \frac{\Delta\omega_r}{2}\frac{2\eta}{1+2\eta} \quad \text{for } \mu \ll 1,$$

$$\Delta\omega_a = \frac{\Delta\omega_r}{2}\frac{2\eta}{2+2\eta} \quad \text{for } \mu \gg 1, \quad \mu \gg \eta. \tag{4.5}$$

From this it is seen that, depending on detuning, the width of the spectrum of amplitude fluctuations can change by a factor of two at small powers. This can perhaps explain the discrepancy between the experimental data and those calculated in[6] for 3.39-μ waves. At higher powers, the dependence of $\Delta\omega_a$ on detuning is weaker. We remark that to develop the full dependence of the width of the spectrum of amplitude fluctuations on power (or η), it is necessary to eliminate $\Delta\omega_r$ from (4.4) and (4.5) by means of (2.15). Of course, this cannot be done if the experimental conditions are such that $\Delta\omega_r$ remains constant as the power is changed.

Integrating (4.3) over frequency, we obtain an expression for the mean square deviation of the field amplitude:

$$\langle \delta E^2\rangle = \frac{\omega_0^2}{2\Delta\omega_a}\langle \xi_1^2\rangle_{\omega_0}.$$

Using the asymptotic expressions for the noise intensity (3.17) we obtain from this in the two limiting cases:

$$\langle \delta E^2\rangle = \frac{N_0^2}{\beta\eta a}, \quad \eta \ll 1; \tag{4.6}$$

$$\langle \delta E^2\rangle = \frac{2\pi\hbar\omega_0}{V}\left(Z - \frac{1}{2}\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\right)\eta, \quad \eta \gg 1. \tag{4.7}$$

In (4.6) we have used the notation

$$N_0 = (N^{(0)}\omega_0 a / \alpha d)^{1/2}, \tag{4.8}$$

where

$$N^{(0)} = N_\omega|_{\eta=0} = \frac{4\pi\hbar d}{V}\left\{\bar{n} + \frac{1}{2} + \frac{1}{2}Z\right\} \tag{4.9}$$

is the noise intensity in zeroth approximation with respect to field.

Thus, considering that $\eta = a\bar{E}_0^2/4$, we obtain the following expressions for the relative amplitude fluctuations:

$$\langle \delta E^2\rangle / \bar{E}_0^2 = N_0^2/4\eta^2, \quad \eta \ll 1; \tag{4.10}$$

$$\langle \delta E^2\rangle / \bar{E}_0^2 = \frac{\pi\hbar\omega_0\beta a}{2V}\left(Z - \frac{1}{2}\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\right) \quad \eta \gg 1. \tag{4.11}$$

We remark that Eqs. (4.7) and (4.11) remain valid also when the power is changed by the pump, if we set Z = 1 in them. Equations (4.6) and (4.10), on the other hand, are independent of the means of changing the power.

It follows from (4.10) that the above linearization method is applicable under the condition $N_0 \ll 2\eta$ (calculation shows that $N_0 \ll 1$). For small values of the excess, we can make use of the method of the Fokker-Planck equation to calculate the amplitude and phase fluctuations. For this, we at once set $aE_0^2 \ll 1$ in Eq. (2.6). Then this equation can be written

$$\frac{dE_0}{dt} = \frac{\omega_0 d}{2}\alpha\left(\eta - \frac{1}{4}\beta a E_0^2\right)E_0 + \omega_0\xi_1(t).$$

The corresponding Fokker-Planck equation for the probability density of the amplitude (see[19,20]) is

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial E_0}\left\{\left[\frac{\omega_0 d}{2}\alpha\left(\eta - \frac{1}{4}\beta a E_0^2\right)E_0 + \frac{\langle\xi_1^2\rangle_\omega\omega_0^2}{2E_0}\right]w\right\} + \frac{\langle\xi_1^2\rangle_\omega\omega_0^2}{2}\frac{\partial^2 w}{\partial E_0^2}. \tag{4.12}$$

One needs the nonstationary solution to Eq. (4.12) to calculate the spectrum of fluctuations. However, we give here only the stationary solution, which is sufficient for determining the moments of the distribution. It has the form (see[19,20])

$$w(E_0) = \sqrt{\frac{2}{\pi}}\frac{\beta a}{2N_0}\left[1 + \Phi\left(\frac{\eta}{N_0}\right)\right]^{-1} E_0 \exp\left\{-\frac{\beta^2 a^2}{32 N_0^2}\left(E_0^2 - \frac{4\eta}{\beta a}\right)^2\right\}. \tag{4.13}$$

From this we find the moments

$$\langle E_0\rangle = \sqrt{\frac{2N_0}{\beta a}}\left[1 + \Phi\left(\frac{\eta}{N_0}\right)\right]^{-1}\exp\left\{-\frac{\eta^2}{4N_0^2}\right\}D_{-1/2}\left(-\frac{\eta}{N_0}\right),$$

$$\langle E_0^2\rangle = \frac{4\eta}{\beta a}\left\{1 + \sqrt{\frac{2}{\pi}}\frac{N_0}{\eta}\left[1 + \Phi\left(\frac{\eta}{N_0}\right)\right]^{-1}\exp\left(-\frac{\eta^2}{2N_0^2}\right)\right\}.$$

Here $D_\nu(z)$ is the function of a parabolical cylinder.

We consider two limiting cases:

(1) $N_0 \ll \eta \ll 1$ (correlation approximation valid). Here

$$\langle E_0\rangle \approx 2\sqrt{\eta/\beta a}(1 - N_0^2/8\eta^2), \quad \langle E_0^2\rangle = (4\eta/\beta a)[1 + o(\exp(-\eta^2/2N_0^2))].$$

Consequently, $\langle \delta E^2\rangle = N_0^2/\beta a\eta$, which agrees with (4.6), which was calculated by the correlation approximation.

(2) $\eta \ll N$. Then

$$\langle E_0\rangle = \frac{2(8\pi^2)^{1/4}}{\Gamma(1/4)}\sqrt{\frac{N_0}{\beta a}}\left[1 - \left(\sqrt{\frac{2}{\pi}} - \frac{\Gamma^2(1/4)}{4\pi}\right)\frac{\eta}{N_0}\right],$$

$$\langle E_0^2\rangle = 4\sqrt{\frac{2}{\pi}}\frac{N_0}{\beta a} + 4\frac{\eta}{\beta a}\left(1 - \frac{2}{\pi}\right), \tag{4.14}$$

whence

$$\langle \delta E^2\rangle / \langle E_0^2\rangle = 1 - \frac{2\pi\sqrt{\pi}}{\Gamma^2(1/4)} - \left[\sqrt{2} - \frac{2\sqrt{\pi}}{\Gamma^2(1/4)}(\pi+2)\right]\sqrt{\frac{\pi}{2}}\frac{\eta}{N_0} \approx 0{,}16 - 0{,}05\frac{\eta}{N_0}. \tag{4.14'}$$

It is interesting that the relative dispersion of the amplitude as $\eta/N_0 \to 0$ tends to a definite finite limit that is independent of the magnitude of noise and equals approximately 0.16. A typical graph of the dependence of the relative dispersion of the amplitude on the ratio of excess to noise is given in Fig. 1 by the solid line. The dashed lines show the asymptotes represented by (4.11) and (4.14′).

Finally, we write the distribution for the number of photons $n_{ph} = \beta E_0^2 V/16\pi\hbar\omega_0$, which follows directly from Eq. (4.13):

$$w(n_{ph}) = \sqrt{\frac{2}{\pi}}\frac{4\pi\hbar\omega_0 a}{VN_0}\left[1 + \Phi\left(\frac{\eta}{N_0}\right)\right]^{-1}\exp\left\{-\frac{(n_{ph} - n_0)^2}{2\sigma^2}\right\}. \tag{4.15}$$

Here

$$\sigma^2 = \frac{V}{4\pi a\hbar\omega_0 a}\left\{\bar{n} + \frac{1}{2} + \frac{1}{2}Z\right\}; \quad n_0 = \frac{\eta}{4\pi\hbar\omega_0 a}V. \tag{4.16}$$

It is interesting to note that the difference in the distributions w for the standing wave and traveling wave regimes appears only through the parameter α, which depends on the detuning μ. When $\mu = 0$, we have $\alpha = 1$, and the distribution w is the same for both regimes.

Let us compare these results with those of[9,11,12,14,15]. Haken,[9] Risken,[12] and Lax[11] give distributions that agree in form with (4.13). However, the expressions for the parameters of the distribution in these papers either are not calculated or do not take into account the dependence of the noise intensity on detuning.[9] These papers also do not give explicit ex-

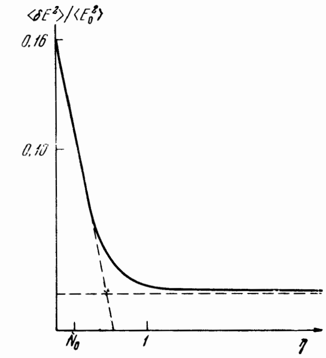

FIG. 1

pressions for the moments of the distribution. Willis[14] obtained an expression for $w(n_{ph})$ which agrees with (4.15) as far as its dependence on n_{ph} is concerned, but the values of the parameters σ^2 and n_0 differ from (4.16). In addition, the normalization of the distribution $w(n_{ph})$ is carried out incorrectly in[14] (without taking into account that n_{ph} can only be positive). The distribution for $w(n_{ph})$ was also considered by Lamb and Scully[15]; however, the model they chose to use does not correspond completely with ours, and hence comparison of the parameters of the distribution is made difficult. Finally, we note that in these papers no concrete results are given for the case of large fields, when saturation exists.

5. AMPLITUDE FLUCTUATIONS IN A RING LASER

In an analogous fashion it is possible to carry out a calculation of the amplitude fluctuations in a ring laser. One of the essential differences in this case is that the amplitudes of the opposite waves are in general different. It is natural to expect that the intensities of the opposite waves are correlated between fluctuations. This was found experimentally by Zaĭtsev.[3]

We present here the results of a calculation of the correlation coefficient for the intensities of the opposite waves, which we define in the following way:

$$\rho_\omega = \frac{\langle(\delta(E_1^2)+\delta(E_2^2))^2\rangle_\omega - \langle(\delta(E_1^2)-\delta(E_2^2))^2\rangle_\omega}{\langle(\delta(E_1^2)+\delta(E_2^2))^2\rangle_\omega + \langle(\delta(E_1^2)-\delta(E_2^2))^2\rangle_\omega}. \tag{5.1}$$

Here $\delta(E^2)$ is the intensity fluctuation. Assuming small field, we obtain the following expression for ρ_ω:

$$\rho_\omega = -2\frac{1+\mu^2}{1+(1+\mu^2)^2}\frac{(\Delta\omega)^2}{\omega^2+(\Delta\omega)^2}, \quad \mu \gg \gamma/ku, \tag{5.2}$$

$$\Delta\omega = \frac{\omega_0\eta}{Q}\frac{\sqrt{1+(1+\mu^2)^2}}{2+\mu^2}. \tag{5.3}$$

Equations (5.2) and (5.3) are valid in the absence of correlation between sources of noise in the equations for the intensities of the opposite waves. In the presence of correlation of noise, there appear in Eq. (5.2) additional terms, which, as $\omega \to \infty$, make ρ_ω tend, not to zero, as it does in (5.2), but to the correlation coefficient between the noise sources.

The function ρ_ω was measured by Zaĭtsev.[3] The frequency dependence of the correlation coefficient he observed agrees with that given by (5.2). Let us consider ρ_ω as a function of the detuning μ when $\omega = 0$. It follows from (5.2) that as μ increases the quantity $|\rho_0|$ monotonically falls from 1 to 0. When $\mu \sim 1$, we obtain $|\rho_0| \sim 0.8$; but if $\mu \gg 1$, then $|\rho_0| \sim 1/\mu^2$. Unfortunately, the dependence of ρ_0 on μ was not investigated in[3].

6. PHASE FLUCTUATIONS. LINE WIDTH.

We now consider phase fluctuations. It follows from Eq. (2.7) that for arbitrary noise intensity and for sufficiently small values of the amplitude the condition for gradual change of phase no longer applies. Such a structure for the equation for phase makes the mean square value of the phase accumulation $\langle(\varphi(t+\tau)-\varphi(t))^2\rangle$ proportional to the quantity $\langle 1/E^2\rangle$, which is determined from the divergence integral. In[19] the quantity $\langle 1/E^2\rangle$ is replaced by the finite value of $1/\langle E^2\rangle$. Of course, such a procedure has no foundation, particularly near threshold. However, it evidently makes it possible to describe the phase fluctuations qualitatively even at small excesses. This conclusion is based on the fact that the result depends weakly on the means of regularizing the divergence integral. For example, it is easy to show that replacing $\langle 1/E^2\rangle$ by $\langle 1/E\rangle^2$ gives very nearly the same result.

Proceeding in this fashion, we obtain from (2.7) the well-known results[19,20] for the traveling wave. In particular, the phase buildup after a sufficiently long time equals

$$\langle\delta\varphi_\tau^2\rangle = 2D\tau, \quad D = \frac{\omega_0^2}{2\langle E_0^2\rangle}\langle\xi_2^2\rangle_\omega. \tag{6.1}$$

If $\eta \gg N_0$, then

$$D = \frac{\omega_0^2 V\Delta\omega_r}{16\pi P}N_\omega(P). \tag{6.2}$$

And if $\eta \ll N_0$, then, using (4.14), we obtain

$$D = \frac{N_0\omega_0 d}{4\sqrt{2/\pi}}\left[1-\sqrt{\frac{\pi}{2}}\left(1-\frac{2}{\pi}\right)\frac{\eta}{N_0}\right]. \tag{6.3}$$

For the standing wave, because of the dependence of the equations for amplitude and phase, the phase diffusion coefficient is, for $\eta \gg N_0$

$$D = \frac{\omega_0^2 V\Delta\omega_r}{16\pi P}\left[1+\mu^2\left(\frac{F-2\pi f^2}{1+F+4\mu^2 f^2}\right)^2\right]N_\omega(P); \tag{6.4}$$

for $\eta \ll N_0$

$$D \approx \frac{N_0\omega_0 d}{8\sqrt{2/\pi}}\frac{2+\mu^2}{1+\mu^2}\left\{1+\frac{\mu^2}{(\mu^2+2)^2}\frac{2}{\pi}\left(1-\frac{2\pi\sqrt{\pi}}{\Gamma^2(1/4)}\right)\right.$$
$$\left.-\left[\sqrt{\frac{\pi}{2}}\left(1-\frac{2}{\pi}\right)-\sqrt{\frac{2}{\pi}}\frac{\mu^2}{(\mu^2+2)^2}\left(1-\frac{2}{\pi}+\frac{8\sqrt{\pi}}{\Gamma^2(1/4)}-\sqrt{2}\right)\right]\frac{\eta}{N_0}\right\}. \tag{6.5}$$

Using (6.2), (6.4), (3.17), and (2.15), we write an explicit expression for D as a function of power for $N_0 \ll \eta \ll 1$ and for $\eta \gg 1$:

$$D = \frac{\hbar\omega_0^3 d^2}{4P}\left[1+\frac{1-\alpha}{\alpha(\mu^2+2)}\right]\left\{\bar{n}+\frac{1}{2}+\frac{1}{2}Z\right.$$
$$\left.-\alpha\left(\bar{n}+\frac{1}{2}+\frac{1}{4}Z+\frac{1}{4}\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\right)\frac{8\pi a P}{V\omega_0 d}\right\}, \quad N_0 \ll \eta \ll 1; \tag{6.6}$$

$$D = \frac{\hbar\omega_0^3 d^2}{32P}\beta\left(Z-\frac{1}{2}\frac{\gamma_b-\gamma_a}{\gamma_b+\gamma_a}\right), \quad \eta \gg 1. \tag{6.7}$$

It follows from (6.6) and (6.7) that in the region $N_0 \ll \eta \ll 1$, the quantity D decreases with increasing power faster than $1/P$, and when $\eta \gg 1$, as $1/P$.

It is interesting that the character of the variation of D at high powers depends significantly on the experimental conditions. In fact, upon a change of power due to the pump with unchanged $\Delta\omega_r$, we obtain the following expression for D from (6.2), (6.4), and (3.18):

$$D = \frac{\pi\hbar\omega_0\Delta\omega_r a}{8V}\beta\frac{3\gamma_a+\gamma_b}{\gamma_a+\gamma_b}, \quad \eta \gg 1. \tag{6.8}$$

Thus, in this case the phase diffusion coefficient tends to a constant value with increasing power.

In the usual way (see[19,20]) we obtain for the field spectrum in the laser

$$\langle E^2\rangle_\omega = \langle E_0\rangle^2\frac{D}{(\omega_0-\omega)^2+D^2} + \frac{1}{2}\langle\delta E\rangle^2\frac{2D+\Delta\omega_a}{(\omega_0-\omega)^2+(2D+\Delta\omega_a)^2/4}. \tag{6.9}$$

The first term here is determined by the fluctuations of the phase accumulation and is a narrow line with half-width $\Delta\omega_{ph} = D$ and intensity $\langle E_0\rangle^2/2$; the second term is determined by the amplitude fluctuations and has half-width $\Delta\omega_A = D + \Delta\omega_a/2$. The intensity of the second term is small and equals $\langle\delta E^2\rangle/2$. Since at any excess we have $\langle\delta E^2\rangle < \langle E_0\rangle^2$, the total width of the spectral line of emission is determined by the width of the phase fluctuations, i.e., $\Delta\omega_{ph}$.

Note that in the experimental papers an expression for the spectral density of the frequency fluctuations W_ν is used instead of $\Delta\omega_{ph}$. These two quantities are connected by the relation

$$\Delta\omega_{ph} = \pi^2 W_\nu.$$

Thus, the formulas we have given for D determine the width of the line due to phase fluctuations. Let us first compare these results with Eq. (1.1). To do this, we rewrite (6.6) in the zeroth approximation with respect to field:

$$D = \frac{\hbar\omega_0(\Delta\omega_p)^2}{4P}\left[\bar{n}+\frac{1}{2}+\frac{1}{2}\frac{\rho_a^{(0)}+\rho_b^{(0)}}{\rho_a^{(0)}-\rho_b^{(0)}}\right]\left(1+\frac{1-\alpha}{\alpha(\mu^2+2)}\right), \quad N_0 \ll \eta \ll 1. \tag{6.10}$$

Here we have substituted in the value of Z and taken into account that when $\eta \ll 1$ we have $\omega_0 d = \Delta\omega_r$. It follows from (6.10) that Eq. (1.1) does not take into account the dependence on detuning for the standing wave regime even in the zeroth approximation with respect to η (for $\eta \gg N_0$).

We now compare our results with experiment. Figure 2 shows three curves: curve 1 is drawn from the experimental data of Zaĭtsev and Stepanov,[5] curve 2 is constructed from the theoretical data of[5],1) curve 3 is the hyperbola that results from (6.7) for high powers. The parameter of the hyperbola was chosen so as to superpose it on the experimental point at the highest value of the power. It is seen from these curves that at the lowest values of P, there is a deviation from the hyperbolic dependence of just the character predicted by (6.4). The insufficiency of experimental data, particularly in the region of low powers (for $\eta \ll 1$) does not permit us to determine the necessary parameters and construct the theoretical curve for all values of power.

1)Unfortunately, the authors of [5] do not indicate how they obtained the results of the theory based on the formula they give for $\Delta\omega_{ph}$.

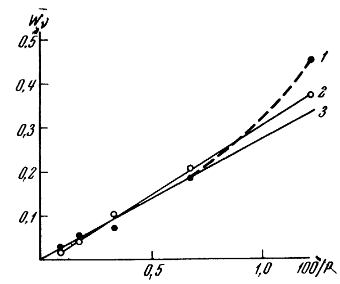

FIG. 2

In conclusion, we estimate the limiting value of the line width near the threshold of excitation. Setting $\eta = 0$, $\mu = 0$ in (6.5) and using (4.6), (4.8), and (4.9), we obtain

$$\frac{\Delta\omega_{ph}}{\Delta\omega_r} = \frac{\pi}{2\sqrt{2}}\left[\frac{\hbar}{V}\omega_0 a\left(\bar{n}+\frac{1}{2}+\frac{1}{2}Z\right)\right]^{1/2}.$$

From a comparison of the theoretical and experimental data at relatively high powers, we can estimate Z, which turns out to be of the order of 4 to 10. Setting $\gamma_a \approx \gamma_b \approx \gamma_{ab} \sim 10^8$ sec^{-1}, $\omega_0 \approx 3 \times 10^{15}$ sec^{-1}, we find $a \sim 10^2$ (in CGSE units) and $\Delta\omega_{ph}/\Delta\omega_r \sim 10^{-4}$, i.e., when $\Delta\nu_r = 10^7$ Hz, $\Delta\nu_{ph} \sim 1$ kHz. These estimates agree with the experimental data of[4].

The results of this paper were discussed with I. L. Bershteĭn, I. A. Andronova, and Yu. I. Zaĭtsev. We take this opportunity to thank them.

Note added in proof (December 2, 1968).– It is of interest to determine the magnitudes of the coefficients of the diffusion of phase D of each of the opposite waves and of the diffusion of phase difference D_Φ. These coefficients characterize the spectral density of the fluctuactions in the frequency of each of the waves and in the difference between the frequencies of the opposite waves:

$$D = \frac{\omega_0^2 N_\omega}{2\langle E_0^2\rangle}\left\{1+\frac{\mu^2[(\mu^2+1)^2+1]}{(\mu^2+1)^3(\mu^2+2)}\left[\frac{\mu^2}{\mu^2+1}-\frac{\gamma^2}{(ku)^2}\right]^{-1}\right\}, \quad \mu \geqslant \frac{\gamma}{ku},$$

$$D_\Phi = \frac{\omega_0^2 N_\omega}{\langle E_0^2\rangle}\left\{1+\frac{\mu^2}{4(\mu^2+1)(\mu^2+2)}\left[\frac{\mu^2}{\mu^2+1}-\frac{\gamma^2}{(ku)^2}\right]\right\}.$$

From this it is seen that the magnitude of D increases without limit as the stability boundary is approached (see [17]), when $\mu = \gamma/ku$. But the magnitude of D_Φ is finite at all values.

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Translated by L. M. Matarrese
37